Meromorphic solutions to the q-Painlevé equations around the origin

Yousuke Ohyama

Osaka University, Japan

17:00-17:25, Tuesday 15 July

XXXth International Colloquium on Group Theoretical Methods in Physics

q-Painlevé equations: q-P_{VI}, q-P_{V}, q-P_{III}

$y = y(t), z = z(t),\ \bar{y} = y(qt), \bar{z} = z(qt).$

$a_1, a_2, a_3, a_4, b_1,\ b_2,\ b_3,\ b_4$: parameter

$$q\text{-}P_{\mathrm{VI}} : \frac{y\bar{y}}{a_3 a_4} = \frac{(\bar{z} - b_1 t)(\bar{z} - b_2 t)}{(\bar{z} - b_3)(\bar{z} - b_4)}, \quad \frac{z\bar{z}}{b_3 b_4} = \frac{(y - a_1 t)(y - a_2 t)}{(y - a_3)(y - a_4)},$$

$$\frac{b_1 b_2}{b_3 b_4} = q \frac{a_1 a_2}{a_3 a_4}.$$

$$q\text{-}P_{\mathrm{V}} : \frac{y\bar{y}}{a_3 a_4} = -\frac{(\bar{z} - b_1 t)(\bar{z} - b_2 t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{(y - a_1 t)(y - a_2 t)}{a_4 (y - a_3)},$$

$$\frac{b_1 b_2}{b_3} = q \frac{a_1 a_2}{a_3 a_4}.$$

$$q\text{-}P_{\mathrm{III}} : \frac{y\bar{y}}{a_3 a_4} = -\frac{\bar{z}(\bar{z} - b_2 t)}{\bar{z} - b_3}, \quad \frac{z\bar{z}}{b_3} = -\frac{y(y - a_1 t)}{a_4 (y - a_3)}.$$

Taking a **limit** $q \to 1$, q-Painlevé equations go to differential Painlevé equations.

[Classical case]

In general, solutions to the Painlevé equations has a **wild singularity** at $x = 0$, but it is known that

- There exist special Painlevé transcendents with **tempered orders** at the fixed singularity
- Such tempered Painlevé transcendents **play important roles in physics**
- We can **determine monodromy/Stokes data** for tempered Painlevé transcendents

[q-analogue]

We consider **q-analogue of tempered Painlevé transcendents** for the q-cases.

Basic notations

0) **q-shifted factorial**: **(Pochhammer never used!)**

$$(a_1, \ldots, a_r; q)_n = \prod_{i=1}^{n} (a_i; q)_n, \quad (a; q)_n = (1-a)(1-qa)\cdots(1-q^{n-1}a).$$

1) **generalized q-hypergeometric series**:

$$\begin{aligned} & {}_r\varphi_s\left(a_1, \ldots, a_r; b_1, \ldots, b_s; q, z\right) \\ & \qquad = \sum_{n=0}^{\infty} \frac{(a_1, \ldots, a_r; q)_n}{(b_1, \ldots, b_s; q)_n (q; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n. \end{aligned}$$

2) **Theta function**:

$$\theta_q(x) := \sum_{n=-\infty}^{\infty} q^{n(n-1)/2} x^n = (q, -x, -q/x; q)_\infty.$$

$$x\theta_q(xq) = \theta_q(x).$$

We set

$$e_c(x) := \frac{\theta(x)}{\theta(cx)},$$

for $c \in \mathbb{C}^\times$. We have

$$\theta(xq) = \frac{1}{x}\theta(x), \qquad \theta(1/x) = \frac{1}{x}\theta(x).$$

$$e_c(xq) = ce_c(x).$$

The reason why we use $e_c(x)$:
The first order q-difference equation

$$u(xq) = cu(x)$$

has many solutions:

- $u = e_c(x)$. single-valued, but has poles $x = -q^{\mathbb{Z}}/c$
- $u = x^\gamma$, where $c = q^\gamma$. no poles but multi-valued

It depends on the constant d: $u(xe^{2\pi i}) = du(x)$

We consider single-valued functions, i.e. $d = 1$.

***q*-difference equation** of size r:

$$Y(qx) = A(x)Y(x),$$

$$A(x) = A_0 + xA_1 + \cdots + x^N A_N$$

We assume that the **eigenvalues** of A_0 is $\rho_1, ..., \rho_r (\neq 0)$ **[regular singular]** and

$$\rho_j/\rho_k \notin q^{\mathbb{Z}}$$

if $j \neq k$. We also assume that

$$A_N = \begin{bmatrix} \kappa_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \kappa_r \end{bmatrix}.$$

$\kappa_1, ..., \kappa_r (\neq 0)$ and $\kappa_j/\kappa_k \notin q^{\mathbb{Z}}$ $(j \neq k)$.

Local solutions and the Connection

A standard form of **local solutions** around $x = 0$ and $x = \infty$:

$$Y_0(x) = L(x) \begin{bmatrix} e_{\rho_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\rho_r}(x) \end{bmatrix},$$

$$Y_\infty(x) = \theta(x)^{-N} R(x) \begin{bmatrix} e_{\kappa_1}(x) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e_{\kappa_r}(x) \end{bmatrix}.$$

Here

$$L(x) = \sum_{n=0}^{\infty} L_n x^n, \qquad R(x) = \sum_{n=0}^{\infty} R_n x^{-n}$$

We assume that $R_0 = I_r$ and $\det L_0 = 1$.

Definition. We define the **connection matrix** $P(x)$ as

$$Y_\infty(x) = Y_0(x) P(x).$$

$\star$ In the q-difference case, $P(x)$ **is an elliptic function**, not constants.

Examples of connections

In general, it is difficult to determine connection matrices $P(x)$.

Basic hypergeometric series ${}_2\varphi_1(a, b; c; x)$:

$$(c - abqx)u(xq^2) - [c + q - (a + b)qx]u(qx) + q(1 - x)u(x) = 0.$$

Local solutions around $x = 0$:

$$u_1 = {}_2\varphi_1\left(a, b; c; x\right), \quad u_2 = e_{q/c}(x){}_2\varphi_1\left(qa/c, qb/c; q^2/c; x\right).$$

Local solutions around $x = \infty$:

$$v_1 = \frac{1}{e_a(-x)}{}_2\varphi_1\left(a, aq/c; aq/b; cq/abx\right), \ v_2 = (a \leftrightarrow b).$$

Theorem (Watson, 1911) Connection formula for ${}_2\varphi_1$:

$$u_1 = \frac{(b, c/a; q)_\infty}{(c, b/a; q)_\infty} v_1 + \frac{(a, c/b; q)_\infty}{(c, a/b; q)_\infty} v_2,$$

$$u_2 = \frac{(qb/c, q/a; q)_\infty}{(q^2/c, b/a; q)_\infty} \frac{e_a(-x)e_{q/c}(x)}{e_{qa/c}(-x)} v_1 + \frac{(qa/c, q/b; q)_\infty}{(q^2/c, a/b; q)_\infty} \frac{e_b(-x)e_{q/c}(x)}{e_{qb/c}(-x)} v_2.$$

q-confluent hypergeometric equation

When **some exponents ρ_j or κ_i are zero**, it is **irregular singular**. In cases of confluence of ${}_{r+1}\varphi_r$, we may determine the **Stokes coefficients** (Zhang, Morita).

q-confluent hypergeometric equation

$$(1-abqx)u(q^2x)-\{1-(a+b)qx\}\,u(qx)-qxu(x)=0.$$

Local solutions around $x=0$:

$$u_1(x)={}_2\varphi_0(a,b;-;q,x),\quad u_2(x)=\frac{(abx;q)_\infty}{\theta(-qx)}{}_2\varphi_1\left(\frac{q}{a},\frac{q}{b};0;q,abx\right)$$

Local solutions around $x=\infty$:

$$v_1(\lambda,x)=\frac{\theta\left(\frac{qax}{\lambda}\right)}{\theta\left(\frac{qx}{\lambda}\right)}\,{}_2\varphi_1\left(a,0;\frac{aq}{b};q,\frac{q}{abx}\right),\quad v_2(\lambda,x)=(a\leftrightarrow b)$$

q-Stokes coefficients

$x = 0$ is an **irregular singular point** and u_1 is a divergent series.
We set $\tilde{u}_1(\lambda, x)$ is a resummation of u_1 on $x \in \mathbb{C}^* \setminus (-\lambda)q^{\mathbb{Z}}$.

Theorem ***q*-Stokes coefficients** are given by as follows:

$$\tilde{u}_1(\lambda, x) = \frac{(b;q)_\infty}{\left(\frac{b}{a};q\right)_\infty}\frac{\theta(a\lambda)}{\theta(\lambda)}v_1(\lambda, x) + \frac{(a;q)_\infty}{\left(\frac{a}{b};q\right)_\infty}\frac{\theta(b\lambda)}{\theta(\lambda)}v_2(\lambda, x),$$

$$u_2(x) = \frac{(q/a;q)_\infty}{(b/a;q)_\infty}v_1(-1, x) + \frac{(q/b;q)_\infty}{(a/b;q)_\infty}v_2(-1, x).$$

Remark.
The first formula is given by Zhang (2002), the second formula is given by Morita (2013).
In the q-analogue case, some solutions $(u_2(x))$ may be **convergent** even at an irregular singular point.
The Stokes region is an **open dense** subset in $\mathbb{C}^*$ (outside of a union of ***q*-spirals** $(\lambda_j)q^{\mathbb{Z}}$).

Meromorphic solutions to q-P_{VI}, q-P_{V} and q-P_{III}

Theorem 1. For generic parameters, all of meromorphic solutions to q-P_{VI}, q-P_{V} and q-P_{III} around the origin are holomorphic.

Theorem 2. For generic parameters, there exit **four holomorphic solutions** to q-P_{VI} around the origin:

$$\mathrm{I)}\quad y(t)=\frac{a_3b_3-a_4b_4}{b_3-b_4}+O(t),\ z(t)=\frac{a_3b_3-a_4b_4}{a_3-a_4}+O(t),$$

$$\mathrm{II)}\quad y(t)=\frac{a_4b_3-a_3b_4}{b_3-b_4}+O(t),\ z(t)=\frac{a_3b_4-a_4b_3}{a_3-a_4}+O(t),$$

$$\mathrm{III)}\quad y(t)=\frac{a_1a_2(b_1-b_2)}{a_2b_1-a_1b_2}t+O(t^2),\quad z(t)=-\frac{b_1b_2(a_1-a_2)}{(a_2b_1-a_1b_2)q}t+O(t^2),$$

$$\mathrm{IV)}\quad y(t)=\frac{a_1a_2(b_1-b_2)}{a_1b_1-a_2b_2}t+O(t^2),\quad z(t)=\frac{b_1b_2(a_1-a_2)}{(a_1b_1-a_2b_2)q}t+O(t^2).$$

Theorem 3. There exit **three holomorphic solutions** to q-P_{V} around the origin

$$\text{I)} \quad y(t) = (a_3 - a_4 b_3) + O(t), \qquad z(t) = \left(b_3 - \frac{a_3}{a_4}\right) + O(t),$$

$$\text{II)} \quad y(t) = \frac{a_1 a_2 (b_1 - b_2)}{a_2 b_1 - a_1 b_2} t + O(t^2), \quad z(t) = \frac{b_1 b_2 (a_1 - a_2)}{(a_1 b_2 - a_2 b_1) q} t + O(t^2),$$

$$\text{III)} \quad y(t) = \frac{a_1 a_2 (b_1 - b_2)}{a_1 b_1 - a_2 b_2} t + O(t^2), \quad z(t) = \frac{b_1 b_2 (a_1 - a_2)}{(a_1 b_1 - a_2 b_2) q} t + O(t^2).$$

Theorem 4. There exit **two holomorphic solutions** to q-P_{III} around the origin:

$$\text{I)} \quad y(t) = (a_3 - a_4 b_3) + O(t), \qquad z(t) = \left(b_3 - \frac{a_3}{a_4}\right) + O(t),$$

$$\text{II)} \quad y(t) = \frac{a_1 a_3 a_4 b_2^2}{a_3 a_4 b_2^2 - a_1^2 b_3 q} t + O(t^2), \quad z(t) = \frac{a_1^2 b_2 b_3}{-a_3 a_4 b_2^2 + a_1^2 b_3 q} t + O(t^2).$$

q-Briot-Bouquet Theorem

Theorem 5. (Poincaré) For $f_j(t;y) = f_j(t, y_1, y_2, ..., y_n)$ $(j = 1, 2, ..., n)$, we assume that

∘ f_j is holomorphic around $(t, y) = (0, 0)$ and $f_j(0; 0) = 0$

∘ the eigenvalues of matrix

$$\left(\frac{\partial f_j}{\partial y_k}(0;0)\right)_{1 \leqq j,k \leqq n}$$

are not q^n $(n = 1, 2, 3, ...)$.

Then q-difference equation

$$y_j(qt) = f_j(t;y) \qquad (j = 1, 2, ..., n)$$

has a **unique convergent solution** of the type

$$y_j = \sum_{k=1}^{\infty} a_{jk} t^k \qquad (j = 1, 2, ..., n).$$

Proof of Theorem 1.2.3.4.

1) By direct calculations, we can easily checked that q-P_{J} (J=III, V, VI) has formal solutions shown in Theorem 2.3.4.

2) We can show the formal solutions are **convergent by q-Briot-Bouquet theorem**.

Remark.

In differential cases, P_{VI}, P_{V} and P_{III} have **the same number of meromorphic solutions** around the origin as in Theorem 2,3,4,

Connection Preserving Deformation

CPD : by Jimbo-Sakai, M. Murata

$$Y(qx,t) = A(x,t)Y(x,t),$$
$$Y(x,qt) = B(x,t)Y(x,t).$$

$$A(x,t) = A_0(t) + xA_1(t) + x^2A_2,$$

$$B(x,t) = \frac{x}{(x-a_1qt)(x-a_2qt)}(xI + B_0(t)), \qquad q\text{-}P_{\mathrm{VI}}, q\text{-}P_{\mathrm{V}}$$
$$= \frac{x}{x-a_1qt}(xI + B_0(t)), \qquad q\text{-}P_{\mathrm{III}}$$

auxiliary variable $y = y(t)$, $z_i = z_i(t)$ $(i = 1,2)$:

$$A_{12}(y,t) = 0, \quad A_{11}(y,t) = \kappa_1 z_1, \quad A_{22}(y,t) = z_2,$$

depending variable z:

$$z_2 = \kappa_1\kappa_2 qz(y - a_3).$$

q-$P_{\mathbf{VI}}$

Normalization: $A_2 = \mathrm{diag}(\kappa_1, \kappa_2)$,
Eigenvalues of $A_0(t)$: $\rho_1 = \theta_1 t, \rho_2 = \theta_2 t$,
$\det A(x,t) = \kappa_1\kappa_2(x - a_1 t)(x - a_2 t)(x - a_3)(x - a_4)$.

$$\kappa_1\kappa_2 a_1 a_2 a_3 a_4 = \theta_1\theta_2.$$

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha)+z_1) & \kappa_2 w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2((x-y)(x-\beta)+z_2) \end{pmatrix}.$$

Here

$$\alpha = \frac{[y^{-1}((\rho_1+\rho_2)t - \kappa_1 z_1 - \kappa_2 z_2) - \kappa_2((a_1+a_2)t + a_3 + a_4 - 2y)]}{\kappa_1 - \kappa_2},$$

$$\beta = \frac{[-y^{-1}((\rho_1+\rho_2)t - \kappa_1 z_1 - \kappa_2 z_2) + \kappa_1((a_1+a_2)t + a_3 + a_4 - 2y)]}{\kappa_1 - \kappa_2},$$

$$\gamma = z_1 + z_2 + (y+\alpha)(y+\beta) + (\alpha+\beta)y - a_1 a_2 t^2 - (a_1+a_2)(a_3+a_4)t - a_3 a_4,$$

$$\delta = y^{-1}(a_1 a_2 a_3 a_4 t^2 - (\alpha y + z_1)(\beta y + z_2)),$$

$$b_1 = \frac{a_1 a_2}{\rho_1}, \quad b_2 = \frac{a_1 a_2}{\rho_2}, \quad b_3 = \frac{1}{\kappa_1 q}, \quad b_4 = \frac{1}{\kappa_2}.$$

q-P_{V}

Normalization: $A_2 = \mathrm{diag}(\kappa_1, 0)$,
Eigenvalues of $A_0(t)$: $\rho_1 = \theta_1 t, \rho_2 = \theta_2 t$,
$\det A(x,t) = \kappa_1\kappa_2(x - a_1 t)(x - a_2 t)(x - a_3)$.

$$-\kappa_1\kappa_2 a_1 a_2 a_3 = \theta_1\theta_2.$$

$$z_1 z_2 = \kappa_2(y - a_1 t)(y - a_2 t)(y - a_3).$$

Therefore z_1 is represented by z:

$$z_1 = \frac{(y - a_1 t)(y - a_2 t)}{\kappa_1 q z}$$

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha) + z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x-y) + z_2 \end{pmatrix}.$$

$$\alpha = \frac{1}{\kappa_1}[y^{-1}((\theta_1 + \theta_2)t - \kappa_1 z_1 - z_2) + \kappa_2],$$

$$\gamma = z_2 - \kappa_2((2y + \alpha) - (a_1 + a_2)t - a_3),$$

$$\delta = y^{-1}(-\kappa_2 a_1 a_2 a_3 t^2 - (\alpha y + z_1)(-\kappa_2 y + z_2)).$$

q-P_{III}

Normalization: $A_2 = \mathrm{diag}(\kappa_1, 0)$,
Eigenvalues of $A_0(t)$: $\theta_1 t, 0$,
$\det A(x,t) = \kappa_1\kappa_2 x(x - a_1 t)(x - a_3)$.

$$-\kappa_1\kappa_2 a_1 a_2 a_3 = \theta_1\theta_2.$$

$$z_1 z_2 = \kappa_2 (y - a_1 t)(y - a_2 t).$$

Therefore z_1 is represented by z:

$$z_1 = \frac{y(y - a_1 t)}{\kappa_1 q z}.$$

$$A(x,t) = \begin{pmatrix} \kappa_1((x-y)(x-\alpha) + z_1) & w(x-y) \\ \kappa_1 w^{-1}(\gamma x + \delta) & \kappa_2(x-y) + z_2 \end{pmatrix}.$$

Here

$$\alpha = \frac{1}{\kappa_1}[y^{-1}(\theta_1 t - \kappa_1 z_1 - z_2) + \kappa_2],$$
$$\gamma = z_2 - \kappa_2(2y + \alpha - t(a_1 + a_2)),$$
$$\delta = -y^{-1}(\alpha y + z_1)(-\kappa_2 y + z_2).$$

Theorem

For **solutions (I-IV) to q-P_{VI} and the solution (I) of q-P_{V}**, $A(x,0)$ reduces to basic hypergeometric function ${}_2\varphi_1(t)$.

For **solutions (I, II) of q-P_{III} and for solutions (II),(III) of q-P_{V}** , $A(x,0)$ reduces to basic hypergeometric function ${}_1\varphi_1(t)$.

In this sense, we can **determine the connection problem of linearized q-Painlevé equations**. (q-analogue of [KO])

${}_2\varphi_1(t)$: Watson's formula (1910)

${}_1\varphi_1(t)$: Zhang's and Morita's formula (2003,2013),

$$
{}_2\varphi_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} x^n,
$$

$$
{}_1\varphi_1(a,-,c;x) = \sum_{n=0}^{\infty} \frac{(a;q)_n}{(c;q)_n (q;q)_n} \left(-q^{n(n-1)/2}\right) x^n,
$$

Here $(a;q)_n = (a)_n = \prod_{j=0}^{n-1}(1-aq^j)$. And $(a;q)_\infty = (a)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$.

We explain the cases of q-$P_{\rm V}$ **(I) and (II)**

q-$P_{\rm V}$ (I)

For

$$Y(qx) = A(x,t)Y(x),$$

we set $x = \xi t$. Then

$$Y(q\xi t) = (A_0 + A_1\xi t + t^2 A_2 \xi^2)Y(\xi t).$$

We set $Y(\xi t) = \xi^{\mathrm{lq}\, t} Z(\xi)$. (Here $\mathrm{lq}\, t = \log t / \log q$)

$$Z(q\xi) = (A_0/t + A_1\xi + tA_2\xi^2)Z(\xi)$$

Taking the limit $t \to 0$, we have (Eigenvalues of $A_0(t)$ are $\theta_1 t, \theta_2 t$)

$$Z(q\xi) = (A_0/t + A_1\xi)Z(\xi)$$

$$A_0/t = \begin{pmatrix} \frac{a_1 a_2 (b_1+b_2)}{b_1 b_2} & a_3 \left(\frac{{a_4}^2 b_1 b_2}{a_1 a_2 q} - 1 \right) w'(0) \\ \frac{{a_1}^3 {a_2}^3 q}{a_3 b_1 b_2 (a_1 a_2 q - a_4^2 b_1 b_2) w'(0)} & 0 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} -\frac{a_1 a_2}{a_4 b_1 b_2} & 0 \\ \frac{a_1^2 a_2^2 (a_1 + a_2 - a_4 b_1 - a_4 b_2) q}{a_3 b_1 b_2 (a_1 a_2 q - a_4^2 b_1 b_2) w'(0)} & -a_4 \end{pmatrix}$$

Therefore it reduces to hypergeometric: ${}_2\varphi_1 \left(\frac{a_4 b_2}{a_2}, \frac{a_1 q}{a_4 b_1}; \frac{b_2 q}{b_1}; \frac{\xi}{a_1} \right)$

q-P_{V} (II)

$$A(x,0)=\tilde{A}_1x+\tilde{A}_2x^2,$$

Here $\tilde{A}_1$ is

$$\begin{pmatrix} \frac{a_3b_1b_2+a_4b_1b_2-a_2b_1-a_1b_2}{b_1b_2(-b_3q)} & w(0) \\ \frac{(a_1-a_4b_1)(a_4b_2-a_2)(a_4b_1-a_1b_3q)(a_4b_2-a_2b_3q)}{a_4^2b_1^2b_2^2(1-b_3q)^2w(0)} & \frac{a_3a_4b_2^2+a_2^2b_3q-a_2a_3b_2-a_2a_4b_2}{a_2b_2(1-b_3q)} \end{pmatrix}$$

$$\tilde{A}_2=A_2=\mathrm{diag}\,(\kappa_1,0)$$

Solution to $Y(qx)=A(x,0)Y(x)$ around $x=0$ are given by

$$Y^{(0)}(x)=\frac{q^{u(u-1)/2}}{(x/a_3)_\infty}\begin{pmatrix} y_{11}^{(0)} & y_{12}^{(0)} \\ y_{21}^{(0)} & y_{22}^{(0)} \end{pmatrix}x^{D_0}$$

is represented by

$$
\begin{aligned}
y_{11}^{(0)} &= C_{11} \cdot {}_1\varphi_1\left(\frac{a_4 b_2}{a_2 b_3}, -, \frac{a_1 b_2}{a_2 b_1} q; \frac{x}{a_4}\right), \\
y_{12}^{(0)} &= C_{12} \cdot {}_1\varphi_1\left(\frac{a_4 b_1}{a_1 b_3}, -, \frac{a_2 b_1}{a_1 b_2} q; \frac{x}{a_4}\right), \\
y_{21}^{(0)} &= C_{21} \cdot {}_1\varphi_1\left(\frac{a_1 b_4}{a_3 b_1}, -, \frac{a_1 b_2}{a_2 b_1} q; \frac{x}{a_4}\right), \\
y_{22}^{(0)} &= C_{22} \cdot {}_1\varphi_1\left(\frac{a_2 b_4}{a_3 b_2}, -, \frac{a_2 b_1}{a_1 b_2} q; \frac{x}{a_4}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
C_{11} &= b_2(a_1 a_2 - a_3 a_4 b_1 b_2), \\
C_{12} &= b_1(a_1 a_2^2 - a_3 a_4 b_1 b_2) w(0), \\
C_{21} &= a_1(a_2 - a_3 b_2)(a_2 - a_4 b_2)/w(0), \\
C_{22} &= a_2(a_1 - a_3 b_1)(a_1 - a_4 b_1).
\end{aligned}
$$

which reduced to ***q*-confluent hypergeometric** equations.

Summary

- q-P_{VI}, q-P_{V}, q-P_{III} has a singular point of **q-Briot-Bouquet type** at $x = 0$.
- For q-Painlevé equations, there exist **finite number of holomorphic solutions** around singular points of q-Briot-Bouquet type.
- For such solutions, we can **determine connection/Stokes coefficients** of linear difference equations.

Future problems

- How about other q-Painlevé equations? Higher order cases?
- How about singular points of **non q-Briot-Bouquet type**? (**q-Boutroux solution**?)
- How about generic solutions (**two parameter solutions**)?

Search "ohyama home"

References

[JS] Jimbo, M.; Sakai, H., A q-analog of the sixth Painleve equation. *Lett. Math. Phys.* **38** (1996), 145–154.

[KO] K. Kaneko and Y. Ohyama, Meromorphic Painlevé transcendents at a fixed singularity, *Math. Nachr.* **286** (2013), 861–875.

[Mo] Morita, T., A connection formula for the q-confluent hypergeometric function, *SIGMA* **9** (2013), 050, 13 pages.

[Mu] Murata, M,, Lax forms of the q-Painlevé equations, *J. Phys. A: Math. Theor.* **42** (2009), 115201.

[W] Watson, G. N., The continuation of functions defined by generalized hypergeometric series, *Trans. Cambridge Phil. Soc.*, **21** (1910), 281–299.

[Z] Zhang, C., Une sommation discrete pour des equations aux q-differences lineaires et a coefficients analytiques: theorie generale et exemples. *Differential equations and the Stokes phenomenon*, 309–329, World Sci. Publ., 2002.