Particular solutions of q-Painlevé equations and q-hypergeometric equations

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Abstract. We study a degeneration diagram of linear q-difference equations of hypergeometric type, which are second order q-difference equations whose coefficients are linear functions. We obtain seven q-hypergeometric equations, including two types of q-Bessel equations and two types of q-Airy functions. We explain how our degeneration scheme corresponds to a degeneration diagram of q-Painlevé equations.

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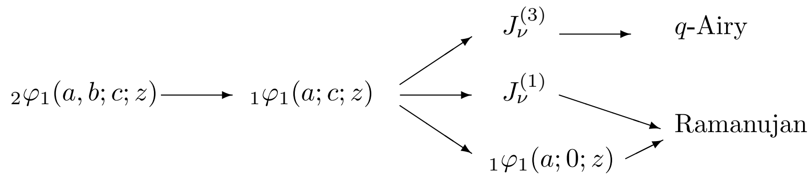
1. Introduction

We give a unified theory for q-special functions, which come from degeneration of the basic hypergeometric functions ${}_2\varphi_1(a, b, c; q; x)$. We obtain seven types of q-special functions. We have two different the q-Bessel functions. We also have two q-Airy equations, which are essentially equivalent.



We also show that a relation to hypergeometric type of q-Painlevé equations and our classification of q-special functions. See [6] for details.

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2. q-difference equation of the hypergeometric type

We call a q-difference equation of the second order with the linear coefficients

$$(a_0 + b_0 x)u(xq^2) + (a_1 + b_1 x)u(xq) + (a_2 + b_2 x)u(x) = 0 \tag{2.1}$$

it the hypergeometric type. We denote the solution space of the above equation as

$$\Phi \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix}; x \Bigg]$$

following [1]. We set $\mathrm{lq}\, x = \log x / \log q$.

Theorem 2.1. *A q-difference equation of the hypergeometric type has transformations which keep the hypergeometric type:*

(A) Change $x \to cx$:

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; cx\right] = \Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ cb_1 & cb_2 & cb_3 \end{matrix}; x\right]$$

(B) Change $u \to x^\gamma u$ ($c = q^\gamma$)

$$x^\gamma\ \Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; x\right] = \Phi \left[\begin{matrix} c^2 a_1 & ca_2 & a_3 \\ c^2 b_1 & cb_2 & b_3 \end{matrix}; x\right]$$

(C) Change $x \to 1/x$

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; \frac{1}{x}\right] = \Phi \left[\begin{matrix} b_3 & b_2 & b_1 \\ q^2 a_3 & q^2 a_2 & a_1 \end{matrix}; x\right]$$

(D) Change $u \to (ax;q)_\infty/(bx;q)_\infty u$:

$$\Phi \left[\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{matrix}; x\right] = x^s \frac{(-b_3 x/a_3;q)_\infty}{(-b_1 x/a_1 q;q)_\infty}\ \Phi \left[\begin{matrix} a_3 & a_2 & a_1 \\ qb_3 & b_2 & q^{-1} b_1 \end{matrix}; x\right]$$

where $s = \mathrm{lq}(a_3/a_1)$.

We classify q-difference equations of the hypergeometric type up to the transformations in Theorem 1. Then we obtain seven classes of q-difference equations:

Theorem 2.2. *A q-difference equation* (2.1) *of the hypergeometric type reduces to one of the following equation by transforms in theorem* 2.1. *($p = \sqrt{q}$)*

1) When $a_1 a_3 b_1 b_3 \neq 0$, Heine's hypergeometric ${}_2\varphi_1(a,b;c;q;x)$:

$$(c - abqx)u(xq^2) - [c + q - (a+b)qx]u(qx) + q(1-x)u(x) = 0.$$

2) When $b_3 = 0$, $a_1 a_3 b_1 b_2 \neq 0$, ${}_1\varphi_1(a;c;q;x)$:

$$(c - aqx)u(xq^2) - (c + q - qx)u(qx) + qu(x) = 0.$$

3-1) When $b_1 = b_2 = 0$, $a_3 \cdot a_2 a_1 b_3 \neq 0$, Jackson's Bessel functions $J_\nu^{(1)}(x;q)$:

$$u(xp^2) - (p^\nu + p^{-\nu})u(xp) + (1 + x/4)u(x) = 0.$$

3-2) When $b_1 = b_3 = 0$, $a_2 \cdot a_3 a_1 b_2 \neq 0$, Hahn-Exton's Bessel functions $J_\nu^{(3)}(x;q)$:

$$u(xp^2) + [-(p^\nu + p^{-\nu}) + p^{2-\nu} x]u(xp) + u(x) = 0.$$

3-3) When $b_3 = a_1 = 0$, $a_2 b_2 \cdot a_3 b_1 \neq 0$, *q-Hermite-Weber* ${}_1\varphi_0(a; -; q; x)$

$$axu(xq^2) + (1-x)u(xq) - u(x) = 0.$$

4-1) When $b_1 = a_2 = b_3 = 0$, *q-Airy* $Ai_q(x) = {}_1\varphi_1(0; -q; q; -x)$ *:*

$$u(xq^2) + xu(xq) - u(x) = 0.$$

4-2) When $a_1 = b_2 = b_3 = 0$, *the Ramanujan function* ${}_0\varphi_1(-; 0; q; -tq)$*:*

$$qxu(xq^2) - u(xq) + u(x) = 0.$$

In the study of differential equations, shearing transformations are useful to study irregular singular points whose Poincaré rank is non-integer.

Shearing transformations are also useful for q-differential equations when a slope of the Newton diagram is non-integer.

For a q-difference equation

$$a(x)u(xq^2) + b(x)u(xq) + c(x)u(x) = 0,$$

a *shearing transformation* is the following transformation:

$$x = t^2,\ p = \sqrt{q},\ v(t) = u(x).$$

Then we have

$$a(t^2)v(tp^2) + b(t^2)v(tp) + c(t^2)v(t) = 0.$$

Lemma 2.3. *In Theorem* 2.2*, (4-2) is equivalent to (4.1) by shearing transformation.*

By connection formula of the q-Airy function by T. Morita, we obtain a relation between the q-Airy function and the Ramanujan function [5]:

$$A_{q^2}\left(-\frac{q^3}{x^2}\right) = -\frac{1}{(q,-1;q)_\infty}\left\{\theta\left(\frac{x}{q}\right)\mathrm{Ai}_q(-x) + \theta\left(-\frac{x}{q}\right)\mathrm{Ai}_q(x)\right\}.$$

3. Hypergeometric solutions of the q-Painlevé equations

As the same as the Painlevé differential equations have particular solutions represented by (confluent) hypergeometric functions, the q-Painlevé equations also have special solutions written by q-hypergeometric functions.

In [3], they has studies q-hypergeometric solutions of the q-Painlevé equations. The degeneration diagram of q-hypergeometric solutions of the q-Painlevé equations is as follows:

q-**P**	q-P_{VI}	$\to$	q-P_{V}	$\to$	q-P_{IV} q-P_{III}	$\to$	q-P_{II}	$\to$	q-P_{I}
HG	${}_2\varphi_1$	$\to$	${}_1\varphi_1$	$\to$	${}_1\varphi_1(a;0;z)$ ${}_1\varphi_1(0;b;z)$	$\to$	${}_1\varphi_1(0;-q;z)$	$\to$	none
	(1)	$\to$	(2)	$\to$	(3-3) (3-2)	$\to$	(4-1)	$\to$	none

Comparing our list in Theorem 2.2, we do not have (3-1) and (4-2). The equation (4-2) is related to (4-1) by a shearing transformation. The equation (3-1) appears in another form of q-$P_{\rm III}$.

It is know that there are several types of the q-Painlevé equations. For q-$P_{\rm III}$, one is called q-$P_{\rm III}(A_5^{(1)})$ by Sakai [8]:

$$\frac{y\bar{y}}{a_3a_4}=-\frac{\bar{z}(\bar{z}-b_2t)}{\bar{z}-b_3},\quad \frac{z\bar{z}}{b_3}=-\frac{y(y-a_1t)}{a_4(y-a_3)}.$$

Another one is known by Ramani, Grammaticos and Hietarinta [7]:

$$\frac{\overline{w}\underline{w}}{a_3a_4}=\frac{(w-a_1s)(w-sa_2)}{(w-a_1)(w-a_4)},\tag{3.1}$$

which is a symmetric specialization of q-$P_{\rm VI}$ found by Jimbo and Sakai [2]. And $J_\nu^{(1)}(x;q)$, a solution of (3-1), is a special solution of (3.1) shown by [4].

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