BOREL-PLUS-POWERS MONOMIAL IDEALS

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Abstract. Let $S = K[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field K. In this paper, we show that the lex-plus-powers ideal has the largest graded Betti numbers among all Borel-plus-powers monomial ideals with the same Hilbert function. In addition in the case of characteristic 0, by using this result, we prove the lex-plus-powers conjecture for graded ideals containing $x_1^p, \dots, x_n^p$, where p is a prime number.

1. Introduction

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with each $\deg x_i = 1$. In this paper, we study the graded Betti numbers of Borel-plus-powers ideals introduced by Mermin, Peeva and Stillman [11], and show the lex-plus-powers conjecture for these ideals.

A monomial ideal I of S is said to be *strongly stable* if $ux_j \in I$ implies $ux_i \in I$ for all $1 \leq i < j \leq n$. Also a monomial ideal I of S is said to be *lexsegment* if, for all monomials $u \in I$ and $v >_{\mathrm{lex}} u$ with $\deg v = \deg u$, it follows that $v \in I$, where $<_{\mathrm{lex}}$ is the degree lexicographic order induced by $x_1 > \cdots > x_n$. Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a sequence of integers or ∞ satisfying $2 \leq a_1 \leq a_2 \leq \cdots \leq a_n$. A monomial ideal I of S is said to be $\mathbf{a}$-*Borel-plus-powers* (respectively $\mathbf{a}$-*lex-plus-powers*) if there exists a strongly stable (respectively lexsegment) ideal J of S such that $I = J + (x_1^{a_1}, \dots, x_n^{a_n})$, where $x_i^{\infty} = 0$. For a finitely generated graded S-module M, let $\beta_{ij}(M)$ be the graded Betti numbers of M.

Let I be a graded ideal of S containing $x_1^{a_1}, \dots, x_n^{a_n}$. The Clements–Lindström Theorem [2] says that there exists the unique $\mathbf{a}$-lex-plus-powers ideal with the same Hilbert function as I. In [11], Mermin, Peeva and Stillman conjectured that the $\mathbf{a}$-lex-plus-powers ideal has the largest graded Betti numbers among all $\mathbf{a}$-Borel-plus-powers ideals with the same Hilbert function, and proved it in the case when $a_1 = \cdots = a_n = 2$. In this paper, we prove this conjecture, that is to say, we show

Theorem 1.1. *Let I be an $\mathbf{a}$-Borel-plus-powers ideal of S and L the $\mathbf{a}$-lex-plus-powers ideal with the same Hilbert function as I. Then $\beta_{ij}(I) \leq \beta_{ij}(L)$ for all i and j.*

The above theorem is a special case of the lex-plus-powers conjecture (see [4, 5]), which states that the $\mathbf{a}$-lex-plus-powers ideal has the largest graded Betti numbers among all graded ideals containing a regular sequence of homogeneous elements of degrees $a_1, \cdots, a_n$ with the same Hilbert function. By using the special case when

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$a_1 = \cdots = a_n = 2$ of Theorem 1.1, Mermin, Peeva and Stillman [11, Theorem 1.1] proved the lex-plus-powers conjecture for graded ideals containing $x_1^2, \dots, x_n^2$ in characteristic 0. By using their method together with Theorem 1.1, we prove the lex-plus-powers conjecture for graded ideals containing $x_1^p, \dots, x_n^p$ in characteristic 0, where p is a prime number (Theorem 3.4).

We also study the graded Betti numbers of monomial ideals containing the squares of the variables. If $a_1 = \cdots = a_n = 2$, $\mathbf{a}$-Borel-plus-powers (respectively $\mathbf{a}$-lex-plus-powers) ideals are called *Borel-plus-squares* (respectively *lex-plus-squares*) ideals. First we show that the graded Betti numbers of a Borel-plus-squares ideal I are equal to those of the lex-plus-squares ideal with the same Hilbert function as I if and only if I is the lex-plus-squares ideal. Second, we extend [11, Theorem 1.1] in arbitrary characteristic, that is, we show that the lex-plus-squares ideal L has the largest graded Betti numbers among all graded ideals containing $x_1^2, \dots, x_n^2$ in arbitrary characteristic. To prove this fact, we show that the graded Betti numbers of $I_\Delta + (x_1^2, \dots, x_n^2)$ increase by combinatorial shifting (see [9, §8]), where I_Δ is the Stanley–Reisner ideal of a simplicial complex Δ.

This paper is organized as follows: In §2, we describe the graded Betti numbers of Borel-plus-powers ideals. In §3, the proof of Theorem 1.1 is given. In §4, we study the graded Betti numbers of monomial ideals containing $x_1^2, \dots, x_n^2$. In §5, we show that the graded Betti numbers of an $\mathbf{a}$-Borel-plus-powers ideal I are obtained from those of the $\mathbf{a}$-lex-plus-powers ideal with the same Hilbert function as I by consecutive cancellations, and give an affirmative answer to [11, Problem 1.3].

2. Betti numbers of Borel-plus-powers ideals

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with each $\deg x_i = 1$. Let $\mathbf{a} = (a_1, a_2, \dots, a_n)$ be a sequence of integers or ∞ satisfying $2 \leq a_k$ for $k = 1, 2, \dots, n$. Set $\bar{\mathbf{a}} = (a_1 - 1, a_2 - 1, \dots, a_n - 1)$.

A monomial $u = x_1^{b_1} x_2^{b_2} \cdots x_n^{b_n} \in S$ is called an $\mathbf{a}$-*monomial* if $b_k \leq a_k - 1$ for $k = 1, 2, \dots, n$. Let V be a set of $\mathbf{a}$-monomials of S. We say that V is *strongly* $\mathbf{a}$-*stable* if, for any $\mathbf{a}$-monomial $ux_j \in V$, one has $ux_i \in V$ for all $i \in \{k : k < j \text{ and } ux_k \text{ is an } \mathbf{a}\text{-monomial of } S\}$. Also, V is said to be $\mathbf{a}$-*lexsegment* if, for all $\mathbf{a}$-monomials $u \in V$ and $v >_{\mathrm{lex}} u$ with $\deg v = \deg u$, it follows that $v \in V$. A monomial ideal generated by $\mathbf{a}$-monomials is called an $\mathbf{a}$-*ideal.* An $\mathbf{a}$-ideal I is said to be *strongly* $\mathbf{a}$-*stable* (respectively $\mathbf{a}$-*lexsegment*) if the set of all $\mathbf{a}$-monomials in I is strongly $\mathbf{a}$-stable (respectively $\mathbf{a}$-lexsegment).

In the rest of this section, we assume $a_1 \leq \cdots \leq a_n$, and write $P = (x_1^{a_1}, \dots, x_n^{a_n})$ and $N = \max\{k : a_k \neq \infty\}$, where $N = 0$ if $a_1 = \cdots = a_n = \infty$. For a subset $\sigma \subset [n] = \{1, 2, \dots, n\}$, let $x_\sigma^{\mathbf{a}} = \prod_{i \in \sigma} x_i^{a_i}$ and $x_\sigma^{\bar{\mathbf{a}}} = \prod_{i \in \sigma} x_i^{a_i - 1}$. Also we write $|\sigma^{\mathbf{a}}| = \deg x_\sigma^{\mathbf{a}}$ and $|\sigma^{\bar{\mathbf{a}}}| = \deg x_\sigma^{\bar{\mathbf{a}}}$.

For any $\mathbf{a}$-monomial $u = x_{i_1}^{b_1} \cdots x_{i_t}^{b_t}$ with $i_1 > \cdots > i_t$ and with $b_k > 0$ for $k = 1, 2 \dots, t$, let

$$\kappa^{\mathbf{a}}(u) = \begin{cases} \min\{k : b_k \neq a_{i_k} - 1\}, & \text{if } u \neq x_{\{i_1, \dots, i_t\}}^{\bar{\mathbf{a}}}, \\ t + 1, & \text{if } u = x_{\{i_1, \dots, i_t\}}^{\bar{\mathbf{a}}}, \end{cases}$$

and let

$$A_p^{\mathbf{a}}(u) = \sum_{j=1}^{\kappa^{\mathbf{a}}(u)} \binom{i_j - 1}{p - j} \text{ for } p = 0, 1, 2, \dots,$$

where $i_{t+1} = 0$ and where $\binom{-1}{p} = 0$ for all p. Remark that the ordering of $i_1, i_2, \dots, i_t$ is not the usual ordering of the variables. (Usually we write a monomial in the form $u = x_1 x_2 x_4^2$, but we consider the ordering of the variables in the form $x_4^2 x_2 x_1$ for the definition of $\kappa^{\mathbf{a}}(u)$ and $A_p^{\mathbf{a}}(u)$.) This ordering of the variables will be used several times in this paper. For any $\mathbf{a}$-ideal I of S, write $\mathcal{M}^{\mathbf{a}}(I)$ for the set of all $\mathbf{a}$-monomials in I. The aim of this section is to show the next proposition.

Proposition 2.1. *Let I be a strongly $\mathbf{a}$-stable ideal of S. Then*

$$\beta_{pp+s}(S/(I+P)) = \sum_{\substack{u \in \mathcal{M}^{\mathbf{a}}(I) \\ \deg u = s+1}} A_p^{\mathbf{a}}(u) - \sum_{\substack{u \in \mathcal{M}^{\mathbf{a}}(I) \\ \deg u = s}} \left\{ \binom{n}{p} - A_{p+1}^{\mathbf{a}}(u) \right\} + \beta_{pp+s}(S/P)$$

for all $p \geq 0$ and $s \geq 0$.

To prove the above statement, we first recall the relation between the graded Betti numbers of an $\mathbf{a}$-ideal I of S and those of $I + P$ given in [11, Theorem 6.1]. For a finite set V we write $|V|$ for its cardinality.

Theorem 2.2 (Mermin–Peeva–Stillman)**.** *Let I be an $\mathbf{a}$-ideal of S and let $\mathcal{F}_\sigma = S/(I : x_\sigma^{\bar{\mathbf{a}}})$ for each $\sigma \subset [N]$. We have the long exact sequence*

$$(1) \quad 0 \longrightarrow \bigoplus_{\substack{\sigma \subset [N] \\ |\sigma| = N}} \mathcal{F}_\sigma \xrightarrow{\varphi_N} \cdots \longrightarrow \bigoplus_{\substack{\sigma \subset [N] \\ |\sigma| = 1}} \mathcal{F}_\sigma \xrightarrow{\varphi_1} \bigoplus_{\substack{\sigma \subset [N] \\ |\sigma| = 0}} \mathcal{F}_\sigma = S/I \longrightarrow S/(I+P) \longrightarrow 0$$

with maps φ_i the Koszul maps for the sequence $x_1^{a_1}, \dots, x_N^{a_N}$. Moreover, $S/(I+P)$ is minimally resolved by the iterated mapping cones from (1).

The above theorem immediately implies the next corollary.

Corollary 2.3. *Let I be an $\mathbf{a}$-ideal of S. Then*

$$\beta_{pp+s}(S/(I+P)) = \sum_{i=0}^{N} \left\{ \sum_{\sigma \subset [N],\ |\sigma| = i} \beta_{p-i, p+s-|\sigma^{\mathbf{a}}|}(S/(I : x_\sigma^{\bar{\mathbf{a}}})) \right\} \quad \text{for all } p \text{ and } s.$$

Next, we recall how we can obtain the graded Betti numbers of strongly $\mathbf{a}$-stable ideals. For any nonunit $\mathbf{a}$-monomial u of S, we write $\max u = \max\{j : x_j \text{ divides } u\}$ and

$$m^{\mathbf{a}}(u) = |\{j \in [n] : j < \max u \text{ and } ux_j \text{ is an } \mathbf{a}\text{-monomial}\}|.$$

Also, we set $\max 1 = -1$ and $m^{\mathbf{a}}(1) = -1$. The following result was shown in [7, Corollary 2.3].

Theorem 2.4 (Gasharov–Hibi–Peeva)**.** *Let I be a strongly $\mathbf{a}$-stable ideal of S and $G(I)$ the set of minimal monomial generators of I. Then*

$$\beta_{pp+s}(S/I) = \sum_{u \in G(I),\ \deg u = s+1} \binom{m^{\mathbf{a}}(u)}{p-1} \quad \text{for } p > 0 \text{ and } s \geq 0.$$

We will prove Proposition 2.1 by using Theorems 2.2 and 2.4. For each $\sigma \subset [n]$ with $[n] \setminus \sigma = \{j_1, \dots, j_t\}$, let $S[\sigma] = K[x_{j_1}, \dots, x_{j_t}]$. Let $\sigma \subset [N]$. If I is an $\mathbf{a}$-ideal of S, then it is easy to see that $(I : x_\sigma^{\bar{\mathbf{a}}})$ is generated by $\mathbf{a}$-monomials in $S[\sigma]$. Set $I[\sigma] = (I : x_\sigma^{\bar{\mathbf{a}}}) \cap S[\sigma]$. Then

$$\beta_{ij}(S/(I : x_\sigma^{\bar{\mathbf{a}}})) = \beta_{ij}(S[\sigma]/I[\sigma]) \quad \text{for all } i \text{ and } j. \tag{2}$$

Here $\beta_{ij}(S/(I : x_\sigma^{\bar{\mathbf{a}}}))$ are the graded Betti numbers over S and $\beta_{ij}(S[\sigma]/I[\sigma])$ are the graded Betti numbers over $S[\sigma]$. Also, since $S[\sigma]/I[\sigma] = 0$ if and only if $x_\sigma^{\bar{\mathbf{a}}} \in I$ and since $\beta_{0s}(S[\sigma]/I[\sigma]) = \binom{0}{s}$ if $S[\sigma]/I[\sigma] \neq 0$, Corollary 2.3 says

$$\begin{aligned} \beta_{pp+s}(S/(I+P)) \;=\; & \sum_{i=0}^{p-1} \left\{ \sum_{\sigma \subset [N],\ |\sigma| = i} \beta_{p-i,(p-i)+(s-|\sigma^{\bar{\mathbf{a}}}|)}(S[\sigma]/I[\sigma]) \right\} \\ & + \left\{ \sum_{\sigma \subset [N]} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|} - \sum_{\sigma \subset [N],\ x_\sigma^{\bar{\mathbf{a}}} \in I} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|} \right\} \end{aligned} \tag{3}$$

for $p \geq 0$ and $s \geq 0$, where $\delta_{a,b} = \binom{0}{a-b}$ for all integers a and b.

Suppose that I is strongly $\mathbf{a}$-stable. For $\sigma \subset [N]$ with $[n] \setminus \sigma = \{j_1, \dots, j_t\}$, where $j_1 < \cdots < j_t$, write $\mathbf{a}_\sigma = (a_{j_1}, \cdots, a_{j_t})$. Then $I[\sigma]$ is a strongly $\mathbf{a}_\sigma$-stable ideal of $S[\sigma]$. Thus we can compute the graded Betti numbers of $I[\sigma]$ by using Theorem 2.4. For each $\sigma \subset [N]$ and for each nonunit $\mathbf{a}$-monomial $u \in S[\sigma]$, we let

$$m^{\mathbf{a}}(u;\sigma) = |\{j \in [n] \setminus \sigma : j < \max u,\ ux_j \text{ is an } \mathbf{a}\text{-monomial}\}|$$

and let $m^{\mathbf{a}}(1;\sigma) = -1$. Then Theorem 2.4 says that

$$\beta_{pp+s}(S[\sigma]/I[\sigma]) = \sum_{u \in G(I[\sigma]),\ \deg u = s+1} \binom{m^{\mathbf{a}}(u;\sigma)}{p-1} \quad \text{for } p > 0 \text{ and } s \geq 0.$$

Let $\mathcal{M}^{\mathbf{a}}(I[\sigma])$ be the set of all $\mathbf{a}$-monomials belonging to $I[\sigma]$. For each $\sigma \subset [N]$ and for each integer $s \geq 0$, set

$$\mathcal{N}_I(\sigma; s) = \{u \in \mathcal{M}^{\mathbf{a}}(I[\sigma]) : \deg(ux_\sigma^{\bar{\mathbf{a}}}) = s\}$$

and

$$\mathcal{S}_I(\sigma; s) = \{x_j u \in \mathcal{M}^{\mathbf{a}}(I[\sigma]) : u \in \mathcal{M}^{\mathbf{a}}(I[\sigma]),\ \deg(x_j u x_\sigma^{\bar{\mathbf{a}}}) = s\}.$$

Clearly $\{u \in G(I[\sigma]) : \deg(ux_\sigma^{\bar{\mathbf{a}}}) = s\} = \mathcal{N}_I(\sigma; s) \setminus \mathcal{S}_I(\sigma; s)$. Hence we have

$$\beta_{pp+s}(S[\sigma]/I[\sigma]) \;=\; \sum_{u \in \mathcal{N}_I(\sigma; s+1+|\sigma^{\bar{\mathbf{a}}}|)} \binom{m^{\mathbf{a}}(u;\sigma)}{p-1} - \sum_{u \in \mathcal{S}_I(\sigma; s+1+|\sigma^{\bar{\mathbf{a}}}|)} \binom{m^{\mathbf{a}}(u;\sigma)}{p-1}$$

for $p > 0$ and $s \geq 0$. Then the above equation and (3) say

$$\begin{aligned} & \beta_{pp+s}(S/(I+P)) - \left\{ \sum_{\sigma \subset [N]} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|} - \sum_{\sigma \subset [N],\ x_\sigma^{\bar{\mathbf{a}}} \in I} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|} \right\} \\ & = \sum_{\sigma \subset [N]} \left\{ \sum_{u \in \mathcal{N}_I(\sigma; s+1)} \binom{m^{\mathbf{a}}(u;\sigma)}{p-|\sigma|-1} - \sum_{u \in \mathcal{S}_I(\sigma; s+1)} \binom{m^{\mathbf{a}}(u;\sigma)}{p-|\sigma|-1} \right\} \end{aligned} \tag{4}$$

for all $p \geq 0$ and $s \geq 0$. (Note that the second line of (4) vanishes if $|\sigma| \geq p$.)

Next, we study the structure of $\mathcal{N}_I(\sigma; s)$ and that of $\mathcal{S}_I(\sigma; s)$. For each $\sigma \subset [N]$ let e_σ be a symbol. For an $\mathbf{a}$-monomial $u \in S$, let

$$\Phi(u) = \{(u/x_\sigma^{\bar{\mathbf{a}}})e_\sigma : \sigma \subset [N] \text{ and } x_\sigma^{\bar{\mathbf{a}}} \text{ divides } u\}$$

and

$$\Psi(u) = \{(vx_j)e_\sigma : ve_\sigma \in \Phi(u),\ j \geq \max v \text{ and } ux_j \text{ is an } \mathbf{a}\text{-monomial}\}.$$

Lemma 2.5. *Let I be a strongly $\mathbf{a}$-stable ideal of S, and let $s \geq 0$ be an integer. Then*

$$\bigcup_{\sigma \subset [N]} \{ue_\sigma : u \in \mathcal{N}_I(\sigma; s)\} \ = \ \dot{\bigcup}_{u \in \mathcal{M}^{\mathbf{a}}(I),\ \deg u = s} \Phi(u) \tag{5}$$

and

$$\bigcup_{\sigma \subset [N]} \{ue_\sigma : u \in \mathcal{S}_I(\sigma; s)\} \ = \ \dot{\bigcup}_{u \in \mathcal{M}^{\mathbf{a}}(I),\ \deg u = s-1} \Psi(u), \tag{6}$$

where $\dot{\bigcup}$ denotes a disjoint union.

Proof. In both (5) and (6), the lefthand side contains the righthand side since $u \in \mathcal{M}^{\mathbf{a}}(I[\sigma])$ if and only if $ux_\sigma^{\bar{\mathbf{a}}} \in \mathcal{M}^{\mathbf{a}}(I)$. On the other hand, the righthand side is a disjoint union since $ve_\sigma \in \Phi(u)$ implies $vx_\sigma^{\bar{\mathbf{a}}} = u$ and $ve_\sigma \in \Psi(u)$ implies $(v/x_{\max v})x_\sigma^{\bar{\mathbf{a}}} = u$. In each of the cases (5) and (6), we will show that the righthand side contains the lefthand side.

Let $v \in \mathcal{N}_I(\sigma; s)$. Then $v \in S[\sigma]$, $\deg(vx_\sigma^{\bar{\mathbf{a}}}) = s$ and $vx_\sigma^{\bar{\mathbf{a}}} \in \mathcal{M}^{\mathbf{a}}(I)$. Hence $ve_\sigma \in \Phi(vx_\sigma^{\bar{\mathbf{a}}})$ and $\deg(vx_\sigma^{\bar{\mathbf{a}}}) = s$ as desired.

Let $v \in \mathcal{S}_I(\sigma; s)$. Then $vx_\sigma^{\bar{\mathbf{a}}}$ is an $\mathbf{a}$-monomial of degree s in I and there exists an integer j such that $v/x_j \in I[\sigma]$. Since $I[\sigma]$ is strongly $\mathbf{a}_\sigma$-stable, we may assume $j = \max v$. Then $(v/x_{\max v})x_\sigma^{\bar{\mathbf{a}}} \in \mathcal{M}^{\mathbf{a}}(I)$ and $(v/x_{\max v})e_\sigma \in \Phi((v/x_{\max v})x_\sigma^{\bar{\mathbf{a}}})$. Hence we have $ve_\sigma \in \Psi((v/x_{\max v})x_\sigma^{\bar{\mathbf{a}}})$ and $\deg((v/x_{\max v})x_\sigma^{\bar{\mathbf{a}}}) = s - 1$ as desired. □

Now, by using Lemma 2.5 and (4), for any strongly $\mathbf{a}$-stable ideal I of S, one has

$$\begin{aligned}
\beta_{pp+s}(S/(I+P)) \ = \ & \sum_{\substack{u \in \mathcal{M}^{\mathbf{a}}(I) \\ \deg u = s+1}} \left\{ \sum_{ve_\sigma \in \Phi(u)} \binom{m^{\mathbf{a}}(v;\sigma)}{p - |\sigma| - 1} \right\} \\
& - \sum_{\substack{u \in \mathcal{M}^{\mathbf{a}}(I) \\ \deg u = s}} \left\{ \sum_{ve_\sigma \in \Psi(u)} \binom{m^{\mathbf{a}}(v;\sigma)}{p - |\sigma| - 1} \right\} - \sum_{\sigma \subset [N],\ x_\sigma^{\bar{\mathbf{a}}} \in I,\ \deg x_\sigma^{\bar{\mathbf{a}}} = s} \delta_{p,|\sigma|} \\
& + \sum_{\sigma \subset [N]} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|}
\end{aligned}$$

for all $p \geq 0$ and $s \geq 0$. A routine computation implies $\sum_{\sigma \subset [N]} \delta_{p,|\sigma|}\delta_{s,|\sigma^{\bar{\mathbf{a}}}|} = \beta_{pp+s}(S/P)$. Then Proposition 2.1 follows from Lemma 2.6 stated below.

Lemma 2.6. *Let $u = x_{i_1}^{b_1} \cdots x_{i_t}^{b_t}$ be an* $\mathbf{a}$*-monomial with $i_1 > \cdots > i_t$ and with $b_k > 0$ for $k = 1, 2, \dots, t$. Then, for all $p \geq 0$, one has*

$$\sum_{ve_\sigma \in \Phi(u)} \binom{m^{\mathbf{a}}(v;\sigma)}{p - |\sigma| - 1} = A_p^{\mathbf{a}}(u) \tag{7}$$

and

$$\sum_{ve_\sigma \in \Psi(u)} \binom{m^{\mathbf{a}}(v;\sigma)}{p - |\sigma| - 1} = \begin{cases} \binom{n}{p} - A_{p+1}^{\mathbf{a}}(u) - \binom{0}{p-t}, & \text{if } u = x_{\{i_1,\dots,i_t\}}^{\bar{\mathbf{a}}}, \\ \binom{n}{p} - A_{p+1}^{\mathbf{a}}(u), & \text{otherwise.} \end{cases} \tag{8}$$

Before proving Lemma 2.6, we write a few fundamental facts on binomial coefficients, which will appear many times in the proof. The next statement easily follows from the well-known relation $\binom{a+1}{b} = \binom{a}{b} + \binom{a}{b-1}$ of binomial coefficients.

Lemma 2.7. *Let $a, \ell, s \in \mathbb{Z}_{\geq 0}$, and let $b \in \mathbb{Z}$. Suppose $s \leq a$.*

(a) $\binom{a+\ell}{b} = \binom{a}{b} + \sum_{k=1}^{\ell} \binom{a+k-1}{b-1}$;

(b) $\binom{a}{b} = \binom{a-s}{b-s} + \sum_{k=1}^{s} \binom{a-k}{b-(k-1)}$.

Proof of Lemma 2.6. Let $i_{t+1} = 0$, $\varepsilon = \kappa^{\mathbf{a}}(u)$ and $\varepsilon' = \min\{\varepsilon, t\}$. We write

$$\Phi_j(u) = \{ve_\sigma \in \Phi(u) : \max v = i_j\} \quad \text{for } j = 1, 2, \dots, \varepsilon',$$

and write $\Phi_{t+1}(u) = \{e_{\{i_1,\dots,i_t\}}\}$ if $\varepsilon = t+1$. Note that $\Phi(u) = \bigcup_{j=1}^{\varepsilon} \Phi_j(u)$. Let

$$\tau = \{i_k : b_k = a_{i_k} - 1\}$$

and

$$\tau_j = \{i_k : b_k = a_{i_k} - 1 \text{ and } i_k < i_j\} \quad \text{for } j = 1, 2, \dots, t.$$

Then we have $\sigma \subset \tau$ for any $ve_\sigma \in \Phi(u)$ since $a_k \geq 2$ for $k = 1, 2, \dots, n$. Thus, for any $ve_\sigma \in \Phi_j(u)$ with $1 \leq j \leq t$, we have

$$m^{\mathbf{a}}(v;\sigma) = |\{1, 2, \dots, i_j - 1\} \setminus \tau_j|. \tag{9}$$

For a subset $F \subset [n]$, we write $\binom{F}{p} = \binom{|F|}{p}$.

First, we show (7). Since we assume $i_1 > \cdots > i_t$, it is clear that $ve_\sigma \in \Phi_j(u)$ if and only if $\{i_1, \dots, i_{j-1}\} \subset \sigma$ and $i_j \notin \sigma$. This fact says that

$$\Phi_j(u) = \{(u/x_{\sigma' \cup \{i_1,\dots,i_{j-1}\}}^{\bar{\mathbf{a}}}) e_{\sigma' \cup \{i_1,\dots,i_{j-1}\}} : \sigma' \subset \tau_j\}$$

for $j = 1, 2, \dots, \varepsilon'$. Thus (9) says that if $1 \leq j \leq \varepsilon'$ then

$$\begin{aligned} \sum_{ve_\sigma \in \Phi_j(u)} \binom{m^{\mathbf{a}}(v;\sigma)}{p - |\sigma| - 1} &= \sum_{\sigma' \subset \tau_j} \binom{\{1, \dots, i_j - 1\} \setminus \tau_j}{p - (|\sigma'| + j - 1) - 1} \\ &= \sum_{q=0}^{|\tau_j|} \binom{\tau_j}{q} \binom{\{1, \dots, i_j - 1\} \setminus \tau_j}{p - j - q} \\ &= \binom{\tau_j \cup (\{1, \dots, i_j - 1\} \setminus \tau_j)}{p - j} = \binom{i_j - 1}{p - j} \end{aligned} \tag{10}$$

for all p. Since $m^{\mathbf{a}}(1;\sigma)=-1$ and $i_{t+1}=0$, the lefthand side of (7) is equal to

$$\sum_{j=1}^{\varepsilon}\left\{\sum_{ve_\sigma\in\Phi_j(u)}\binom{m^{\mathbf{a}}(v;\sigma)}{p-|\sigma|-1}\right\}=\sum_{j=1}^{\varepsilon}\binom{i_j-1}{p-j}=A_p^{\mathbf{a}}(u)$$

for all p, as required.

Second we consider (8). For $j=1,2,\ldots,t+1$, let

$$\eta_j=\{k\in[n]:k\geq i_j\text{ and }ux_k\text{ is an }\mathbf{a}\text{-monomial}\}=\{k\in[n]:k\geq i_j\text{ and }k\notin\tau\}$$

and $\ell_j=|\eta_j|$. For each $ve_\sigma\in\Phi_j(u)$, we write

$$\Theta(ve_\sigma)=\{(vx_k)e_\sigma:k\in\eta_j\}.$$

Then $\Psi(u)=\bigcup_{ve_\sigma\in\Phi(u)}\Theta(ve_\sigma)$. For the proof of (8), we will show two equations (11) and (12) stated in Steps 1 and 2 below.

Step 1: We claim that if $1\leq j\leq\varepsilon'$ then, for all $p\geq 0$, one has

$$\sum_{ve_\sigma\in\Phi_j(u)}\left\{\sum_{we_\sigma\in\Theta(ve_\sigma)}\binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1}\right\}+\binom{i_j-1}{p+1-j}=\binom{|\{1,\ldots,i_j-1\}\cup\eta_j|}{p+1-j}. \tag{11}$$

Fix an integer $1\leq j\leq\varepsilon'$. Let $\eta_j=\{m_1,\ldots,m_{\ell_j}\}$ with $m_1<\cdots<m_{\ell_j}$. Thus

$$\Theta(ve_\sigma)=\{(vx_{m_k})e_\sigma:k=1,2,\ldots,\ell_j\}$$

for any $ve_\sigma\in\Phi_j(u)$. Let $ve_\sigma\in\Phi_j(u)$. Since $\sigma\subset\tau$, we have

$$m^{\mathbf{a}}(vx_{m_k};\sigma)=|\{k':k'<m_k,\ k'\notin\tau\}|=|(\{1,\ldots,i_j-1\}\setminus\tau_j)\cup\{m_1,\ldots,m_{k-1}\}|$$

for $k=1,\ldots,\ell_j$. Thus, in the same way as (10), we have

$$\begin{aligned}
&\sum_{ve_\sigma\in\Phi_j(u)}\left\{\sum_{we_\sigma\in\Theta(ve_\sigma)}\binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1}\right\}\\
&=\sum_{ve_\sigma\in\Phi_j(u)}\left\{\sum_{k=1}^{\ell_j}\binom{(\{1,\ldots,i_j-1\}\setminus\tau_j)\cup\{m_1,\ldots,m_{k-1}\}}{p-|\sigma|-1}\right\}\\
&=\sum_{q=0}^{|\tau_j|}\binom{\tau_j}{q}\left\{\sum_{k=1}^{\ell_j}\binom{(\{1,\ldots,i_j-1\}\setminus\tau_j)\cup\{m_1,\ldots,m_{k-1}\}}{p-j-q}\right\}\\
&=\sum_{k=1}^{\ell_j}\left\{\sum_{q=0}^{|\tau_j|}\binom{\tau_j}{q}\binom{(\{1,\ldots,i_j-1\}\setminus\tau_j)\cup\{m_1,\ldots,m_{k-1}\}}{p-j-q}\right\}\\
&=\sum_{k=1}^{\ell_j}\binom{\{1,\ldots,i_j-1\}\cup\{m_1,\ldots,m_{k-1}\}}{p-j}
\end{aligned}$$

for all p. Then Lemma 2.7 (a) says that the lefthand side of (11) is equal to

$$\begin{aligned}&\left\{\sum_{k=1}^{\ell_j}\binom{\{1,\dots,i_j-1\}\cup\{m_1,\dots,m_{k-1}\}}{p-j}\right\}+\binom{i_j-1}{p+1-j}\\ &=\binom{|\{1,\dots,i_j-1\}\cup\{m_1,\dots,m_{\ell_j-1}\}|+1}{p+1-j}\\ &=\binom{|\{1,\dots,i_j-1\}\cup\eta_j|}{p+1-j}\end{aligned}$$

for all p, as desired.

Step 2: We claim that if $\varepsilon=t+1$ then, for all $p\geq 0$, one has

$$\sum_{ve_\sigma\in\Phi_\varepsilon(u)}\left\{\sum_{we_\sigma\in\Theta(ve_\sigma)}\binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1}\right\}+\binom{0}{p-t} = \binom{n-t}{p-t}. \tag{12}$$

Since $\varepsilon=t+1$, we have $u=x^{\bar{\mathbf{a}}}_{\{i_1,\dots,i_t\}}$ and $\Phi_\varepsilon(u)=\{e_{\{i_1,\dots,i_t\}}\}$. It is clear that

$$\Theta(e_{\{i_1,\dots,i_t\}})=\{x_k e_{\{i_1,\dots,i_t\}}:k\in[n]\setminus\{i_1,\dots,i_t\}\}.$$

Thus we have

$$\sum_{ve_\sigma\in\Phi_\varepsilon(u)}\left\{\sum_{we_\sigma\in\Theta(ve_\sigma)}\binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1}\right\} = \sum_{k=1}^{n-t}\binom{k-1}{p-t-1}\quad\text{for all } p\geq 0.$$

Then (12) follows since $\sum_{k=1}^{n-t}\binom{k-1}{p-t-1}+\binom{0}{p-t}=\binom{n-t}{p-t}$ by Lemma 2.7 (a).

Now we will show (8). Recall that $i_j\in\tau$ for $j<\varepsilon$ and $i_\varepsilon\notin\tau$. Then, by the definition of η_j, we have

$$|\{1,\dots,i_j-1\}\cup\eta_j|=|[n]\setminus\{k\in\tau:k\geq i_j\}|=\begin{cases} n-j, & \text{if } 1\leq j<\varepsilon,\\ n-j+1, & \text{if } j=\varepsilon.\end{cases} \tag{13}$$

Suppose $\varepsilon\leq t$. Then $u\neq x^{\bar{\mathbf{a}}}_{\{i_1,\dots,i_t\}}$. Also (11) and (13) say that

$$\begin{aligned}&\sum_{j=1}^{\varepsilon}\left[\sum_{ve_\sigma\in\Phi_j(u)}\left\{\sum_{we_\sigma\in\Theta(ve_\sigma)}\binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1}\right\}\right]+\sum_{j=1}^{\varepsilon}\binom{i_j-1}{p+1-j}\\ &=\sum_{j=1}^{\varepsilon-1}\binom{n-j}{p+1-j}+\binom{n-\varepsilon+1}{p+1-\varepsilon}=\binom{n}{p}\end{aligned} \tag{14}$$

for all p (we use Lemma 2.7 (b) for the last equality). Since

$$\Psi(u)=\bigcup_{ve_\sigma\in\Phi(u)}\Theta(ve_\sigma)=\bigcup_{j=1}^{\varepsilon}\left\{\bigcup_{ve_\sigma\in\Phi_j(u)}\Theta(ve_\sigma)\right\}$$

and since $A^{\mathbf{a}}_{p+1}(u)=\sum_{j=1}^{\varepsilon}\binom{i_j-1}{p+1-j}$, (14) is equivalent to the desired equality (8).

Next, suppose $\varepsilon = t+1$. Then $u = x^{\bar{\mathbf{a}}}_{\{i_1,\dots,i_t\}}$ and $A^{\mathbf{a}}_{p+1}(u) = \sum_{j=1}^{t} \binom{i_j - 1}{p+1-j}$. Also (11), (12) and (13) say

$$\sum_{j=1}^{\varepsilon} \left[\sum_{ve_\sigma \in \Phi_j(u)} \left\{ \sum_{we_\sigma \Theta(ve_\sigma)} \binom{m^{\mathbf{a}}(w;\sigma)}{p-|\sigma|-1} \right\} \right] + A^{\mathbf{a}}_{p+1}(u) + \binom{0}{p-t}$$
$$= \sum_{j=1}^{t} \binom{n-j}{p+1-j} + \binom{n-t}{p-t} = \binom{n}{p}$$

for all p. This is equivalent to the desired equality (8). □

Example 2.8. Let $a_1 = a_2 = 3$, $a_3 = 4$ and $I = (x_1^2x_2, x_1x_2^2, x_1^2x_3^2) \subset K[x_1,x_2,x_3]$. Then I is strongly $\mathbf{a}$-stable. The set of $\mathbf{a}$-monomials of degree 3 in I is

$$\{x_1^2x_2, x_1x_2^2\}$$

and that of degree 4 is

$$\{x_1^2x_2^2, x_1^2x_2x_3, x_1x_2^2x_3, x_1^2x_3^2\}.$$

Also

$A^{\mathbf{a}}_p(x_1^2x_2) = \binom{1}{p-1}$, $A^{\mathbf{a}}_p(x_1x_2^2) = \binom{1}{p-1} + \binom{0}{p-2}$, $A^{\mathbf{a}}_p(x_1^2x_2^2) = \binom{1}{p-1} + \binom{0}{p-2} + \binom{-1}{p-3}$
and $A^{\mathbf{a}}_p(x_1^2x_2x_3) = A^{\mathbf{a}}_p(x_1^2x_3^2) = A^{\mathbf{a}}_p(x_1x_2^2x_3) = \binom{2}{p-1}$.

Let $S = K[x_1,x_2,x_3]$ and $R = S/(I + (x_1^3, x_2^3, x_3^4))$. Proposition 2.1 says

$$\begin{aligned} \beta_{pp+3}(R) &= 3\binom{2}{p-1} + \binom{1}{p-1} + \binom{0}{p-2} + \binom{-1}{p-3} \\ &\quad - \left\{ 2\binom{3}{p} - 2\binom{1}{p} - \binom{0}{p-1} \right\} + \beta_{pp+3}(S/(x_1^3, x_2^3, x_3^4)). \end{aligned}$$

Thus $\beta_{0,3}(R) = 0$, $\beta_{1,4}(R) = 2$, $\beta_{2,5}(R) = 2$ and $\beta_{3,6}(R) = 1$.

3. Comparison of Borel-plus-powers ideals and lex-plus-powers ideals

In this section, we will prove Theorem 1.1. Let $\mathbf{a} = (a_1,\dots,a_n)$ be a sequence of integers or ∞ satisfying $2 \le a_k$ for $k = 1,2,\dots,n$. Let $<_{\mathrm{rev}}$ be the degree reverse lexicographic order induced by $x_1 > \cdots > x_n$. We use the following fact proved in [8, Theorem 2.1]. (We do not assume $a_1 \le \cdots \le a_n$ in Theorem 3.1 and Corollary 3.2 below.)

Theorem 3.1 (Gasharov)**.** *Let $s \ge 0$ be an integer. Let $B \subset K[x_1,\dots,x_n]$ be a strongly $\mathbf{a}$-stable set of $\mathbf{a}$-monomials of degree s, and let $L \subset K[x_1,\dots,x_n]$ be the $\mathbf{a}$-lexsegment set of $\mathbf{a}$-monomials of degree s with $|L| = |B|$. Then*

$$|B \cap K[x_1,\dots,x_{n-1}]| \ge |L \cap K[x_1,\dots,x_{n-1}]|.$$

The above theorem implies the next fact.

Corollary 3.2. *With the same notation as in Theorem 3.1, let* $B = \{u_1, \dots, u_t\}$ *and* $L = \{v_1, \dots, v_t\}$ *where* $u_i >_{\mathrm{rev}} u_{i+1}$ *and* $v_i >_{\mathrm{rev}} v_{i+1}$ *for* $i = 1, 2, \dots, t-1$. *Then*

$$u_k \geq_{\mathrm{rev}} v_k \quad \text{for } k = 1, 2, \dots, t.$$

Proof. We use induction on n and s. If $n = 1$ or $s = 0$ then the statement is obvious. Suppose $n > 1$ and $s > 0$. Set

$$\begin{array}{lcl} B' = B \cap K[x_1, \dots, x_{n-1}], & & L' = L \cap K[x_1, \dots, x_{n-1}], \\ B'' = \{u : ux_n \in B\} & \text{and} & L'' = \{v : vx_n \in L\}. \end{array}$$

Theorem 3.1 says that $|B'| \geq |L'|$. Let $|L'| = \ell$ and $|B'| - |L'| = s$. Then $u_k \in B'$ and $v_k \in L'$ for $k \leq \ell$. Since B' is strongly $(a_1, \dots, a_{n-1})$-stable and L' is $(a_1, \dots, a_{n-1})$-lexsegment, the induction hypothesis says that

$$u_k \geq_{\mathrm{rev}} v_k \quad \text{for } k = 1, 2, \dots, \ell. \tag{15}$$

Also, since x_n divides v_k for $k > \ell$ and $u_k \in K[x_1, \dots, x_{n-1}]$ for $k \leq \ell + s$, we have

$$u_k \geq_{\mathrm{rev}} v_k \quad \text{for } k = \ell + 1, \ell +, 2, \dots, \ell + s. \tag{16}$$

Let $B'' = \{u'_1, \dots, u'_{t-\ell-s}\}$ with $u'_1 >_{\mathrm{rev}} \cdots >_{\mathrm{rev}} u'_{t-\ell-s}$. Then $u_{\ell+s+k} = u'_k x_n$ for $k = 1, 2, \dots, t-\ell-s$. Let $L''' = \{v'_1, \dots, v'_{t-\ell-s}\} \subset L''$ be the set of the largest $t-\ell-s$ monomials w.r.t. $<_{\mathrm{lex}}$ in L'', where $v'_1 >_{\mathrm{rev}} \cdots >_{\mathrm{rev}} v'_{t-\ell-s}$. Since $\{vx_n : v \in L'''\} \subset L$ and since $\{v_{\ell+s+1}, \cdots, v_t\}$ is the set of the smallest $t - \ell - s$ monomials w.r.t. $<_{\mathrm{rev}}$ in L, we have $v'_k x_n \geq_{\mathrm{rev}} v_{\ell+s+k}$ for $k = 1, 2, \dots, t - \ell - s$.

Suppose $a_n > 2$. Let $\mathbf{a}' = (a_1, \dots, a_{n-1}, a_n - 1)$. Then B'' is a strongly $\mathbf{a}'$-stable set of $\mathbf{a}'$-monomials of degree $s - 1$ and L''' is the $\mathbf{a}'$-lexsegment set of $\mathbf{a}'$-monomials of degree $s - 1$ with $|L'''| = |B''|$. Thus the induction hypothesis says that

$$u_{\ell+s+k} = u'_k x_n \geq_{\mathrm{rev}} v'_k x_n \geq_{\mathrm{rev}} v_{\ell+s+k} \quad \text{for } k = 1, 2, \dots, t - \ell - s. \tag{17}$$

Then the statement follows from (15), (16) and (17).

On the other hand, if $a_n = 2$ then $B'' \subset K[x_1, \dots, x_{n-1}]$ is strongly $(a_1, \dots, a_{n-1})$-stable and $L''' \subset K[x_1, \dots, x_{n-1}]$ is $(a_1, \dots, a_{n-1})$-lexsegment. Thus the induction hypothesis implies (17) again, and therefore the statement follows. ☐

We also require the following fact.

Lemma 3.3. *Let* u *and* v *be* $\mathbf{a}$*-monomials of* S *with* $\deg u = \deg v$ *and with* $u >_{\mathrm{rev}} v$. *Then* $A_p^{\mathbf{a}}(u) \leq A_p^{\mathbf{a}}(v)$ *for all* $p \geq 0$.

Proof. Let $u = x_{i_1}^{b_1} \cdots x_{i_s}^{b_s}$ with $i_1 > \cdots > i_s$ and with $b_k > 0$ for $k = 1, 2, \dots, s$, and let $v = x_{j_1}^{c_1} \cdots x_{j_t}^{c_t}$ with $j_1 > \cdots > j_t$ and with $c_k > 0$ for $k = 1, 2, \dots, t$. Since $u >_{\mathrm{rev}} v$ and since we assume $i_1 > \cdots > i_s$ and $j_1 > \cdots > j_t$, there exists an integer $1 \leq \ell \leq t$ such that (i) $j_\ell > i_\ell$ and $x_{i_k}^{b_k} = x_{j_k}^{c_k}$ for all $k < \ell$ or (ii) $j_\ell = i_\ell$, $b_\ell < c_\ell$ and $x_{i_k}^{b_k} = x_{j_k}^{c_k}$ for all $k < \ell$.

Case 1: If $b_k \neq a_{i_k} - 1$ for some $1 \leq k \leq \ell - 1$ then $\kappa^{\mathbf{a}}(u) = \kappa^{\mathbf{a}}(v) \leq \ell - 1$. Thus $A_p^{\mathbf{a}}(u) = \sum_{k=1}^{\kappa^{\mathbf{a}}(u)} \binom{i_k - 1}{p-k} = \sum_{k=1}^{\kappa^{\mathbf{a}}(v)} \binom{j_k - 1}{p-k} = A_p^{\mathbf{a}}(v)$ for all p.

Case 2: Next, suppose $b_k = a_{i_k} - 1$ for all $k = 1, 2, \dots, \ell - 1$ and $j_\ell = i_\ell$. Then $b_\ell < c_\ell \leq a_{i_\ell} - 1$. Thus $\kappa^{\mathbf{a}}(v) \geq \kappa^{\mathbf{a}}(u) = \ell$. Since $i_k = j_k$ for all $k \leq \ell$, we have $A_p^{\mathbf{a}}(u) = \sum_{k=1}^{\ell} \binom{i_k - 1}{p-k} = \sum_{k=1}^{\ell} \binom{j_k - 1}{p-k} \leq A_p^{\mathbf{a}}(v)$ for all p, as desired.

Case 3: Finally, suppose $b_k = a_{i_k} - 1$ for all $k = 1, 2, \dots, \ell - 1$ and $j_\ell > i_\ell$. Then $j_\ell \geq s - \ell + 2$ and $i_k \leq j_\ell - (k - \ell + 1)$ for $k = \ell, \dots, s$. Thus

$$\begin{aligned} A_p^{\mathbf{a}}(v) &\geq \sum_{k=1}^{\ell-1} \binom{j_k - 1}{p-k} + \binom{j_\ell - 1}{p - \ell} \\ &= \sum_{k=1}^{\ell-1} \binom{i_k - 1}{p-k} + \sum_{k=\ell}^{s} \binom{j_\ell - 1 - (k - \ell + 1)}{p-k} + \binom{j_\ell - 2 - s + \ell}{p - s - 1} \\ &\geq A_p^{\mathbf{a}}(u) \end{aligned}$$

for all p, as desired (we use Lemma 2.7 (b) for the second line). □

Now we can prove Theorem 1.1.

Proof of Theorem 1.1. Let $P = (x_1^{a_1}, \dots, x_n^{a_n})$. By the assumption, there exist the strongly $\mathbf{a}$-stable ideal I' and the $\mathbf{a}$-lexsegment ideal L' such that $I = I' + P$ and $L = L' + P$. Since $I = I' + P$ and $L = L' + P$ have the same Hilbert function, we have $|\{u \in \mathcal{M}^{\mathbf{a}}(I') : \deg u = s\}| = |\{u \in \mathcal{M}^{\mathbf{a}}(L') : \deg u = s\}|$ for all $s \geq 0$. Thus, by Proposition 2.1, what we must prove is

$$\sum_{u \in \mathcal{M}^{\mathbf{a}}(I'),\ \deg u = s} A_p^{\mathbf{a}}(u) \leq \sum_{u \in \mathcal{M}^{\mathbf{a}}(L'),\ \deg u = s} A_p^{\mathbf{a}}(u) \quad \text{for all } p \geq 0 \text{ and } s \geq 0. \tag{18}$$

Fix an integer $s \geq 0$. Let $\{u \in \mathcal{M}^{\mathbf{a}}(I') : \deg u = s\} = \{u_1, \dots, u_t\}$ and let $\{u \in \mathcal{M}^{\mathbf{a}}(L') : \deg u = s\} = \{v_1, \dots, v_t\}$. By Corollary 3.2, we may assume that $u_k \geq_{\text{rev}} v_k$ for $k = 1, 2, \dots, t$. Then (18) follows from Lemma 3.3. □

Theorem 1.1 implies the following special cases of the lex-plus-powers conjecture.

Theorem 3.4. *Let p be a prime number and $a_1 = \cdots = a_n = p$. Suppose* $\operatorname{char}(K) = 0$ *or* $\operatorname{char}(K) = p$. *Let $I \subset S$ be a graded ideal containing $x_1^p, x_2^p, \dots, x_n^p$ and L the* $\mathbf{a}$-*lex-plus-powers ideal with the same Hilbert function as I. The graded Betti numbers of L are larger than or equal to those of I.*

Proof. We sketch the proof since it is essentially the same as the proof of [11, Theorem 5.1]. First, we may assume that I is a monomial ideal since the initial ideal of I (w.r.t. any term order) contains $x_1^p, \dots, x_n^p$. Second, we may assume $\operatorname{char}(K) = p$ since the graded Betti numbers of any $\mathbf{a}$-lex-plus-powers ideal do not depend on the characteristic of the base field and since the graded Betti numbers of any monomial ideal J of $K[x_1, \dots, x_n]$ over a field K of characteristic p are larger than or equal to those of the monomial ideal J' of $K'[x_1, \dots, x_n]$ over a field K' of characteristic 0 with $G(J') = G(J)$. Then the generic initial ideal of I (w.r.t. any term order) contains $x_1^p, \dots, x_n^p$, and therefore is $\mathbf{a}$-Borel-plus-powers by [3, Theorem 15.23]. Thus the statement follows from Theorem 1.1. □

4. Monomial ideals containing the squares of the variables

If $a_1 = \cdots = a_n = 2$, $\mathbf{a}$-Borel-plus-powers (respectively $\mathbf{a}$-lex-plus-powers) ideals are called *Borel-plus-squares* (respectively *lex-plus-squares*) ideals. In this section, we will study the graded Betti numbers of monomial ideals containing $x_1^2, \dots, x_n^2$.

First we will show that the graded Betti numbers of a Borel-plus-squares ideal I are equal to those of the lex-plus-squares ideal L with the same Hilbert function as I if and only if $I = L$. Second, we will extend the result of Mermin–Peeva–Stillman [11, Theorem 1.1] in arbitrary characteristic, that is to say, we will show that the lex-plus-squares ideal has the largest graded Betti numbers among all graded ideals containing $x_1^2, x_2^2, \dots, x_n^2$ with the same Hilbert function in arbitrary characteristic.

Throughout this section, we write $Q = (x_1^2, \dots, x_n^2)$. If $a_1 = \cdots = a_n = 2$, then $\mathbf{a}$-ideals are squarefree monomial ideals. Also strongly $\mathbf{a}$-stable (respectively $\mathbf{a}$-lexsegment) ideals are said to be *squarefree strongly stable* (respectively *squarefree lexsegment*) if $a_1 = \cdots = a_n = 2$.

Let Δ be a simplicial complex on $[n]$. Thus Δ is a family of subsets of $[n]$ satisfying that if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$ (we do not assume $\{i\} \in \Delta$ for all $i \in [n]$). For a subset $F = \{i_1, \dots, i_k\} \subset [n]$, we write $x_F = x_{i_1} \cdots x_{i_k}$, where $x_F = 1$ if $F = \emptyset$. The *Stanley–Reisner ideal* I_Δ *of* Δ is the ideal of S generated by all squarefree monomials $x_F \in S$ with $F \notin \Delta$. A simplicial complex Δ is said to be *shifted* if $F \in \Delta$ and $i \in F$ imply $(F \setminus \{i\}) \cup \{j\} \in \Delta$ for all $i < j \leq n$. Thus Δ is shifted if and only if I_Δ is squarefree strongly stable.

For any squarefree monomial $x_F = x_{i_1} x_{i_2} \cdots x_{i_t}$ with $i_1 > \cdots > i_t$, let

$$A_p^*(x_F) = \sum_{j=1}^{t} \binom{i_j - 1}{p - j} \quad \text{for } p = 0, 1, \dots.$$

The next statement is a special case of Proposition 2.1.

Proposition 4.1. *Let Δ be a shifted simplicial complex on $[n]$. Then*

$$\beta_{pp+s}(S/(I_\Delta + Q)) = \sum_{\substack{x_F \in I_\Delta, \\ \deg x_F = s+1}} A_p^*(x_F) - \sum_{\substack{x_F \in I_\Delta, \\ \deg x_F = s}} \left\{ \binom{n}{p} - A_{p+1}^*(x_F) \right\} + \beta_{pp+s}(S/Q)$$

for all $p \geq 0$ and $s \geq 0$.

For any simplicial complex Δ on $[n]$, let Δ^{lex} be the simplicial complex on $[n]$ whose Stanley–Reisner ideal $I_{\Delta^{\mathrm{lex}}}$ is the squarefree lexsegment ideal with the same Hilbert function as I_Δ. Then $I_{\Delta^{\mathrm{lex}}} + Q$ is the lex-plus-squares ideal with the same Hilbert function as $I_\Delta + Q$. In Theorem 3.4 we proved $\beta_{ij}(I_\Delta + Q) \leq \beta_{ij}(I_{\Delta^{\mathrm{lex}}} + Q)$ for all i and j. It would be natural to ask which simplicial complex Δ satisfies $\beta_{ij}(I_\Delta + Q) = \beta_{ij}(I_{\Delta^{\mathrm{lex}}} + Q)$ for all i and j. We consider this problem for shifted complexes.

Lemma 4.2. *Let x_F and x_G be squarefree monomials of S with $\deg x_F = \deg x_G$ and with $x_F >_{\mathrm{rev}} x_G$. Then there exists an integer $p > 0$ such that $A_p^*(x_F) < A_p^*(x_G)$.*

Proof. Let $x_F = x_{i_1} \cdots x_{i_t}$ and $x_G = x_{j_1} \cdots x_{j_t}$, where $i_1 > \cdots > i_t$ and $j_1 > \cdots > j_t$. Since $x_F >_{\mathrm{rev}} x_G$ there exits an integer $1 \leq \ell \leq t$ such that $i_\ell < j_\ell$ and $i_k = j_k$ for all $k < \ell$. Note that $j_\ell \geq t - \ell + 2$ and $i_k \leq j_\ell - (k - \ell + 1)$ for $k = \ell, \ell + 1, \dots, t$.

Then, by Lemma 2.7 (b), we have

$$\begin{aligned} A_p^*(x_G) &\geq \sum_{k=1}^{\ell-1}\binom{j_k-1}{p-k}+\binom{j_\ell-1}{p-\ell} \\ &= \sum_{k=1}^{\ell-1}\binom{i_k-1}{p-k}+\sum_{k=\ell}^{t}\binom{j_\ell-1-(k-\ell+1)}{p-k}+\binom{j_\ell-2-t+\ell}{p-t-1} \\ &\geq A_p^*(x_F)+\binom{j_\ell-2-t+\ell}{p-t-1}. \end{aligned}$$

Thus if $p=t+1$ then $A_p^*(x_G) > A_p^*(x_F)$ since $\binom{j_\ell-2-t+\ell}{p-t-1} > 0$. □

Theorem 4.3. *Let Δ be a shifted simplicial complex on $[n]$. Then the graded Betti numbers of $I_\Delta + Q$ are equal to those of $I_{\Delta^{\mathrm{lex}}} + Q$ if and only if $\Delta = \Delta^{\mathrm{lex}}$.*

Proof. It suffices to show that if $I_\Delta \neq I_{\Delta^{\mathrm{lex}}}$ then $\beta_{ij}(S/(I_\Delta+Q)) < \beta_{ij}(S/(I_{\Delta^{\mathrm{lex}}}+Q))$ for some i and j. Suppose that there exists a squarefree monomial x_F of degree s satisfying $x_F \in I_\Delta$ and $x_F \notin I_{\Delta^{\mathrm{lex}}}$. Let $B=\{x_{F_1},\dots,x_{F_t}\}$ be the set of squarefree monomials of degree s in I_Δ, and let $L=\{x_{G_1},\dots,x_{G_t}\}$ be the set of squarefree monomials of degree s in $I_{\Delta^{\mathrm{lex}}}$. Then by Corollary 3.2 we may assume that $x_{F_k} \geq_{\mathrm{rev}} x_{G_k}$ for all $k=1,2,\dots,t$. Also by the assumption there exists an integer $1 \leq \ell \leq t$ such that $x_{F_\ell} >_{\mathrm{rev}} x_{G_\ell}$. Then Lemma 4.2 says $A_p^*(x_{F_\ell}) < A_p^*(x_{G_\ell})$ for some $p>0$, and therefore $\sum_{x_F \in I_\Delta,\ \deg x_F = s} A_p^*(x_F) < \sum_{x_G \in I_{\Delta^{\mathrm{lex}}},\ \deg x_G = s} A_p^*(x_G)$ by Lemma 3.3. Then Proposition 4.1 and (18) imply $\beta_{p,p+s-1}(S/(I_\Delta+Q)) < \beta_{p,p+s-1}(S/(I_{\Delta^{\mathrm{lex}}}+Q))$. □

Remark 4.4. Theorem 4.3 does not hold for all **a**-Borel-plus-powers ideals. Indeed, if $a_1=\cdots=a_n=\infty$ then **a**-Borel-plus-powers ideals are strongly stable ideals and **a**-lex-plus-powers ideals are lexsegment ideals. It is known that the graded Betti numbers of a graded ideal I are equal to those of the lexsegment ideal L with the same Hilbert function as I if and only if I is Gotzmann, that is, $\beta_1(S/I)=\beta_1(S/L)$ (see [10]), and there are many Gotzmann strongly stable ideals. On the other hand, we are not sure that there exists a simplicial complex Δ which is not isomorphic to Δ^{lex} such that the graded Betti numbers of $I_\Delta+Q$ are equal to those of $I_{\Delta^{\mathrm{lex}}}+Q$.

Next, we will show that the lex-plus-squares ideal has the largest graded Betti numbers among all graded ideals containing $x_1^2, x_2^2, \dots, x_n^2$ in arbitrary characteristic. To prove this fact, we require combinatorial shifting and the Hochster's formula.

Combinatorial shifting. Let Δ be a simplicial complex on $[n]$. For integers $1 \leq i < j \leq n$, let $\mathrm{Shift}_{ij}(\Delta)=\{C_{ij}^\Delta(F) : F \in \Delta\}$ be the family of subsets of $[n]$ defined by

$$C_{ij}^\Delta(F)=\begin{cases} (F\setminus\{i\})\cup\{j\}, & \text{if } i\in F,\ j\notin F \text{ and } (F\setminus\{i\})\cup\{j\}\notin\Delta, \\ F, & \text{otherwise.}\end{cases}$$

This $\mathrm{Shift}_{ij}(\Delta)$ is indeed a simplicial complex. It is not hard to see that there exists a sequence of pairs of integers $(i_1,j_1),(i_2,j_2),\dots,(i_p,j_p)$, where $i_k<j_k$ for all k, such that

$$\mathrm{Shift}_{i_pj_p}(\cdots(\mathrm{Shift}_{i_2j_2}(\mathrm{Shift}_{i_1j_1}(\Delta)))\cdots)$$

is shifted (see e.g., [9, §8]). Such a shifted complex is called a *combinatorial shifted complex of* Δ, and will be denoted by Δ^c. The operation $\Delta \to \Delta^c$ is called *combinatorial shifting*. The following fact was proved in [12, Lemma 3.3].

Lemma 4.5. *Let Δ be a simplicial complex on $[n]$ and let $1 \leq i < j \leq n$. Then*

$$\beta_{pq}(S/I_\Delta) \leq \beta_{pq}(S/I_{\mathrm{Shift}_{ij}(\Delta)}) \quad \textit{for all } p \textit{ and } q.$$

Hochster's formula. For a simplicial complex Δ on $[n]$, let

$$0 \longrightarrow C_{n-1}(\Delta) \xrightarrow{\partial_{n-1}} \cdots \longrightarrow C_1(\Delta) \xrightarrow{\partial_1} C_0(\Delta) \xrightarrow{\partial_0} C_{-1}(\Delta) \longrightarrow 0$$

be the augmented oriented chain complex of Δ over K. Thus each $C_{k-1}(\Delta)$ is the K-vector space with basis $\{e_F : F \in \Delta \text{ and } |F| = k\}$ and each $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ is the K-linear map induced by

$$\partial_k(e_{\{i_0,i_1,\dots,i_k\}}) = \sum_{j=0}^{k} (-1)^j e_{\{i_0,\dots,i_{j-1},i_{j+1},\dots,i_k\}},$$

where $i_0 < i_1 < \cdots < i_k$. The *k-th reduced homology group $\tilde{H}_k(\Delta;K)$ of Δ over K* is the k-th homology group of the augmented oriented chain complex of Δ over K.

Graded Betti numbers of Stanley–Reisner ideals can be computed from reduced homology groups by using the Hochster's formula [1, Theorem 5.5.1]. For a subset $W \subset [n]$, let $\Delta_W = \{F \subset W : F \in \Delta\}$.

Theorem 4.6 (Hochster)**.** *Let Δ be a simplicial complex on $[n]$. Then*

$$\beta_{pp+q}(S/I_\Delta) = \sum_{W \subset [n],\ |W|=p+q} \dim_K \tilde{H}_{q-1}(\Delta_W;K) \quad \textit{for all } p \textit{ and } q.$$

We will show

Theorem 4.7. *Let Δ be a simplicial complex on $[n]$ and let $\Gamma = \mathrm{Shift}_{ij}(\Delta)$, where $1 \leq i < j \leq n$. Then we have*

$$\beta_{pq}(I_\Delta + Q) \leq \beta_{pq}(I_\Gamma + Q) \quad \textit{for all } p \textit{ and } q.$$

In particular, the graded Betti numbers of $I_{\Delta^c} + Q$ are larger than or equal to those of $I_\Delta + Q$.

Theorems 1.1 and 4.7 immediately imply the next corollary.

Corollary 4.8. *Let K be an arbitrary field. Let I be a graded ideal of S containing $x_1^2, \dots, x_n^2$, and let L be the lex-plus-squares ideal with the same Hilbert function as I. The graded Betti numbers of L are larger than or equal to those of I.*

Proof. In the same way as the proof of Theorem 3.4, we may assume that I is a monomial ideal. Then there exists the simplicial complex Δ on $[n]$ such that $I = I_\Delta + Q$. By the definition of Shift_{ij}, the number of elements F in Δ with $|F| = k$ is equal to that in $\mathrm{Shift}_{ij}(\Delta)$ for all integer $k \geq 0$ and for all $1 \leq i < j \leq n$.

This fact says that $I_\Delta+Q$ and $I_{\Delta^c}+Q$ have the same Hilbert function. Since $I_{\Delta^c}+Q$ is a Borel-plus-squares ideal, Theorems 1.1 and 4.7 say

$$\beta_{pq}(I_\Delta+Q)\le\beta_{pq}(I_{\Delta^c}+Q)\le\beta_{pq}(L)$$

for all p and q. □

To prove Theorem 4.7, we require the following notion. Let Δ be a simplicial complex on $[n]$ and $I=I_\Delta$. The *link* of $\sigma\subset[n]$ in Δ is the simplicial complex

$$\mathrm{lk}_\Delta(\sigma)=\{F\subset[n]\setminus\sigma : F\cup\sigma\in\Delta\}.$$

We regard $\mathrm{lk}_\Delta(\sigma)$ as a simplicial complex on $[n]\setminus\sigma$ and regard $I_{\mathrm{lk}_\Delta(\sigma)}$ as an ideal of $S[\sigma]$. Then it is easy to see that the ideal $I[\sigma]=(I:x_\sigma)\cap S[\sigma]$ defied in §2 is the ideal $I_{\mathrm{lk}_\Delta(\sigma)}$. Thus by Corollary 2.3 and (2) we have

Lemma 4.9. *Let Δ and Γ be simplicial complexes on $[n]$. If $\sum_{|\sigma|=k}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Delta(\sigma)})\le\sum_{|\sigma|=k}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Gamma(\sigma)})$ for all p,q and k, where $\sigma\subset[n]$, then one has*

$$\beta_{pq}(I_\Delta+Q)\le\beta_{pq}(I_\Gamma+Q)\quad \textit{for all } p \textit{ and } q.$$

In the rest of this section we will prove Theorem 4.7 by using the technique developed in [12]. Let Δ be a simplicial complex on $[n]$ and $\Gamma=\mathrm{Shift}_{ij}(\Delta)$, where $1\le i<j\le n$. The next properties easily follow from the definition of Shift_{ij}.

Lemma 4.10. *Let $\sigma\subset[n]\setminus\{i,j\}$.*
- (i) *$\sigma\cup\{j\}\in\Gamma$ if and only if $\sigma\cup\{i\}\in\Delta$ or $\sigma\cup\{j\}\in\Delta$;*
- (ii) *$\sigma\cup\{i\}\in\Gamma$ if and only if $\sigma\cup\{i\}\in\Delta$ and $\sigma\cup\{j\}\in\Delta$;*
- (iii) *For any $F\in\mathrm{lk}_\Delta(\sigma)$, one has $\sigma\cup C_{ij}^{\mathrm{lk}_\Delta(\sigma)}(F)=C_{ij}^{\Delta}(\sigma\cup F)$.*

By virtue of Lemma 4.9, to prove Theorem 4.7, it is enough to show

$$(19)\qquad \sum_{\sigma\subset[n],\ |\sigma|=k}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Delta(\sigma)})\le\sum_{\sigma\subset[n],\ |\sigma|=k}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Gamma(\sigma)})\quad\text{for all } p,q \text{ and } k.$$

We remark following simple facts.

Lemma 4.11. *Let $\sigma\subset[n]$.*
- (i) *If $\{i,j\}\subset\sigma$ then $\mathrm{lk}_\Delta(\sigma)=\mathrm{lk}_\Gamma(\sigma)$;*
- (ii) *If $\{i,j\}\cap\sigma=\emptyset$ then $\mathrm{Shift}_{ij}(\mathrm{lk}_\Delta(\sigma))=\mathrm{lk}_\Gamma(\sigma)$.*

Proof. Statement (i) is clear since, for any $F\subset[n]\setminus\{i,j\}$, one has $F\cup\{i,j\}\in\Delta$ if and only if $F\cup\{i,j\}\in\Gamma$. On the other hand, statement (ii) follows from Lemma 4.10 (iii). □

Now, Lemmas 4.5 and 4.11 say that, to prove (19), it suffices to show

$$(20)\qquad \begin{aligned}&\sum_{\sigma\subset[n]\setminus\{i,j\},\ |\sigma|=k-1}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Delta(\sigma\cup\{i\})})+\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Delta(\sigma\cup\{j\})})\\ \le&\sum_{\sigma\subset[n]\setminus\{i,j\},\ |\sigma|=k-1}\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Gamma(\sigma\cup\{i\})})+\beta_{pq}(S[\sigma]/I_{\mathrm{lk}_\Gamma(\sigma\cup\{j\})})\end{aligned}$$

for all p, q and k. On the other hand, the Hochster's formula says that, for any $\sigma \subset [n] \setminus \{i,j\}$, we have

$$(21)\ \beta_{pp+q}(S[\sigma]/I_{\mathrm{lk}_\Delta(\sigma\cup\{i\})}) = \sum_{\substack{W\subset[n]\setminus(\sigma\cup\{i,j\})\\|W|=p+q}} \dim_K \tilde{H}_{q-1}((\mathrm{lk}_\Delta(\sigma\cup\{i\}))_W;K) + \sum_{\substack{W\subset[n]\setminus(\sigma\cup\{i,j\})\\|W|=p+q-1}} \dim_K \tilde{H}_{q-1}((\mathrm{lk}_\Delta(\sigma\cup\{i\}))_{W\cup\{j\}};K)$$

for all p and q, and have similar equations for $I_{\mathrm{lk}_\Delta(\sigma\cup\{j\})}$, $I_{\mathrm{lk}_\Gamma(\sigma\cup\{i\})}$ and $I_{\mathrm{lk}_\Gamma(\sigma\cup\{j\})}$.

Let $\sigma \subset [n]\setminus\{i,j\}$ and $W \subset [n]\setminus(\sigma\cup\{i,j\})$. Set

$$\begin{aligned} \Delta_1 &= \mathrm{lk}_\Delta(\sigma\cup\{i\})_W, & \Gamma_1 &= \mathrm{lk}_\Gamma(\sigma\cup\{i\})_W,\\ \Delta_2 &= \mathrm{lk}_\Delta(\sigma\cup\{j\})_W, & \Gamma_2 &= \mathrm{lk}_\Gamma(\sigma\cup\{j\})_W,\\ \Delta_3 &= \mathrm{lk}_\Delta(\sigma\cup\{i\})_{W\cup\{j\}}, & \Gamma_3 &= \mathrm{lk}_\Gamma(\sigma\cup\{i\})_{W\cup\{j\}},\\ \Delta_4 &= \mathrm{lk}_\Delta(\sigma\cup\{j\})_{W\cup\{i\}} \quad \text{and} & \Gamma_4 &= \mathrm{lk}_\Gamma(\sigma\cup\{j\})_{W\cup\{i\}}. \end{aligned}$$

Then (21) says that, to prove (20), it is enough to show

$$(22)\quad \dim_K \tilde{H}_k(\Delta_1;K) + \dim_K \tilde{H}_k(\Delta_2;K) \le \dim_K \tilde{H}_k(\Gamma_1;K) + \dim_K \tilde{H}_k(\Gamma_2;K)$$

and

$$(23)\quad \dim_K \tilde{H}_k(\Delta_3;K) + \dim_K \tilde{H}_k(\Delta_4;K) \le \dim_K \tilde{H}_k(\Gamma_3;K) + \dim_K \tilde{H}_k(\Gamma_4;K)$$

for all k.

Case 1: First, we will show (22). The next fact easily follows.

Lemma 4.12. $\Delta_1 \cap \Delta_2 = \Gamma_1$ *and* $\Delta_1 \cup \Delta_2 = \Gamma_2$.

Proof. By Lemma 4.10 (ii), we have

$$\begin{aligned} F \in \Delta_1\cap\Delta_2 &\Leftrightarrow F \subset W,\ F\cup\sigma\cup\{i\} \in \Delta \text{ and } F\cup\sigma\cup\{j\}\in\Delta\\ &\Leftrightarrow F\subset W \text{ and } F\cup\sigma\cup\{i\}\in\Gamma\\ &\Leftrightarrow F \in \mathrm{lk}_\Gamma(\sigma\cup\{i\})_W = \Gamma_1. \end{aligned}$$

Thus $\Delta_1\cap\Delta_2 = \Gamma_1$. On the other hand, $\Delta_1\cup\Delta_2=\Gamma_2$ follows from Lemma 4.10 (i) in the same way. □

By using the Mayer–Vietoris exact sequence of Δ_1 and Δ_2 (see [14, p. 21]), we obtain the exact sequence

$$(24)\quad 0 \to \ker\delta_k \longrightarrow \tilde{H}_k(\Delta_1\cap\Delta_2;K) \xrightarrow{\delta_k} \tilde{H}_k(\Delta_1;K)\bigoplus\tilde{H}_k(\Delta_2;K) \longrightarrow \tilde{H}_k(\Delta_1\cup\Delta_2;K) \longrightarrow \ker\delta_{k-1}\to 0.$$

The above exact sequence together with Lemma 4.12 says

$$\dim_K \tilde{H}_k(\Delta_1;K) + \dim_K \tilde{H}_k(\Delta_2;K) \le \dim_K \tilde{H}_k(\Gamma_1;K) + \dim_K \tilde{H}_k(\Gamma_2;K)$$

for all k, as desired.

Case 2: Next, we will show (23). Set $\Sigma = \mathrm{lk}_\Delta(\sigma \cup \{i,j\})_W$. Lemma 4.11 (i) says $\Sigma = \mathrm{lk}_\Gamma(\sigma \cup \{i,j\})_W$. Then we have

Lemma 4.13.

$$\Delta_3 = \Delta_1 \cup \{F \cup \{j\} : F \in \Sigma\}, \qquad \Gamma_3 = \Gamma_1 \cup \{F \cup \{j\} : F \in \Sigma\},$$
$$\Delta_4 = \Delta_2 \cup \{F \cup \{i\} : F \in \Sigma\} \quad \textit{and} \quad \Gamma_4 = \Gamma_2 \cup \{F \cup \{i\} : F \in \Sigma\}.$$

Proof. We will show $\Delta_3 = \Delta_1 \cup \{F \cup \{j\} : F \in \Sigma\}$. (Other cases can be proved in the same way.) It is clear that $\Delta_3 \supset \Delta_1 \cup \{F \cup \{j\} : F \in \Sigma\}$. Let $F \in \Delta_3$. If $F \subset W$ then $F \in \Delta_1$ is clear. Suppose $F \not\subset W$. Then $j \in F$. Moreover $F \in \Delta_3$ implies $\sigma \cup \{i\} \cup F \in \Delta$. Then $F \setminus \{j\} \in \mathrm{lk}_\Delta(\sigma \cup \{i,j\})_W = \Sigma$ as desired. □

Then Lemmas 4.12 and 4.13 say that $\Delta_3 \cap \Delta_4 = \Gamma_3 \cap \Gamma_4$ and $\Delta_3 \cup \Delta_4 = \Gamma_3 \cup \Gamma_4$. Set $A = \Delta_3 \cap \Delta_4 = \Gamma_3 \cap \Gamma_4$ and $B = \Delta_3 \cup \Delta_4 = \Gamma_3 \cup \Gamma_4$. By using the same exact sequences as (24), we have

$$\dim_K \tilde{H}_k(\Delta_3;K) + \dim_K \tilde{H}_k(\Delta_4;K) = \dim_K \tilde{H}_k(A;K) + \dim_K \tilde{H}_k(B;K) - \{\dim_K \ker \delta_k + \dim_K \ker \delta_{k-1}\} \tag{25}$$

and

$$\dim_K \tilde{H}_k(\Gamma_3;K) + \dim_K \tilde{H}_k(\Gamma_4;K) = \dim_K \tilde{H}_k(A;K) + \dim_K \tilde{H}_k(B;K) - \{\dim_K \ker \delta'_k + \dim_K \ker \delta'_{k-1}\}, \tag{26}$$

where $\delta_k : \tilde{H}_k(\Delta_3 \cap \Delta_4;K) \to \tilde{H}_k(\Delta_3;K) \bigoplus \tilde{H}_k(\Delta_4;K)$ is the map which appears in the Mayer-Vietoris exact sequence of Δ_3 and Δ_4 and $\delta'_k : \tilde{H}_k(\Gamma_3 \cap \Gamma_4;K) \to \tilde{H}_k(\Gamma_3;K) \bigoplus \tilde{H}_k(\Gamma_4;K)$ is that of Γ_3 and Γ_4. Then Lemma 4.14, stated below, implies the desired inequality (23).

Lemma 4.14. $\ker \delta_k \supset \ker \delta'_k$ *for all* k.

Proof. We may assume that $j = i+1$ by a proper permutation of $[n]$. Suppose $[a] \in \ker \delta'_k$ where $a \in C_k(\Gamma_3 \cap \Gamma_4)$. Then $\delta'_k([a]) = ([a],[a]) \in \tilde{H}_k(\Gamma_3;K) \bigoplus \tilde{H}_k(\Gamma_4;K)$ vanishes. Hence there exists $u \in C_{k+1}(\Gamma_3)$ such that $\partial_{k+1}(u) = a$. By Lemma 4.13, u can be written in the form

$$u = \sum_{F \in \Gamma_1,\ |F|=k+2} \alpha_F e_F + \sum_{G \in \Sigma,\ |G|=k+1} \alpha_G e_{G \cup \{j\}}$$

where each $\alpha_F \in K$. Since Lemmas 4.12 and 4.13 say that $\Gamma_3 \subset \Delta_3$, we have $u \in C_{k+1}(\Delta_3)$ and $[a] = [\partial_{k+1}(u)] \in \tilde{H}_k(\Delta_3;K)$ vanishes.

On the other hand, Lemmas 4.12 and 4.13 also say that $\Gamma_1 \cup \{F \cup \{i\} : F \in \Sigma\} \subset \Delta_4$. Hence

$$v = \sum_{F \in \Gamma_1,\ |F|=k+2} \alpha_F e_F + \sum_{G \in \Sigma,\ |G|=k+1} \alpha_G e_{G \cup \{i\}}$$

is an element of $C_{k+1}(\Delta_4)$. Since we assume $j = i+1$, it follows that $\partial_{k+1}(v) = a$. Thus $[a] \in \tilde{H}_k(\Delta_4;K)$ vanishes. Hence $\delta_k([a]) = ([a],[a]) \in \tilde{H}_k(\Delta_3;K) \bigoplus \tilde{H}_k(\Delta_4;K)$ vanishes as required. □

Remark 4.15. Although we proved $\sum_{|\sigma|=k}\beta_{pq}(I_\Delta : x_\sigma) \leq \sum_{|\sigma|=k}\beta_{pq}(I_{\Delta^c} : x_\sigma)$ for all p, q and k in this section, it is not always true that $\sum_{|\sigma|=k}\beta_{pq}(I_\Delta : x_\sigma) \leq \sum_{|\sigma|=k}\beta_{pq}(I_{\Delta^{\mathrm{lex}}} : x_\sigma)$. See [11, Example 3.10].

5. Consecutive cancellations in Betti numbers

Let $\{b_{ij}\}$ and $\{b'_{ij}\}$, where $i,j \in \mathbb{Z}$, be sequences of integers. We say that $\{b'_{ij}\}$ is obtained from $\{b_{ij}\}$ by *a consecutive* p,q*-cancellation* if $\{b'_{ij}\}$ is obtained from $\{b_{ij}\}$ by replacing b_{pq} by $b_{pq}-1$ and by replacing $b_{p-1,q}$ by $b_{p-1,q}-1$. Also, we say that $\{b'_{ij}\}$ is obtained form $\{b_{ij}\}$ by *consecutive cancellations* if there exists a sequence $\{b'_{ij}\} = \{b^{(0)}_{ij}\}, \{b^{(1)}_{ij}\}, \dots, \{b^{(\ell)}_{ij}\} = \{b_{ij}\}$ such that, for each $k=1,2,\dots,\ell$, $\{b^{(k-1)}_{ij}\}$ is obtained from $\{b^{(k)}_{ij}\}$ by a consecutive p_k,q_k-cancellation for some $(p_k,q_k) \in \mathbb{Z}^2$.

Let I and J be graded ideals of S. Then it is not hard to see that the graded Betti numbers of S/I are obtained from those of S/J by consecutive cancellations if and only if there exist integers $c_{ij} \geq 0$ such that

$$\beta_{ij}(S/I) = \beta_{ij}(S/J) - c_{ij} - c_{i+1,j} \quad \text{for all } i \text{ and } j.$$

The above equation says that a consecutive i,j-cancellation occurs c_{ij} times to obtain the graded Betti numbers of S/I from those of S/J. The integer c_{ij} will be called the i,j-th *cancellation number* of I and J. We refer the reader to [8, Example 1.35] and [13] for further details on consecutive cancellations for graded Betti numbers.

It is known that the graded Betti numbers of any graded ideal I are obtained from those of its initial ideal by consecutive cancellations (see [8, Corollary 1.21] or [13]). Moreover, Peeva [13] proved that the graded Betti numbers of I are obtained from those of the lexsegment ideal with the same Hilbert function as I by consecutive cancellations. Proposition 2.1 and (18) imply the following fact.

Theorem 5.1. *Let* $2 \leq a_1 \leq \cdots \leq a_n \leq \infty$*. The graded Betti numbers of any* $\mathbf{a}$*-Borel-plus-powers ideal* I *are obtained from those of the* $\mathbf{a}$*-lex-plus-powers ideal* L *having the same Hilbert function as* I *by consecutive cancellations.*

Indeed, it is easy to see that the $i,i+j$-th cancellation number of I and L is

$$\sum_{u\in\mathcal{M}^{\mathbf{a}}(L'),\ \deg u=j+1} A_i^{\mathbf{a}}(u) - \sum_{u\in\mathcal{M}^{\mathbf{a}}(I'),\ \deg u=j+1} A_i^{\mathbf{a}}(u) \geq 0.$$

Here I' and L' are $\mathbf{a}$-ideals satisfying $I = I' + (x_1^{a_1},\dots,x_n^{a_n})$ and $L = L' + (x_1^{a_1},\dots,x_n^{a_n})$.

In [11, Problem 1.3], it was asked that the graded Betti numbers of a graded ideal I containing $x_1^2,\dots,x_n^2$ are obtained from those of the lex-plus-squares ideal with the same Hilbert function as I by consecutive cancellations. The following refinement of Theorem 3.4 gives an affirmative answer to this problem.

Theorem 5.2. *With the same notation as in Theorem 3.4, the graded Betti numbers of* I *are obtained from those of* L *by consecutive cancellations.*

Proof. Since the graded Betti numbers of any graded ideal J are obtained from those of its initial ideal by consecutive cancellations, Theorem 5.1 and the proof of Theorem 3.4 say that what we must prove is that, for any monomial ideal I of

$K[x_1, \ldots, x_n]$ over a field K of characteristic 0, the graded Betti numbers of I are obtained from those of the monomial ideal I' of $K'[x_1, \ldots, x_n]$ over a field K' of characteristic p with $G(I') = G(I)$ by consecutive cancellations.

Since the graded Betti numbers of any monomial ideal are equal to those of some squarefree monomial ideal (see [1, Lemma 4.2.16]), we may assume that I is a squarefree monomial ideal. Hence there exists a simplicial complex Γ such that $I = I_\Gamma$. Then, by using the Hochster's formula, it is enough to show that the K-dimensions of the reduced homology groups $\tilde{H}_k(\Delta; K)$ and the K'-dimensions of $\tilde{H}_k(\Delta; K')$ differ by consecutive cancellations for any simplicial complex Δ, i.e., there exist integers $c_k \geq 0$, where $k = 0, 1, \ldots$, such that

$$\dim_K \tilde{H}_k(\Delta; K) = \dim_{K'} \tilde{H}_k(\Delta; K') - c_k - c_{k+1} \quad \text{for all } k. \tag{27}$$

We will show the above equation. The dimension of the kernel of $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ over the field K' is larger than or equal to that of $\partial_k : C_k(\Delta) \to C_{k-1}(\Delta)$ over the field K for each $k = 0, 1, \ldots$, since ∂_k is identified with a matrix whose coefficients are 0 or ± 1 and since the rank of a matrix is equal to the maximal size of its nonzero minors. Let c_k be this difference. Since

$$\dim_K \tilde{H}_k(\Delta; K) = \dim_K \ker \partial_k - \{|\{F \in \Delta : |F| = k+2\}| - \dim_K \ker \partial_{k+1}\},$$

the desired equation (27) follows. □

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