

FREE RESOLUTIONS OF LEX IDEALS OVER A KOSZUL TORIC RING

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ABSTRACT. In this paper, we study the minimal free resolution of lex-ideals over a Koszul toric ring. In particular, we study in which toric ring R all lex-ideals are componentwise linear. We give a certain necessity and sufficiency condition for this property, and show that lex-ideals in a strongly Koszul toric ring are componentwise linear. In addition, it is shown that, in the toric ring arising from the Segre product $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$, every Hilbert function of a graded ideal is attained by a lex-ideal and that lex-ideals have the greatest graded Betti numbers among all ideals having the same Hilbert function.

1. INTRODUCTION

Lex ideals, introduced by Macaulay [Ma], have played an important role in the study of Hilbert functions of homogeneous ideals in a polynomial ring. Recently, it is of interest to study lex-ideals in a quotient ring $R = S/I$, where $S = \mathbb{Q}[x_1, \dots, x_n]$ is a polynomial ring and where I is either a monomial or a projective toric ideal (see e.g., [GHP, GMP, MP, MM, MuP]). We call $R = S/I$ a *projective toric ring* if I is a projective toric ideal. In this paper, we study the minimal free resolution of lex-ideals over a Koszul projective toric ring.

Lex ideals over a projective toric ring were introduced by Gasharov, Horwitz and Peeva [GHP] (see section 2 for the definition). They proposed a series of open problems on lex-ideals over a projective toric ring. One of the fundamental problems is the following problem, which was originally suggested by Mermin and Peeva [MP].

Problem 1.1 (Mermin-Peeva). Find a class of either monomial or projective toric ideals I such that every Hilbert function of a graded ideal in $R = S/I$ is attained by a lex-ideal. Moreover, prove that lex-ideals have the greatest graded Betti numbers over R among all graded ideals in R having the same Hilbert function.

The above problem is motivated by Macaulay's Theorem [Ma], which proved that for every homogeneous ideal $J \subset S$ there exists a lex-ideal in S having the same Hilbert function as J , as well as the Bigatti-Hulett-Pardue Theorem [Bi, Hu, Pa], which proved that lex-ideals in S have the greatest graded Betti numbers among all homogeneous ideals in S having the same Hilbert function. Problem 1.1 was

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studied in several cases, including exterior algebras [AHH], Clements-Lindströme rings [CL, MuP] and Veronese subrings of a polynomial ring [GMP].

In order to attack Problem 1.1, it is important to study the following problem proposed by Gasharov, Horwitz and Peeva [GHP, Problem 4.6].

Problem 1.2 (Gasharov-Horwitz-Peeva). Study the structure of minimal free resolutions of lex-ideals in a projective toric ring.

The aim of the first part of this paper is to study in which toric ring R all lex-ideals are componentwise linear, in other words, in which toric ring R all lex-ideals generated by monomials of the same degree have a linear resolution. Such a toric ring must be Koszul (i.e., the minimal free resolution of the residue field over R is a linear resolution) since the maximal ideal in R is a lex-ideal. If all lex-ideals in R are componentwise linear, then the minimal free resolution of lex-ideals seems not to be complicated. Indeed, we give a nice way to compute the graded Betti numbers of lex-ideals in R when all lex-ideals in R are componentwise linear (see Corollary 3.17).

It was observed in [MM, Remark 4.7] that if I is generated by monomials of degree 2 then all lex-ideals in $R = S/I$ are componentwise linear (the result of Fröberg [Fr] guarantees that R is Koszul). While we cannot expect such a nice property for all Koszul toric rings, we give a certain necessity and sufficiency condition for projective toric rings R such that all lex-ideals in R are componentwise linear (Theorem 3.13). In particular, we show that lex-ideals in a strongly Koszul toric ring are componentwise linear (Theorem 3.20).

In the second part of this paper, we study Hilbert functions and graded Betti numbers of graded ideals in the following toric ring: Let r be a positive integer, $U = \{t_{1,i_1} t_{2,i_2} \cdots t_{r,i_r} : i_k \in \{0, 1\} \text{ for } k = 1, 2, \dots, r\}$ and $S' = \mathbb{Q}[x_m : m \in U]$, where $t_{i,j}$ and x_m with $m \in U$ are indeterminates. Let I_W be the kernel of the ring homomorphism

$$\begin{aligned} S' &\rightarrow \mathbb{Q}[t_{1,0}, t_{1,1}, t_{2,0}, t_{2,1}, \dots, t_{r,0}, t_{r,1}] \\ x_m &\mapsto m \end{aligned}$$

and $R_W = S'/I_W$. Thus $I_W \subset S'$ is the defining ideal of the Segre embedding of $\mathbb{P}^1 \times \cdots \times \mathbb{P}^1$ in \mathbb{P}^{2^r-1} . It is known that R_W is strongly Koszul, and therefore all lex-ideals in R_W are componentwise linear.

Theorem 1.3. *For every graded ideal $J \subset R_W$, there exists a lex-ideal $L \subset R_W$ having the same Hilbert function as J . Moreover, the graded Betti numbers of L are greater than or equal to those of J .*

Theorem 1.3 is false for $\mathbb{P}^1 \times \mathbb{P}^2$ (see Example 4.21), and the proof of Theorem 1.3 is not easy. Our proof based on the combinatorial result of Frankl, Füredi and Kalai [FFK], which characterizes the face vectors of colored simplicial complexes.

Finally, we discuss a few results which are obtained from the componentwise linearity of lex-ideals. Let $R = S/I$, where I is either a monomial or a projective toric ideal. Suppose that R satisfies the following conditions:

- (A) For every graded ideal $J \subset R$, there exists a lex-ideal $L \subset R$ having the same Hilbert function as J .
- (B) Every lex-ideal $L \subset R$ has the greatest graded Betti numbers among all graded ideals in R having the same Hilbert function as L .
- (C) Every lex-ideal in R is componentwise linear.

For example, polynomial rings, Veronese subrings of a polynomial ring and the toric ring R_W considered in Theorem 1.3 satisfy the above conditions.

First, we show that the graded Betti numbers of any graded ideal $J \subset R$ are obtained from those of the lex-ideal $L \subset R$ having the same Hilbert function as J by a sequence of consecutive cancellations (see section 5 for the definition of ‘consecutive cancellations’). Second, we show that an analogue of Gotzmann’s persistence theorem [Go] holds for R . Third, we show that the graded Betti numbers of a graded ideal $J \subset R$ are equal to those of the lex-ideal $L \subset R$ having the same Hilbert function as J if and only if J is a Gotzmann ideal, that is, the number of minimal generators of J is equal to that of L .

This paper is organized as follows: In section 2, we recall fundamental properties of toric rings and lex-ideals, and introduce the notion of initial ideals for ideals in a semigroup ring. In sections 3, we study when all lex-ideals in a projective toric ring are componentwise linear. In section 4, by using the results proved in sections 2 and 3, we prove Theorem 1.3. In section 5, we discuss a relation between the componentwise linearity of lex-ideals and Gotzmann’s persistence theorem.

2. LEX IDEALS OVER TORIC RINGS

In this section, we recall fundamental properties about toric rings and lex-ideals. We first introduce the notations that will be used throughout the paper. Let $S = K[x_1, \dots, x_n]$ be a polynomial ring over a field K with each $\deg x_i = 1$, $I \subset S$ a graded ideal and $R = S/I$. Let M be a finitely generated graded R -module. The *Hilbert function* $H(M, -) : \mathbb{Z} \rightarrow \mathbb{Z}$ of M is the numerical function defined by

$$H(M, d) = \dim_K M_d$$

for all $d \in \mathbb{Z}$, where M_d is the graded component of degree d of M . We write $\beta_{i,j}^R(M) = \dim_K \operatorname{Tor}_i^R(M, K)_j$ for the *graded Betti numbers* of M over R . Recall that the graded Betti numbers $\beta_{i,j}^R = \beta_{i,j}^R(M)$ appear in the minimal graded free R -resolution

$$\cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{i,j}^R} \longrightarrow \cdots \longrightarrow \bigoplus_j R(-j)^{\beta_{1,j}^R} \longrightarrow \bigoplus_j R(-j)^{\beta_{0,j}^R} \longrightarrow M \longrightarrow 0$$

of M over R . For a graded ideal $J \subset R$ and for an element $m \in R$, we write $(J : m) = \{f \in R : fm \in J\}$.

2.1. Lex ideals in a toric ring.

Let \mathbb{N} be the set of nonnegative integers and $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$ a subset of $\mathbb{N}^r \setminus \{0\}$, where $\mathbf{a}_i = (a_{i,1}, a_{i,2}, \dots, a_{i,r})$ for $i = 1, 2, \dots, n$. The *toric ideal* $I_{\mathcal{A}} \subset S$ associated to \mathcal{A} is the kernel of the ring homomorphism

$$\begin{aligned} \pi_{\mathcal{A}} : S &\rightarrow K[t_1, \dots, t_r] \\ x_i &\mapsto \mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i,1}} t_2^{a_{i,2}} \cdots t_r^{a_{i,r}}. \end{aligned}$$

Set $T = K[t_1, \dots, t_r]$ and $K[\mathcal{A}] = K[\mathbf{t}^{\mathbf{a}_1}, \dots, \mathbf{t}^{\mathbf{a}_n}]$. The *toric ring* associated to \mathcal{A} is

$$R_{\mathcal{A}} = S/I_{\mathcal{A}} \cong K[\mathcal{A}].$$

We denote by $\varphi_{\mathcal{A}} : R_{\mathcal{A}} \rightarrow K[\mathcal{A}]$ the ring isomorphism induced by $\pi_{\mathcal{A}}$.

A toric ring $R_{\mathcal{A}}$ is said to be *projective* if $I_{\mathcal{A}}$ is a homogeneous ideal. It is well-known that $R_{\mathcal{A}}$ is projective if and only if there exists $c_1, \dots, c_r \in \mathbb{R}$ such that

$$a_{i,1}c_1 + \cdots + a_{i,r}c_r = 1$$

for $i = 1, 2, \dots, n$.

A projective toric ring $R_{\mathcal{A}}$ inherits the grading of S . We define the grading of a semigroup ring $K[\mathcal{A}]$ by $\deg(t_1^{b_1} \cdots t_r^{b_r}) = b_1c_1 + \cdots + b_rc_r$ for any monomial $t_1^{b_1} \cdots t_r^{b_r} \in K[\mathcal{A}]$. Then the isomorphism $\varphi_{\mathcal{A}}$ preserves the grading. A projective toric ring $R_{\mathcal{A}} \cong K[\mathcal{A}]$ is also \mathbb{Z}^r -graded by considering the \mathbb{Z}^r -grading of T .

In the rest of this paper, we assume that $R_{\mathcal{A}}$ is always projective and $I_{\mathcal{A}} \subset (x_1, \dots, x_n)^2$. Note that the latter condition says that $\mathbf{a}_i \neq \mathbf{a}_j$ for all $i \neq j$.

Now, we recall the definition of lex-ideals in a projective toric ring $R_{\mathcal{A}}$ introduced by Gasharov, Horwitz and Peeva [GHP].

Definition 2.1. Let $>_{\text{lex}}$ be the degree lexicographic order on S induced by the ordering $x_1 > \cdots > x_n$. For a monomial m in $K[\mathcal{A}]$, let $\text{top}(m)$ be the lex-greatest monomial in S such that its image in $K[\mathcal{A}]$ is m . Thus

$$\text{top}(m) = \max_{>_{\text{lex}}} \{\mu \in S : \pi_{\mathcal{A}}(\mu) = m\}.$$

Define the total order $>_{\text{lex}_{\mathcal{A}}}$ on the set of monomials in $K[\mathcal{A}]$ by

$$m >_{\text{lex}_{\mathcal{A}}} m' \iff \text{top}(m) >_{\text{lex}} \text{top}(m').$$

Sometimes, it is convenient to consider monomials in the quotient ring $R_{\mathcal{A}} = S/I_{\mathcal{A}}$ instead of monomials in the semigroup ring $K[\mathcal{A}]$. An element $m \in R_{\mathcal{A}}$ is called a *monomial* if $\varphi_{\mathcal{A}}(m)$ is a monomial in $K[t_1, \dots, t_r]$. Thus $m \in R_{\mathcal{A}}$ is a monomial if and only if it is an image of a monomial in S . An ideal in $R_{\mathcal{A}}$ generated by monomials is called a *monomial ideal*. For monomials $\mu, \mu' \in R_{\mathcal{A}}$, we define $\text{top}(\mu) = \text{top}(\varphi_{\mathcal{A}}(\mu))$ and we write $\mu >_{\text{lex}_{\mathcal{A}}} \mu'$ if $\varphi_{\mathcal{A}}(\mu) >_{\text{lex}_{\mathcal{A}}} \varphi_{\mathcal{A}}(\mu')$.

A set of monomials $L \subset R_{\mathcal{A}}$ is said to be *lex-segment* if, for all monomials $m \in L$ and $m' \in R_{\mathcal{A}}$ with $\deg m' = \deg m$ and $m' >_{\text{lex}_{\mathcal{A}}} m$, one has $m' \in L$. A *lex-ideal* $M \subset R_{\mathcal{A}}$ is a monomial ideal such that the set of all monomials in M is lex-segment.

We recall a few easy properties of lex-ideals.

Lemma 2.2 (Gasharov-Horwitz-Peeva [GHP, Theorem 3.4]). *If $L \subset R_{\mathcal{A}}$ is a lex-segment set of monomials of degree d then $\{x_i m \in R_{\mathcal{A}} : m \in L, i = 1, 2, \dots, n\}$ is a lex-segment set of monomials.*

Lemma 2.3. *Let m be a monomial in $R_{\mathcal{A}}$ of degree d and $L \subset R_{\mathcal{A}}$ the monomial ideal generated by all monomials of degree d in $R_{\mathcal{A}}$ which are strictly greater than m with respect to $>_{\text{lex}_{\mathcal{A}}}$. If $1 \leq i < j \leq n$ and $x_i, x_j \notin (L : m)$, then $m x_i >_{\text{lex}_{\mathcal{A}}} m x_j$.*

Proof. It is enough to show that $\text{top}(m x_i) = \text{top}(m) x_i$ if $x_i \notin (L : m)$. Suppose contrary that $\text{top}(m x_i) \neq \text{top}(m) x_i$. Then $\text{top}(m x_i) >_{\text{lex}} \text{top}(m) x_i$ and $\text{top}(m x_i) - \text{top}(m) x_i \in I_{\mathcal{A}}$. Let $u = \text{top}(m x_i)$ and let ℓ be the greatest integer k such that x_k divides u . Since $u >_{\text{lex}} \text{top}(m) x_i$, we have $u/x_{\ell} \geq_{\text{lex}} \text{top}(m)$.

First, suppose $u/x_{\ell} >_{\text{lex}} \text{top}(m)$. Let $v \in R_{\mathcal{A}}$ be the image of u/x_{ℓ} to $R_{\mathcal{A}}$. Then $v \in L$ and $m x_i = v x_{\ell} \in L$. This contradicts the assumption that $x_i \notin (L : m)$. Next, suppose $u/x_{\ell} = \text{top}(m)$. Then $u - \text{top}(m) x_i = \text{top}(m)(x_{\ell} - x_i)$ and $\ell \neq i$. Since $u - \text{top}(m) x_i \in I_{\mathcal{A}}$ and since $I_{\mathcal{A}}$ is prime, it follows that $x_{\ell} - x_i \in I_{\mathcal{A}}$. This contradicts $I_{\mathcal{A}} \subset (x_1, \dots, x_n)^2$. \square

Example 2.4. Let $K[\mathcal{A}] = K[s_0 t_0, s_1 t_0, s_0 t_1, s_1 t_1]$ and

$$\pi : K[x_1, \dots, x_4] \rightarrow K[s_0, s_1, t_0, t_1]$$

be the ring homomorphism defined by $\pi(x_1) = s_0 t_0$, $\pi(x_2) = s_1 t_0$, $\pi(x_3) = s_0 t_1$ and $\pi(x_4) = s_1 t_1$. Then the monomials of degree 2 in $K[\mathcal{A}]$ are ordered as follows:

$$\begin{aligned} s_0^2 t_0^2 >_{\text{lex}_{\mathcal{A}}} s_0 s_1 t_0^2 >_{\text{lex}_{\mathcal{A}}} s_0^2 t_0 t_1 >_{\text{lex}_{\mathcal{A}}} s_0 s_1 t_0 t_1 >_{\text{lex}_{\mathcal{A}}} s_1^2 t_0^2 \\ >_{\text{lex}_{\mathcal{A}}} s_1^2 t_0 t_1 >_{\text{lex}_{\mathcal{A}}} s_0^2 t_1^2 >_{\text{lex}_{\mathcal{A}}} s_0 s_1 t_1^2 >_{\text{lex}_{\mathcal{A}}} s_1^2 t_1^2. \end{aligned}$$

In particular, one observe that Lemma 2.3 does not hold if $x_j \in (L : m)$ since $\pi(x_2^2) = s_1^2 t_0^2 <_{\text{lex}_{\mathcal{A}}} s_0 s_1 t_0 t_1 = \pi(x_2 x_3)$.

2.2. Initial ideals and changes of coordinates.

Initial ideals and changes of coordinates have played an important role in the study of Problem 1.1 when I is a monomial ideal. Those methods were also used in [GMP] to study Problem 1.1 for Veronese toric rings. The aim of this subsection is to introduce initial ideals in a semigroup ring and to study their fundamental properties.

Recall that for a vector $\mathbf{w} = (w_1, \dots, w_r) \in \mathbb{R}^r$, the weight order $\succ_{\mathbf{w}}$ on $T = K[t_1, \dots, t_r]$ is a partial order on the set of monomials in T defined by

$$t_1^{b_1} \cdots t_r^{b_r} \succ_{\mathbf{w}} t_1^{b'_1} \cdots t_r^{b'_r} \iff \sum_{k=1}^r b_k w_k > \sum_{k=1}^r b'_k w_k.$$

For convenience, we write $\mathbf{w}(t_1^{b_1} \cdots t_r^{b_r}) = \sum_{k=1}^r b_k w_k$.

Definition 2.5. Let \succ be either a monomial order on T or a weight order on T . For any polynomial $f \in T$, we write $\text{in}_{\succ}(f)$ for the initial form of f with respect to \succ . The *initial ideal* $\text{in}_{\succ}(J)$ of a graded ideal $J \subset K[\mathcal{A}]$ with respect to \succ is the K -vector space spanned by the initial forms of all polynomials in J .

The following fact can be proved in the same way as polynomial rings (cf. [E, Section 15.8]).

Lemma 2.6. *Let $J \subset K[\mathcal{A}]$ be a graded ideal and \succ either a monomial order on T or a weight order on T . Then $\text{in}_\succ(J)$ is a graded ideal in $K[\mathcal{A}]$ having the same Hilbert function as J . Moreover, for any monomial order \succ' on T , there exists a weight order $\succ_{\mathbf{w}}$ on T such that $\text{in}_{\succ_{\mathbf{w}}}(J) = \text{in}_{\succ'}(J)$.*

The next claim shows that taking initial ideals over a semigroup ring $K[\mathcal{A}]$ corresponds to taking initial ideals over S with respect to a certain weight order.

Proposition 2.7. *Let $J \subset K[\mathcal{A}]$ be a graded ideal and $\succ_{\mathbf{c}}$ a weight order on T . Then there exists a weight order $\succ_{\mathbf{w}}$ on S such that $\text{in}_{\succ_{\mathbf{w}}}(\pi_{\mathcal{A}}^{-1}(J)) = \pi_{\mathcal{A}}^{-1}(\text{in}_{\succ_{\mathbf{c}}}(J))$ and $\text{in}_{\succ_{\mathbf{w}}}(I_{\mathcal{A}}) = I_{\mathcal{A}}$.*

Proof. Define the weight order $\succ_{\mathbf{w}}$ on S by setting $\mathbf{w}(x_i) = \mathbf{c}(\mathbf{t}^{\mathbf{a}_i})$ for $i = 1, 2, \dots, n$. Then, for any monomial $m \in S$, one has that $\mathbf{w}(m) = \mathbf{c}(\pi_{\mathcal{A}}(m))$. Thus, for any binomial generator $u - v \in I_{\mathcal{A}}$, we have $\text{in}_{\succ_{\mathbf{w}}}(u - v) = u - v$ since $\mathbf{w}(u) = \mathbf{c}(\pi_{\mathcal{A}}(u)) = \mathbf{c}(\pi_{\mathcal{A}}(v)) = \mathbf{w}(v)$. Hence $\text{in}_{\succ_{\mathbf{w}}}(I_{\mathcal{A}}) = I_{\mathcal{A}}$.

It remains to show that $\text{in}_{\succ_{\mathbf{w}}}(\pi_{\mathcal{A}}^{-1}(J)) = \pi_{\mathcal{A}}^{-1}(\text{in}_{\succ_{\mathbf{c}}}(J))$. Since J and $\text{in}_{\succ_{\mathbf{c}}}(J)$ have the same Hilbert function, the ideals $\pi_{\mathcal{A}}^{-1}(J)$ and $\pi_{\mathcal{A}}^{-1}(\text{in}_{\succ_{\mathbf{c}}}(J))$ have the same Hilbert function (see [GHP, Lemma 2.3]). Thus it is enough to prove that $\text{in}_{\succ_{\mathbf{w}}}(\pi_{\mathcal{A}}^{-1}(J)) \subset \pi_{\mathcal{A}}^{-1}(\text{in}_{\succ_{\mathbf{c}}}(J))$.

Let $h = \alpha_1 \mu_1 + \dots + \alpha_s \mu_s \in \text{in}_{\succ_{\mathbf{w}}}(\pi_{\mathcal{A}}^{-1}(J))$, where each $\alpha_k \in K \setminus \{0\}$ and each μ_k is a monomial in S . Let $\mathbf{w}(h) = \mathbf{w}(\mu_1)$. Note that $\mathbf{w}(\mu_1) = \dots = \mathbf{w}(\mu_s)$. We will prove $h \in \pi_{\mathcal{A}}^{-1}(\text{in}_{\succ_{\mathbf{c}}}(J))$, that is, $\pi_{\mathcal{A}}(h) \in \text{in}_{\succ_{\mathbf{c}}}(J)$. We may assume that $h \notin I_{\mathcal{A}}$. Then there exists a polynomial $f = h + \gamma_1 m_1 + \dots + \gamma_t m_t \in \pi_{\mathcal{A}}^{-1}(J)$, where each $\gamma_k \in K \setminus \{0\}$ and each m_k is a monomial in S , such that $\text{in}_{\succ_{\mathbf{w}}}(f) = h$. Then $\mathbf{w}(h) > \mathbf{w}(m_k)$ for $k = 1, 2, \dots, t$. Consider $\pi_{\mathcal{A}}(f) = \sum_{k=1}^s \alpha_k \pi_{\mathcal{A}}(\mu_k) + \sum_{k=1}^t \gamma_k \pi_{\mathcal{A}}(m_k)$. Since

$$\mathbf{c}(\pi_{\mathcal{A}}(\mu_1)) = \dots = \mathbf{c}(\pi_{\mathcal{A}}(\mu_s)) = \mathbf{w}(h) > \mathbf{w}(m_k) = \mathbf{c}(\pi_{\mathcal{A}}(m_k))$$

for $k = 1, 2, \dots, t$ and since $\pi_{\mathcal{A}}(h) \neq 0$, $\text{in}_{\succ_{\mathbf{c}}}(\pi_{\mathcal{A}}(f)) = \sum_{k=1}^s \alpha_k \pi_{\mathcal{A}}(\mu_k) = \pi_{\mathcal{A}}(h)$. Since $\pi_{\mathcal{A}}(f) \in J$, we have $\pi_{\mathcal{A}}(h) \in \text{in}_{\succ_{\mathbf{c}}}(J)$ as desired. \square

By applying [GHP, Proposition 2.6] to Proposition 2.7, we get the following result.

Corollary 2.8. *Let $J \subset K[\mathcal{A}]$ be a graded ideal and \succ a monomial order on T . Then $\beta_{i,j}^{K[\mathcal{A}]}(J) \leq \beta_{i,j}^{K[\mathcal{A}]}(\text{in}_\succ(J))$ and $\beta_{i,j}^S(\pi_{\mathcal{A}}^{-1}(J)) \leq \beta_{i,j}^S(\pi_{\mathcal{A}}^{-1}(\text{in}_\succ(J)))$ for all i, j .*

The above result was proved in [GMP] in the special case when $K[\mathcal{A}]$ is a Veronese subring of a polynomial ring. Also, Gasharov, Horwitz and Peeva [GHP, Theorem 2.5] proved a similar result which is applicable to any projective toric rings. However, Corollary 2.8 seems more convenient since it allows us to use any monomial order on T .

Finally, we consider changes of coordinates. Let $GL_r(K)$ be the general linear groups with coefficients in K . Any $\phi = (b_{ij}) \in GL_r(K)$ induces an automorphism

of T , again denoted by ϕ , namely

$$\phi(f(t_1, \dots, t_r)) = f\left(\sum_{i=1}^r b_{i1}t_i, \dots, \sum_{i=1}^r b_{ir}t_i\right) \quad \text{for all } f \in T.$$

Suppose that the restriction of ϕ to $K[\mathcal{A}]$ is an automorphism of the graded K -algebra $K[\mathcal{A}]$ (i.e. $K[\phi(\mathbf{t}^{\mathbf{a}_1}), \dots, \phi(\mathbf{t}^{\mathbf{a}_n})] = K[\mathcal{A}]$). Write

$$\phi(\mathbf{t}^{\mathbf{a}_j}) = \sum_{i=1}^n \alpha_{ij} \mathbf{t}^{\mathbf{a}_i} \quad \text{for } j = 1, 2, \dots, n.$$

Then, we define the automorphism $\bar{\phi}$ of $S = K[x_1, \dots, x_n]$ by

$$\bar{\phi}(x_j) = \sum_{i=1}^n \alpha_{ij} x_i \quad \text{for } j = 1, 2, \dots, n.$$

Clearly one has

$$\phi \circ \pi_{\mathcal{A}} = \pi_{\mathcal{A}} \circ \bar{\phi}.$$

Since ϕ is an automorphism of $K[\mathcal{A}]$, the above equation says that, for any graded ideal $J \subset K[\mathcal{A}]$, the ideals J and $\phi(J)$ have the same graded Betti numbers over $K[\mathcal{A}]$, and that the ideals $\pi_{\mathcal{A}}^{-1}(J)$ and $\pi_{\mathcal{A}}^{-1}(\phi(J)) = \bar{\phi}(\pi_{\mathcal{A}}^{-1}(J))$ have the same graded Betti numbers over S . In particular, by Corollary 2.8, we obtain

Corollary 2.9. *Let $J \subset K[\mathcal{A}]$ be a graded ideal and $\phi \in GL_r(K)$. Suppose that the restriction of ϕ to $K[\mathcal{A}]$ is an automorphism of $K[\mathcal{A}]$. Then, for any monomial order \succ on T , one has that $\beta_{i,j}^{K[\mathcal{A}]}(J) \leq \beta_{i,j}^{K[\mathcal{A}]}(\text{in}_{\succ}(\phi(J)))$ and $\beta_{i,j}^S(\pi_{\mathcal{A}}^{-1}(J)) \leq \beta_{i,j}^S(\pi_{\mathcal{A}}^{-1}(\text{in}_{\succ}(\phi(J))))$ for all i, j .*

3. COMPONENTWISE LINEARITY OF LEX-IDEALS

Let $S = K[x_1, \dots, x_n]$ be a polynomial ring, $I \subset S$ a homogeneous ideal and $R = S/I$. Let $J = (f_1, \dots, f_{\delta})$ be a graded ideal in R . Set $J_j = (f_1, \dots, f_j)$ for $j = 0, 1, \dots, \delta$, where $(f_1, \dots, f_j) = 0$ if $j = 0$. Then, for $j = 1, 2, \dots, \delta$, we have the short exact sequence

$$(1) \quad 0 \longrightarrow R/(J_{j-1} : f_j)(-\deg f_j) \xrightarrow{\times f_j} R/J_{j-1} \longrightarrow R/J_j \longrightarrow 0.$$

If a free R -resolution of $R/(J_{j-1} : f_j)$ and that of R/J_{j-1} are known, one obtains a free R -resolution of R/J_j by taking the mapping cone. In particular, if minimal free R -resolutions of all the $R/(J_{j-1} : f_j)$ are known, we obtain a free R -resolution of J by iterated mapping cones. While such a resolution is not always minimal, it is known that this construction gives the minimal free resolution of several important classes of graded ideals, such as stable ideals and lex-plus-powers ideals [CE, MPS]. The aim of this section is to apply this construction to obtain the minimal free resolution of lex-ideals over a Koszul toric ring.

Throughout this section, we assume that R is *Koszul*, that is, $\beta_{i,j}^R(K) = 0$ for all i, j with $i \neq j$. For a graded ideal $J \subset R$, the (*Castelnuovo-Mumford*) *regularity* of J is the integer

$$\operatorname{reg}_R(J) = \sup\{k : \beta_{i,i+k}^R(J) \neq 0 \text{ for some } i\}.$$

Note that $\operatorname{reg}_R(J) < \infty$ since R is Koszul (cf. [AP]).

3.1. Ideals with linear quotients.

We say that a graded ideal $J \subset R$ has a *k-linear resolution* if $\beta_{i,i+j}^R(J) = 0$ for all i, j with $j \neq k$. Thus we are assuming that the zero ideal has a *k-linear resolution* for all $k \in \mathbb{N}$.

Definition 3.1. A graded ideal $J \subset R$ is said to have *strong linear quotients* if there exists a sequence of elements $f_1, \dots, f_\delta \in R$ satisfying the following conditions:

- (a) J is minimally generated by f_1, \dots, f_δ ;
- (b) The colon ideals $((f_1, \dots, f_{k-1}) : f_k)$ have a 1-linear resolution for $k = 1, \dots, \delta$.

The sequence f_1, \dots, f_δ satisfying (a) and (b) will be called a *shelling* of J .

The above definition is a natural generalization of the notion ‘ideals with linear quotients’ introduced by Herzog and Takayama [HT]. They considered the case when J is a monomial ideal in a polynomial ring. We are using ‘strong linear quotients’ since the name ‘linear quotients’ is usually used when the colon ideals appearing in (b) are generated by variables. Also, the name ‘shelling’ is used since if J is a squarefree monomial ideal in a polynomial ring then it corresponds to a shelling of simplicial complexes via Alexander duality. Indeed, the next lemma comes from the Rearrangement Lemma for (non-pure) shellable simplicial complexes [BW] in Combinatorics.

Lemma 3.2. *Let $J \subset R$ be a graded ideal having strong linear quotients and f_1, \dots, f_δ a shelling of J . Let $f_{i_1}, \dots, f_{i_\delta}$ be the rearrangement of f_1, \dots, f_δ satisfying that $\deg f_{i_1} \leq \dots \leq \deg f_{i_\delta}$ and that if $\deg f_{i_k} = \deg f_{i_\ell}$ and $i_k < i_\ell$ then $k < \ell$. Then*

$$(2) \quad ((f_{i_1}, f_{i_2}, \dots, f_{i_{k-1}}) : f_{i_k}) = ((f_1, f_2, \dots, f_{i_{k-1}}) : f_{i_k})$$

for $k = 1, 2, \dots, \delta$. In particular, $f_{i_1}, \dots, f_{i_\delta}$ is again a shelling of J .

Proof. We first show that the left-hand side of (2) contains the right-hand side. Since f_1, \dots, f_δ is a shelling, the ideal $((f_1, \dots, f_{i_{k-1}}) : f_{i_k})$ is generated by elements of degree 1. Thus, it is enough to show that, for any linear form $y \in R_1$ with $yf_{i_k} \in (f_1, \dots, f_{i_{k-1}})$, one has $yf_{i_k} \in (f_{i_1}, \dots, f_{i_{k-1}})$. Write $yf_{i_k} = \mu_1 f_{j_1} + \dots + \mu_t f_{j_t}$, where each $f_{j_\ell} \in \{f_1, \dots, f_{i_{k-1}}\}$ and where each $\mu_\ell \in R \setminus \{0\}$. Then $\deg \mu_\ell > 0$ for all ℓ since J is minimally generated by f_1, \dots, f_δ . Thus $\deg f_{j_\ell} \leq \deg f_{i_k}$ and $j_\ell < i_k$ for all ℓ . Then by the assumption of the rearrangement, $f_{j_\ell} \in \{f_{i_1}, \dots, f_{i_{k-1}}\}$ for $\ell = 1, 2, \dots, t$. Hence $yf_{i_k} \in (f_{i_1}, f_{i_2}, \dots, f_{i_{k-1}})$ as desired.

Let $d_j = \deg f_j$ for $j = 1, 2, \dots, \delta$. By (1), for $j = 1, 2, \dots, \delta$,

$$\begin{aligned} & H(R/(f_1, \dots, f_j), d) \\ &= H(R/(f_1, \dots, f_{j-1}), d) - H(R/((f_1, \dots, f_{j-1}) : f_j), d - d_j) \end{aligned}$$

for all $d \geq 0$. Arguing inductively, it follows that

$$H(R/J, d) = H(R, d) - \sum_{k=1}^{\delta} H(R/((f_1, \dots, f_{i_{k-1}}) : f_{i_k}), d - d_{i_k}).$$

Similarly, we have

$$H(R/J, d) = H(R, d) - \sum_{k=1}^{\delta} H(R/((f_{i_1}, \dots, f_{i_{k-1}}) : f_{i_k}), d - d_{i_k}).$$

Since we already proved that $((f_{i_1}, \dots, f_{i_{k-1}}) : f_{i_k}) \supset ((f_1, \dots, f_{i_{k-1}}) : f_{i_k})$, the above equations guarantee the desired equation (2). \square

Lemma 3.2 says that if a graded ideal $J \subset R$ has strong linear quotients then there exists a shelling f_1, \dots, f_δ of J satisfying $\deg f_1 \leq \dots \leq \deg f_\delta$. This fact and the following simple observation yield a simple formula of the graded Betti numbers of ideals having strong linear quotients.

Lemma 3.3. *Let $J \subset R$ be a graded ideal. For any $f \in R$ with $\deg f \geq \operatorname{reg}_R(J)$,*

$$\beta_{i,j}^R(J + (f)) = \beta_{i,j}^R(J) + \beta_{i,j-\deg f}^R(R/(J : f)) \quad \text{for all } i, j.$$

Proof. Consider the short exact sequence

$$0 \longrightarrow R/(J : f)(-\deg f) \xrightarrow{\times f} R/J \longrightarrow R/(J + (f)) \longrightarrow 0.$$

Let \mathbf{G} be the graded minimal free R -resolution of $R/(J : f)(-\deg f)$ and let \mathbf{F} be that of R/J . Consider the mapping cone $\mathbf{M}(\alpha)$ of the complex homomorphism $\alpha : \mathbf{G} \rightarrow \mathbf{F}$ that is a lifting of the map $R/(J : f)(-\deg f) \xrightarrow{\times f} R/J$. Note that $\mathbf{M}(\alpha)$ is a free resolution of $R/(J + (f))$ (cf. [E, p. 654]). Thus what we must prove is that the free resolution $\mathbf{M}(\alpha)$ is minimal. By the construction of mapping cones, it is enough to prove that each $\alpha_i : \mathbf{G}_i \rightarrow \mathbf{F}_i$ satisfies $\alpha_i(\mathbf{G}_i) \subset \mathfrak{m}\mathbf{F}_i$, where \mathfrak{m} is the graded maximal ideal of R . This fact is clear since the minimal shift appearing in \mathbf{G}_i is less than or equal to $-i - \deg f$ and the maximal shift appearing in \mathbf{F}_i is greater than or equal to $-i + 1 - \operatorname{reg}_R(J) \geq -i + 1 - \deg f$ by the assumption. \square

The next result is a generalization of [HT, Corollary 1.6] and [SV, Corollary 2.7].

Theorem 3.4. *Let $J \subset R$ be a graded ideal having strong linear quotients and f_1, \dots, f_δ a shelling of J . Then*

$$\beta_{i,i+j}^R(J) = \sum_{\deg(f_k)=j} \beta_i^R(R/((f_1, \dots, f_{k-1}) : f_k)) \quad \text{for all } i, j.$$

Proof. We use induction on δ . If $\delta = 1$ then the statement is obvious since the first syzygy module of $J = (f_1)$ is $(0 : f_1)$ and since $(0 : f_1)$ has a 1-linear resolution by the assumption. Suppose $\delta > 1$. By Lemma 3.2, we may assume $\deg f_1 \leq \cdots \leq \deg f_\delta$. Let $J' = (f_1, \dots, f_{\delta-1})$. Clearly J' has linear quotients. Then, by the induction hypothesis, the graded Betti numbers of J' are given by the desired formula. In particular, $\text{reg}(J') = \deg f_{\delta-1} \leq \deg f_\delta$. Then, by Lemma 3.3,

$$\beta_{i,j}^R(J) = \beta_{i,j}^R(J' + (f_\delta)) = \beta_{i,j}^R(J') + \beta_{i,j-\deg f_\delta}^R(R/(J' : f_\delta)).$$

Since $(J' : f_\delta)$ has a 1-linear resolution, the desired formula follows from the induction hypothesis. \square

Remark 3.5. If a graded ideal $J \subset R$ has a 1-linear resolution then the Betti numbers of J only depends on Hilbert functions. Indeed, it is known that the Hilbert series $\sum_{k \geq 0} H(R/J, k)T^k$ of R/J and that of R determine the Poincare series $\sum_{k \geq 0} \beta_k^R(R/J)T^k$ of R/J by the following relation (cf. [HHR, p. 163]):

$$\left(\sum_{k \geq 0} \beta_k^R(R/J)T^k \right) \left(\sum_{k \geq 0} H(R/J, k)(-T)^k \right) = \sum_{k \geq 0} H(R, k)(-T)^k.$$

Let $J = \bigoplus_{k \geq 0} J_k \subset R$ be a graded ideal. We write $J_{\langle d \rangle} \subset R$ for the ideal generated by all elements of degree d in J and write $J_{\geq d} = \bigoplus_{k \geq d} J_k$. A graded ideal $J \subset R$ is said to be *componentwise linear* if $J_{\langle d \rangle}$ has a d -linear resolution for all integers $d \geq 0$. It is known that any homogeneous ideal $J \subset S$ having linear quotients is componentwise linear [BW, JZ, SV]. We extend this result to any Koszul graded K -algebra R .

Lemma 3.6. *Let $J \subset R$ be a graded ideal. Then, for all integers $j > d$,*

$$\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(J_{\geq d}).$$

Proof. Let \mathbf{K} be the graded minimal free R -resolution of K . Fix an integer d . Consider the complexes $J \otimes \mathbf{K}$ and $J_{\geq d} \otimes \mathbf{K}$. Recall that $\beta_{i,j}^R(J) = \dim_K H_i(J \otimes \mathbf{K})_j$, where $H_i(-)$ denotes the i th homology. Since R is Koszul, the i th free module \mathbf{K}_i of \mathbf{K} is a free module of the form $\mathbf{K}_i = R(-i)^{\beta_i^R(K)}$ for all i . Then we have

$$(J \otimes \mathbf{K}_p)_q \cong (J_{\geq d} \otimes \mathbf{K}_p)_q \quad \text{if } q - p \geq d.$$

(In the above equation, \mathbf{K}_p is the p -th free module of \mathbf{K} and $(J \otimes \mathbf{K}_p)_q$ is the graded component of degree q of the module $J \otimes \mathbf{K}_p$.) Indeed, any element in $(J \otimes \mathbf{K}_p)_q$ can be written as the sum of elements of the form $f \otimes e$ such that $f \in J$ is an element of degree $q - p$ and e is a base element of \mathbf{K}_p . Then it follows that $H_p(J \otimes \mathbf{K})_q \cong H_p(J_{\geq d} \otimes \mathbf{K})_q$ for all $p < q - d$, and therefore

$$\beta_{i,i+j}^R(J) = \dim_K H_i(J \otimes \mathbf{K})_{i+j} = \dim_K H_i(J_{\geq d} \otimes \mathbf{K})_{i+j} = \beta_{i,i+j}^R(J_{\geq d})$$

for all $j > d$. \square

Proposition 3.7. *If a graded ideal $J \subset R$ has strong linear quotients then J is componentwise linear.*

Proof. Let f_1, \dots, f_δ be a shelling of J satisfying $\deg f_1 \leq \dots \leq \deg f_\delta$. We use induction on δ . If $\delta = 1$, by Theorem 3.4, the ideal J has a linear resolution. Suppose $\delta > 1$. Let $J' = (f_1, \dots, f_{\delta-1})$. By the induction hypothesis J' is componentwise linear. Hence for $k < \deg f_\delta$ the ideal $J_{\langle k \rangle} = J'_{\langle k \rangle}$ has a k -linear resolution. Let $k \geq \deg f_\delta$. Then $J_{\langle k \rangle} = J_{\geq k}$. It is clear that $\beta_{i,i+j}^R(J_{\langle k \rangle}) = 0$ for $j < k$. On the other hand, since $k \geq \deg f_\delta$ and since $\operatorname{reg}_R(J) = \deg f_\delta$ by Theorem 3.4, it follows from Lemma 3.6 that $\beta_{i,i+j}^R(J_{\langle k \rangle}) = \beta_{i,i+j}^R(J_{\geq k}) = \beta_{i,i+j}^R(J) = 0$ for all $j > k$. Hence $J_{\langle k \rangle}$ has a k -linear resolution. \square

If a graded ideal $J \subset R$ is componentwise linear, then the minimal free resolution of J has several nice properties. In this paper, we need the next formula.

Lemma 3.8. *Let $J \subset R$ be a graded ideal. Then*

$$\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(J_{\geq j}) + \beta_{i+1,i+j}^R(J_{\geq j-1}) - (\dim_K J_{j-1})\beta_{i+1}^R(K) \quad \text{for all } i, j.$$

Moreover, if J is componentwise linear then

$$\beta_{i,i+j}^R(J) = \beta_i^R(J_{\langle j \rangle}) + \beta_{i+1}^R(J_{\langle j-1 \rangle}) - (\dim_K J_{j-1})\beta_{i+1}^R(K) \quad \text{for all } i, j.$$

Proof. The short exact sequence

$$0 \longrightarrow J_{\geq j} \longrightarrow J_{\geq j-1} \longrightarrow J_{\geq j-1}/J_{\geq j} \longrightarrow 0$$

yields the exact sequence

$$(3) \quad \begin{aligned} \operatorname{Tor}_{i+1}(J_{\geq j}, K)_{i+j} &\longrightarrow \operatorname{Tor}_{i+1}(J_{\geq j-1}, K)_{i+j} \longrightarrow \operatorname{Tor}_{i+1}(J_{\geq j-1}/J_{\geq j}, K)_{i+j} \\ &\longrightarrow \operatorname{Tor}_i(J_{\geq j}, K)_{i+j} \longrightarrow \operatorname{Tor}_i(J_{\geq j-1}, K)_{i+j} \longrightarrow \operatorname{Tor}_i(J_{\geq j-1}/J_{\geq j}, K)_{i+j}. \end{aligned}$$

Then $\operatorname{Tor}_{i+1}(J_{\geq j}, K)_{i+j} = 0$ since $J_{\geq j}$ has no elements of degree $\leq j-1$. Also, since $J_{\geq j-1}/J_{\geq j}$ is isomorphic to the direct sum of copies of K and is generated in degree $j-1$, it follows that

$$\dim_K (\operatorname{Tor}_{i+1}(J_{\geq j-1}/J_{\geq j}, K)_{i+j}) = (\dim_K J_{j-1})(\dim_K \operatorname{Tor}_{i+1}(K, K)_{i+1})$$

and $\operatorname{Tor}_i(J_{\geq j-1}/J_{\geq j}, K)_{i+j} = 0$. Then by (3)

$$\beta_{i,i+j}^R(J_{\geq j-1}) = \beta_{i,i+j}^R(J_{\geq j}) + \beta_{i+1,i+j}^R(J_{\geq j-1}) - (\dim_K J_{j-1})\beta_{i+1}^R(K).$$

Since $\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(J_{\geq j-1})$ by Lemma 3.6, the above equation yields the desired formula.

Next, suppose that J is componentwise linear. It is enough to prove that

$$\beta_{i,i+j}^R(J_{\geq j}) = \beta_i^R(J_{\langle j \rangle})$$

for all j . Consider the short exact sequence

$$0 \longrightarrow J_{\langle j \rangle} \longrightarrow J_{\geq j} \longrightarrow J_{\geq j}/J_{\langle j \rangle} \longrightarrow 0.$$

Since $J_{\geq j}/J_{\langle j \rangle}$ has no elements of degree $\leq j$, one has $\operatorname{Tor}_i(J_{\geq j}/J_{\langle j \rangle}, K)_{i+k} = 0$ for all $k \leq j$. Then, by considering the long exact sequence of $\operatorname{Tor}(-)$ arising from the above short exact sequence, we have $\operatorname{Tor}_i(J_{\langle j \rangle}, K)_{i+j} \cong \operatorname{Tor}_i(J_{\geq j}, K)_{i+j}$. Since $J_{\langle j \rangle}$ has a j -linear resolution, it follows that

$$\beta_{i,i+j}^R(J_{\geq j}) = \beta_{i,i+j}^R(J_{\langle j \rangle}) = \beta_i^R(J_{\langle j \rangle})$$

as desired. \square

Remark 3.9. It is known for specialists that if R is Koszul then $\beta_{i,i+j}^R(J)$ only depends on J_{j-1} and J_j . Indeed, the proof of Lemma 3.8 says that for any graded ideal $J \subset R$ one has

$$\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(J_{\langle j \rangle}) + \beta_{i+1,i+j}^R(J_{\langle j-1 \rangle}) - (\dim_K J_{j-1})\beta_{i+1}^R(K) \quad \text{for all } i, j.$$

3.2. Toric rings in which all lex-ideals are componentwise linear.

Let $R_{\mathcal{A}} = S/I_{\mathcal{A}}$ be a projective toric ring defined in section 2. In this subsection, we study when all the lex-ideals in $R_{\mathcal{A}}$ are componentwise linear. Such a toric ring must be Koszul since the maximal ideal of $R_{\mathcal{A}}$ is a lex-ideal. Thus, in the rest of this section, we assume that $R_{\mathcal{A}}$ is Koszul.

Let $L \subset R_{\mathcal{A}}$ be a lex-ideal. We say that the ordering u_1, \dots, u_{δ} of the minimal monomial generators of L is *natural* if it satisfies that $\deg u_1 \leq \dots \leq \deg u_{\delta}$ and that $\deg u_i = \deg u_j$ and $u_i >_{\text{lex}_{\mathcal{A}}} u_j$ imply $i < j$. The following fact immediately follows from Lemma 2.2.

Lemma 3.10. *Let $L \subset R_{\mathcal{A}}$ be a lex-ideal and u_1, \dots, u_{δ} a natural order of the minimal monomial generators of L . Then, for $j = 1, 2, \dots, \delta$, the ideal $(u_1, \dots, u_j) \subset R_{\mathcal{A}}$ is a lex-ideal.*

To describe the main result, we need the following technical notation.

Definition 3.11. Let $\mathcal{K}(\mathcal{A})$ be the minimal set of monomial ideals in $R_{\mathcal{A}}$ satisfying the following conditions:

- (i) $(0) \in \mathcal{K}(\mathcal{A})$.
- (ii) If $J \in \mathcal{K}(\mathcal{A})$ then for any $x_j \notin J$ one has $(J + (x_1, \dots, x_{j-1}) : x_j) \in \mathcal{K}(\mathcal{A})$.

Note that $|\mathcal{K}(\mathcal{A})| < \infty$.

Example 3.12. Let $K[\mathcal{A}]$ be as in Example 2.4. Then

$$\begin{aligned} (0 : x_1) &= 0, & ((x_1) : x_2) &= (x_1, x_3), \\ ((x_1, x_2) : x_3) &= (x_1, x_2), & ((x_1, x_2, x_3) : x_4) &= (x_1, x_2, x_3), \\ ((x_1, x_3) : x_2) &= (x_1, x_3). \end{aligned}$$

The above computations show $\mathcal{K}(\mathcal{A}) = \{(0), (x_1, x_2), (x_1, x_3), (x_1, x_2, x_3)\}$.

The main result of this section is the following result.

Theorem 3.13. *The following conditions are equivalent.*

- (i) *Every ideal in $\mathcal{K}(\mathcal{A})$ has a 1-linear resolution;*
- (ii) *Every lex-ideal $L \subset R_{\mathcal{A}}$ has strong linear quotients and the natural order of the minimal monomial generators of L is a shelling of L ;*
- (iii) *Every lex-ideal in $R_{\mathcal{A}}$ is componentwise linear.*

In the rest of this subsection, we prove Theorem 3.13 in a series of lemmas. The next lemma proves (i) \Rightarrow (ii).

Lemma 3.14. *Let $L = (u_1, \dots, u_\delta) \subset R_{\mathcal{A}}$ be a lex-ideal, where u_1, \dots, u_δ is a natural order. Then for $j = 1, 2, \dots, \delta$ one has $((u_1, \dots, u_{j-1}) : u_j) \in \mathcal{K}(\mathcal{A})$.*

Proof. By Lemma 3.10, each (u_1, \dots, u_j) is a lex-ideal. Thus it is enough to consider the case $j = \delta$. We use double induction on δ and $\deg u_\delta$. If $\delta = 1$ then $(0 : u_1) = (0) \in \mathcal{K}(\mathcal{A})$. Also, if $\deg u_\delta = 1$ then $L = (x_1, \dots, x_\delta)$. Thus $((u_1, \dots, u_{\delta-1}) : u_\delta) = ((x_1, \dots, x_{\delta-1}) : x_\delta) \in \mathcal{K}(\mathcal{A})$.

Suppose $\delta > 1$ and $\deg u_\delta > 1$. Set $d = \deg u_\delta$. Let

$$\nu = \max_{>_{\text{lex}_{\mathcal{A}}}} \{m \in R_{\mathcal{A}} : \text{there exists } 1 \leq i \leq n \text{ such that } mx_i = u_\delta\}$$

and

$$J = (\{m \in R_{\mathcal{A}} : m \text{ is a monomial of degree } d-1 \text{ such that } m >_{\text{lex}_{\mathcal{A}}} \nu\}).$$

Clearly J is a lex-ideal and $u_\delta \notin J$. Since L is also a lex-ideal, the definition of a natural order shows that there exists an integer $1 \leq t < \delta$ such that

$$\{m \in L \setminus J : m \text{ is a monomial of degree } d\} = \{u_{t+1}, \dots, u_\delta\}.$$

Since $J + (\nu)$ is a lex-ideal and $u_\delta \in J + (\nu)$, it follows that

$$u_{t+k} \in (J + (\nu)) \setminus J = \nu(R_{\mathcal{A}} \setminus (J : \nu))$$

for $k = 1, 2, \dots, \delta - t$. Let $\{x_{i_1}, \dots, x_{i_s}\}$ be the set of variables which are not in $(J : \nu)$, where $i_1 < \dots < i_s$. Then, since $u_{t+1} >_{\text{lex}_{\mathcal{A}}} \dots >_{\text{lex}_{\mathcal{A}}} u_\delta$, Lemma 2.3 implies that for $k = 1, 2, \dots, \delta - t$, we have

$$u_{t+k} = \nu x_{i_k}.$$

We claim that

$$(4) \quad (J + (u_{t+1}, \dots, u_{\delta-1}) : u_\delta) = ((J : \nu) + (x_{i_1}, \dots, x_{i_{\delta-t-1}}) : x_{i_{\delta-t}}).$$

Since $u_{t+k} = \nu x_{i_k}$, it is clear that the left-hand side contains the right-hand side. We show that the right-hand side contains the left-hand side. Let $m \in (J + (u_{t+1}, \dots, u_{\delta-1}) : u_\delta)$. Since $R_{\mathcal{A}}$ is a \mathbb{Z}^r -graded ring such that each \mathbb{Z}^r -graded component is a 1-dimensional K -vector space spanned by a monomial, we may assume that m is a monomial, and therefore $mu_\delta \in J$ or $mu_\delta \in (u_{t+1}, \dots, u_{\delta-1})$. It is clear that if $mu_\delta = m\nu x_{i_{\delta-t}} \in J$ then

$$m \in ((J : \nu) + (x_{i_1}, \dots, x_{i_{\delta-t-1}}) : x_{i_{\delta-t}}).$$

Suppose $mu_\delta = m\nu x_{i_{\delta-t}} \in (u_{t+1}, \dots, u_{\delta-1}) = \nu(x_{i_1}, \dots, x_{i_{\delta-t-1}})$. Since $R_{\mathcal{A}}$ is a domain and $\nu \neq 0$, $mx_{i_{\delta-t}} \in (x_{i_1}, \dots, x_{i_{\delta-t-1}})$. Thus

$$m \in ((J : \nu) + (x_{i_1}, \dots, x_{i_{\delta-t-1}}) : x_{i_{\delta-t}})$$

as desired.

Now we will complete the proof of Lemma 3.14. Since $J + (\nu)$ is a lex-ideal and $\deg \nu = d - 1$, the induction hypothesis guarantees $(J : \nu) \in \mathcal{K}(\mathcal{A})$. Then, since

$(J + (u_{t+1}, \dots, u_{\delta-1}))_{\geq d} = (u_1, \dots, u_{\delta-1})_{\geq d}$, it follows that

$$\begin{aligned} ((u_1, \dots, u_{\delta-1}) : u_{\delta}) &= (J + (u_{t+1}, \dots, u_{\delta-1}) : u_{\delta}) \\ &= ((J : \nu) + (x_{i_1}, \dots, x_{i_{\delta-t-1}}) : x_{i_{\delta-t}}) \\ &= ((J : \nu) + (x_1, \dots, x_{i_{\delta-t-1}}) : x_{i_{\delta-t}}) \in \mathcal{K}(\mathcal{A}) \end{aligned}$$

as desired. (We use (4) for the second equality. For the third equality we use the fact that $\{x_{i_1}, \dots, x_{i_s}\}$ is the set of variables which are not in $(J : \nu)$.) \square

Lemma 3.15. *For any $J \in \mathcal{K}(\mathcal{A})$, there exists a lex-ideal $L = (u_1, \dots, u_{\delta})$, where u_1, \dots, u_{δ} is a natural order, such that $(u_1, \dots, u_{\delta-1} : u_{\delta}) = J$.*

Proof. By the construction of $\mathcal{K}(\mathcal{A})$, there exists a sequence of integers i_1, \dots, i_t and a sequence of ideals $(0) = J_0, J_1, \dots, J_t = J$ such that

$$J_k = (J_{k-1} + (x_1, \dots, x_{i_{k-1}}) : x_{i_k})$$

and $x_{i_k} \notin J_{k-1}$ for $k = 1, 2, \dots, t$. We construct lex-ideals $L_k = (u_{k,1}, \dots, u_{k,\delta_k})$, where $u_{k,1}, \dots, u_{k,\delta_k}$ is a natural order, such that $((u_{k,1}, \dots, u_{k,\delta_k-1}) : u_{k,\delta_k}) = J_k$ for $k = 1, 2, \dots, t$ inductively.

For $k = 1$, the ideal $L_1 = (x_1, \dots, x_{i_1})$ satisfies the desired condition. Suppose that there exists a lex-ideal $L_k = (u_{k,1}, \dots, u_{k,\delta_k})$ satisfying the desired condition. Let V be the set of variables which are not in $((u_{k,1}, \dots, u_{k,\delta_k-1}) : u_{k,\delta_k}) = J_k$. Note that $x_{i_{k+1}} \in V$ by the assumption. Set

$$L_{k+1} = (u_{k,1}, \dots, u_{k,\delta_k-1}) + u_{k,\delta_k}(\{x_{\ell} \in V : \ell \leq i_{k+1}\}).$$

Then L_{k+1} is a lex-ideal. Also, in the same way as the proof of equation (4), it follows that

$$\begin{aligned} &((u_{k,1}, \dots, u_{k,\delta_k-1}) + u_{k,\delta_k}(\{x_{\ell} \in V : \ell < i_{k+1}\}) : u_{k,\delta_k}x_{i_{k+1}}) \\ &= ((u_{k,1}, \dots, u_{k,\delta_k-1} : u_{k,\delta_k}) + (\{x_{\ell} \in V : \ell < i_{k+1}\}) : x_{i_{k+1}}) \\ &= (J_k + (x_1, \dots, x_{i_{k+1}-1}) : x_{i_{k+1}}) \\ &= J_{k+1} \end{aligned}$$

as desired. \square

Lemma 3.16. *Let $R = S/I$ be a Koszul graded K -algebra and $J \subset R$ a componentwise linear ideal. Then $\text{reg}_R(J) = \max\{k : \beta_{0,k}^R(J) \neq 0\}$.*

Proof. Let $d = \max\{k : \beta_{0,k}^R(J) \neq 0\}$. It is clear that $\text{reg}_R(J) \geq d$. We also have $\text{reg}_R(J) \leq d$ since Lemma 3.6 implies $\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(J_{\geq d}) = \beta_{i,i+j}^R(J_{\langle d \rangle}) = 0$ for all $j > d$. \square

Proof of Theorem 3.13. The implication (i) \Rightarrow (ii) is Lemma 3.14 and (ii) \Rightarrow (iii) is Proposition 3.7. We show (iii) \Rightarrow (i). Let $J \in \mathcal{K}(\mathcal{A})$. By Lemma 3.15 there exists a lex-ideal $L = (u_1, \dots, u_{\delta})$, where u_1, \dots, u_{δ} is a natural order, such that $J = ((u_1, \dots, u_{\delta-1}) : u_{\delta})$. By Lemma 3.10, $(u_1, \dots, u_{\delta-1})$ is a lex-ideal. Since $\deg u_1 \leq \dots \leq \deg u_{\delta}$ and since every lex-ideal in $R_{\mathcal{A}}$ is componentwise linear,

Lemma 3.16 implies $\operatorname{reg}_{R_{\mathcal{A}}}((u_1, \dots, u_{\delta-1})) = \deg u_{\delta-1} \leq \deg u_{\delta}$. Then by Lemma 3.3

$$\beta_{i,i+j}^{R_{\mathcal{A}}}(L) = \beta_{i,i+j}^{R_{\mathcal{A}}}((u_1, \dots, u_{\delta-1})) + \beta_{i,i+j-\deg u_{\delta}}^{R_{\mathcal{A}}}(R_{\mathcal{A}}/J).$$

Since L is a lex-ideal, $\operatorname{reg}_{R_{\mathcal{A}}}(L) = \deg u_{\delta}$ by Lemma 3.16. Then the above equation implies $\operatorname{reg}_{R_{\mathcal{A}}}(R_{\mathcal{A}}/J) \leq \operatorname{reg}_{R_{\mathcal{A}}}(L) - \deg u_{\delta} = 0$. Hence J has a 1-linear resolution. \square

Theorems 3.4 and 3.13 give the following formula of the graded Betti numbers of lex-ideals.

Corollary 3.17. *Suppose that every lex-ideal in $R_{\mathcal{A}}$ is componentwise linear. Let $L = (u_1, \dots, u_{\delta}) \subset R_{\mathcal{A}}$ be a lex-ideal, where u_1, \dots, u_{δ} is a natural order. Then*

$$\beta_{i,i+j}^{R_{\mathcal{A}}}(L) = \sum_{\deg u_k=j} \beta_i^{R_{\mathcal{A}}}(R_{\mathcal{A}}/((u_1, \dots, u_{k-1}) : u_k)) \quad \text{for all } i, j.$$

3.3. Some applications and examples.

In this subsection, we consider a few applications of Theorem 3.13. First, we show that if $R_{\mathcal{A}}$ is strongly Koszul then all lex-ideals in $R_{\mathcal{A}}$ are componentwise linear.

A toric ring $R_{\mathcal{A}}$ is said to be *strongly Koszul* if for any sequence $1 \leq i_1 < \dots < i_t \leq n$ and for all $j = 1, 2, \dots, t$ the colon ideal $((x_{i_1}, \dots, x_{i_{j-1}}) : x_{i_j}) \subset R_{\mathcal{A}}$ is generated by variables. Strongly Koszul algebras were introduced by Herzog, Hibi and Restuccia [HHR]. They proved the following fact.

Lemma 3.18 (Herzog-Hibi-Restuccia). *The following conditions are equivalent.*

- (i) $R_{\mathcal{A}}$ is strongly Koszul;
- (ii) The ideal $((x_i) : x_j)$ is generated by variables for all $i \neq j$;
- (iii) Any ideal in $R_{\mathcal{A}}$ generated by variables has a 1-linear resolution.

While (iii) \Rightarrow (ii) did not appear in [HHR], this implication is obvious since if $((x_i) : x_j)$ has a generator of degree ≥ 2 then the first syzygy module of (x_i, x_j) has a generator f with $\deg f \geq 3$.

Lemma 3.19. *If $R_{\mathcal{A}}$ is strongly Koszul then for any ideal $J \subset R_{\mathcal{A}}$ generated by variables and for any $x_j \notin J$ the colon ideal $(J : x_j)$ has a 1-linear resolution.*

Proof. Let $J = (x_{i_1}, \dots, x_{i_{\delta}})$. Since condition (ii) of Theorem 3.18 does not depend on the ordering of the variables, we may assume $i_1 < \dots < i_{\delta} < j$. Then by the definition of the strongly Koszul property the ideal $(J : x_j)$ is generated by variables. Then the statement follows from Lemma 3.18(iii). \square

If $R_{\mathcal{A}}$ is strongly Koszul, then by applying the above lemma to the construction of $\mathcal{K}(\mathcal{A})$ it follows that any ideal in $\mathcal{K}(\mathcal{A})$ has a 1-linear resolution. Hence by Theorem 3.13, we get

Theorem 3.20. *If $R_{\mathcal{A}}$ is strongly Koszul then every lex-ideal in $R_{\mathcal{A}}$ is componentwise linear.*

Remark 3.21. Properties of lex-ideals depend on the ordering of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. Indeed, if we change the ordering then the set $\mathcal{K}(\mathcal{A})$ changes. However if $R_{\mathcal{A}}$ is strongly Koszul then all lex-ideals in $R_{\mathcal{A}}$ are componentwise linear with respect to any ordering of $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. Such a strong property is only true for strongly Koszul toric rings since if such a property is true then any ideal generated by variables has a 1-linear resolution.

Example 3.22. It is not an easy problem to check whether an ideal $J \subset R_{\mathcal{A}}$ has a linear resolution even if J is generated by variables. On the other hand, there is a nice combinatorial way, called Koszul filtrations, to prove that any ideal in $\mathcal{K}(\mathcal{A})$ has a 1-linear resolution. Koszul filtrations were introduced by Conca, Trung and Valla [CTV]. We consider the following special type of Koszul filtrations: A *simple Koszul filtration* of $R_{\mathcal{A}}$ is a family \mathcal{F} of ideals in $R_{\mathcal{A}}$ generated by variables such that

- (i) $(x_1, \dots, x_n) \in \mathcal{F}$.
- (ii) If $(x_{i_1}, \dots, x_{i_p}) \in \mathcal{F}$ then there exists an integer $1 \leq j \leq p$ such that $(x_{i_1}, \dots, \hat{x}_{i_j}, \dots, x_{i_p}) \in \mathcal{F}$ and $((x_{i_1}, \dots, \hat{x}_{i_j}, \dots, x_{i_p}) : x_{i_j}) \in \mathcal{F}$.

It was proved in [CTV, Proposition 1.2] that if \mathcal{F} is a Koszul filtration of $R_{\mathcal{A}}$ then any ideal in \mathcal{F} has a 1-linear resolution. This fact says that if there exists a Koszul filtration \mathcal{F} of $R_{\mathcal{A}}$ with $\mathcal{F} \supset \mathcal{K}(\mathcal{A})$ then one can conclude that all lex-ideals in $R_{\mathcal{A}}$ are componentwise linear.

For example, consider

$$R_{\mathcal{A}} = K[x_1, \dots, x_8]/I_{\mathcal{A}} \cong K[t_1t_2, t_1t_3, t_1t_4, t_1t_5, t_2t_3, t_2t_4, t_2t_5, t_3t_4].$$

Then by using the computer algebra system Macaulay 2 [GS] we compute

$$\mathcal{K}(\mathcal{A}) = \left\{ \begin{array}{l} (0), (x_1, x_6), (x_1, x_2, x_5), (x_1, x_2, x_3, x_5, x_6), (x_1, \dots, x_4), \\ (x_1, \dots, x_5), (x_1, \dots, x_6), (x_1, \dots, x_7), (x_1, x_2, x_5, x_6, x_7), \\ (x_1, x_2, x_3, x_4, x_6), (x_1, x_2, x_3, x_5, x_6, x_7) \end{array} \right\}.$$

On the other hand, the following set \mathcal{F} is a Koszul filtration of $R_{\mathcal{A}}$

$$\mathcal{F} = \left\{ \begin{array}{l} (0), (x_1), (x_1, x_2), (x_1, x_2, x_3), \dots, (x_1, \dots, x_8), (x_1, x_6), (x_1, x_2, x_5), \\ (x_1, x_2, x_3, x_5), (x_1, x_2, x_6), (x_1, x_2, x_5, x_6), (x_1, x_2, x_5, x_6, x_7), \\ (x_1, x_2, x_3, x_4, x_6), (x_1, x_2, x_3, x_5, x_6), (x_1, x_2, x_3, x_5, x_6, x_7) \end{array} \right\}.$$

(To show that \mathcal{F} is a Koszul filtration of $R_{\mathcal{A}}$, prove that, for each $(x_{i_1}, \dots, x_{i_p}) \in \mathcal{F}$ with $i_1 < \dots < i_p$, $(x_{i_1}, \dots, x_{i_{p-1}}) \in \mathcal{F}$ and $((x_{i_1}, \dots, x_{i_{p-1}}) : x_{i_p}) \in \mathcal{F}$.) Then, we have $\mathcal{K}(\mathcal{A}) \subset \mathcal{F}$, and therefore all lex-ideals in $R_{\mathcal{A}}$ are componentwise linear. Note that $R_{\mathcal{A}}$ is not strongly Koszul since $((x_1) : x_4) = (x_1, x_5x_6)$.

Remark 3.23. There exists a Koszul toric ring $R_{\mathcal{A}}$ such that for any ordering of $\mathbf{a}_1, \dots, \mathbf{a}_n$ there exists a lex-ideal which is not componentwise linear. The Pinched Veronese $K[t_1^3, t_1^2t_2, t_1^2t_3, t_1t_2^2, t_1t_3^2, t_2^3, t_2^2t_3, t_2t_3^2, t_3^3]$ is such an example (this semigroup ring is Koszul by the result of Caviglia [Ca]). To see this, we verify that there exist no orderings x_{i_1}, \dots, x_{i_9} such that $(x_{i_1}, \dots, x_{i_p})$ has a linear resolution for all $p = 1, 2, \dots, 9$ by using the computer algebra system Macaulay 2 [GS].

4. TORIC RINGS OF SEGRE TYPE

In this section, we study lex-ideals in the toric ring arising from the Segre product $\mathbb{P}^{e_1} \times \cdots \times \mathbb{P}^{e_r}$, where e_1, \dots, e_r are positive integers. Since $R_{\mathcal{A}}$ is isomorphic to $K[\mathcal{A}]$ as a ring, considering ideals in $R_{\mathcal{A}}$ and considering ideals in $K[\mathcal{A}]$ are same. In this section we mainly consider ideals in $K[\mathcal{A}]$. Thus we consider the semigroup ring

$$Q = K[\{t_{1,i_1} t_{2,i_2} \cdots t_{r,i_r} : 0 \leq i_k \leq e_k \text{ for } k = 1, 2, \dots, r\}]$$

where $t_{i,j}$ are indeterminates. As we considered in subsection 2.1, we define a grading on Q by $\deg(\prod_{i,j} t_{i,j}^{\alpha_{ij}}) = \frac{1}{r}(\sum_{i,j} \alpha_{ij})$.

4.1. Betti numbers of Borel ideals.

In this subsection, we study the Betti numbers of the following monomial ideals.

Definition 4.1. A monomial ideal (or a set of monomials) $B \subset Q$ is said to be *colored Borel* if, for $i = 1, 2, \dots, r$, $ut_{i,j} \in B$ implies $ut_{i,\ell} \in B$ for all $0 \leq \ell < j$.

For any monomial $m = u_1 \cdots u_r \in Q$, where each $u_k \in K[t_{k,0}, \dots, t_{k,e_k}]$, we write $\max_k(m)$ for the maximal integer ℓ such that $t_{k,\ell}$ divides m . Let $G(M)$ be the set of minimal monomial generators of a monomial ideal $M \subset Q$.

Lemma 4.2. *Let $M \subset Q$ be a colored Borel ideal. For any monomial $m \in M$ there exists the unique monomial $u = u_1 \cdots u_r \in G(M)$, where $u_k \in K[t_{k,0}, \dots, t_{k,e_k}]$ for $k = 1, 2, \dots, r$, satisfying the following conditions*

- (i) u divides m .
- (ii) for any variable $t_{i,j}$ which divides $\frac{m}{u}$, one has $j \geq \max_i(u)$.

The proof of the above lemma is easy and is the same as the proof of [EK, Lemma 1.1]. Thus we omit the proof.

Lemma 4.3. *If $M \subset Q$ is a colored Borel ideal then there exists an ordering u_1, \dots, u_δ of the monomials in $G(M)$ such that for $j = 1, 2, \dots, \delta$*

$$((u_1, \dots, u_{j-1}) : u_j) = (\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u_j) \text{ for some } k\}).$$

Proof. Choose an ordering u_1, \dots, u_δ such that (u_1, \dots, u_j) is colored Borel for $j = 1, 2, \dots, \delta$. By Lemma 4.2, the ideals (u_1, \dots, u_j) can be decomposes into a direct sum

$$(u_1, \dots, u_j) = \bigoplus_{\ell=1}^j u_\ell K[t_{1,i_1} \cdots t_{r,i_r} : \max_k(u_\ell) \leq i_k \leq e_k \text{ for } k = 1, 2, \dots, r]$$

as K -vector spaces. Then it follows that

$$\begin{aligned} ((u_1, \dots, u_{j-1}) : u_j) &= Q \setminus K[t_{1,i_1} \cdots t_{r,i_r} : \max_k(u_j) \leq i_k \leq e_k \text{ for } k = 1, 2, \dots, r] \\ &= (\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u_j) \text{ for some } k\}) \end{aligned}$$

as desired. □

Theorem 4.4. *Every colored Borel ideal $M \subset Q$ has strong linear quotients and*

$$\beta_{i,i+j}^Q(M) = \sum_{u \in G(M), \deg u=j} \beta_i^Q(Q/(\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u) \text{ for some } k\})).$$

Proof. It follows from [HHR, Proposition 2.3] that Q is strongly Koszul. Thus any ideal in Q generated by a subset of $\{t_{1,i_1} t_{2,i_2} \cdots t_{r,i_r} : 0 \leq i_k \leq e_k \text{ for } k = 1, 2, \dots, r\}$ has a 1-linear resolution over Q . Thus, by Lemma 4.3 and Theorem 3.4, the ideal M has strong linear quotients and the graded Betti numbers of M are given by the desired formula. \square

Example 4.5. Let $Q = K[s_0 t_0, s_1 t_0, s_0 t_1, s_1 t_1]$. Then $M = (s_0^2 t_0^2, s_0^2 t_0 t_1, s_0^2 t_1^2, s_0 s_1 t_0^2)$ is colored Borel. The ideal M has linear quotients and

$$\begin{aligned} ((s_0^2 t_0^2) : s_0^2 t_0 t_1) &= (s_0 t_0, s_1 t_0), \\ ((s_0^2 t_0^2, s_0^2 t_0 t_1) : s_0^2 t_1^2) &= (s_0 t_0, s_1 t_0), \\ ((s_0^2 t_0^2, s_0^2 t_0 t_1, s_0^2 t_1^2) : s_0 s_1 t_0^2) &= (s_0 t_0, s_0 t_1). \end{aligned}$$

Thus M has a 2-linear resolution and

$$\beta_{i,i+2}^Q(M) = \beta_i^Q(Q) + 2\beta_i^Q(Q/(s_0 t_0, s_1 t_0)) + \beta_i^Q(Q/(s_0 t_0, s_0 t_1)).$$

4.2. When is every Hilbert function attained by a lex-ideal?

In this subsection, we show that if $e_1 = \cdots = e_r = 1$ then every Hilbert function of a graded ideal in Q is attained by a lex-ideal. Let

$$U = \{t_{1,i_1} t_{2,i_2} \cdots t_{r,i_r} : i_k \in \{0, 1\} \text{ for } k = 1, 2, \dots, r\}.$$

Consider the semigroup ring

$$W = K[U] = K[t_{1,i_1} t_{2,i_2} \cdots t_{r,i_r} : i_k \in \{0, 1\} \text{ for } k = 1, 2, \dots, r].$$

We first define lex-ideals in W .

Definition 4.6. For any monomial $m = \prod_{k=1}^r (t_{k,0}^{p_k} t_{k,1}^{q_k}) \in W$, let

$$H(m) = \{1 + r(q_1 - 1), 2 + r(q_2 - 1), \dots, r + r(q_r - 1)\} \subset \mathbb{Z}_{\geq -r+1},$$

where $\mathbb{Z}_{\geq s} = \{k \in \mathbb{Z} : k \geq s\}$, and

$$\ell(m) = \max_1(m) + \cdots + \max_r(m).$$

Note that $\max_k(m) = 0$ if $q_k = 0$ and $\max_k(m) = 1$ if $q_k \neq 0$.

Let $>_{\text{rev}}$ be the degree reverse lexicographic order on the set of finite subsets of \mathbb{Z} . Thus, for finite subsets F, G of \mathbb{Z} , one has $F >_{\text{rev}} G$ if $|F| > |G|$ or $|F| = |G|$ and the greatest integer in the symmetric difference $(F \setminus G) \cup (G \setminus F)$ of F and G belongs to G . For monomials $m, m' \in W$ of the same degree, we define $m >_{\text{lex}_W} m'$ if $H(m) >_{\text{rev}} H(m')$. A set of monomials $L \subset W$ is said to be *lex-segment* if, for all monomials $u, v \in W$ of the same degree, $u \in L$ and $v >_{\text{lex}_W} u$ imply $v \in L$. A *lex-ideal* $J \subset W$ is a monomial ideal such that the set of all monomials in J is lex-segment.

Remark 4.7. One may feel strange that the above order $>_{\text{lex}_W}$ is called ‘lex’ since we essentially consider the reverse lexicographic order. However, the order $>_{\text{lex}_W}$ coincides with the lex order $>_{\text{lex}_A}$ defined in section 2 (with respect to a proper ordering of the variables). This fact is not obvious. But we omit the proof since we do not use this fact in the rest of this paper.

For any set $V \subset W$ of monomials of the same degree, the set

$$\text{Shadow}^\uparrow(V) = \{mv : m \in U, v \in V\}$$

will be called the *upper shadow* of V .

Lemma 4.8. *If $L \subset W$ is a lex-segment set of monomials of the same degree, then $\text{Shadow}^\uparrow(L)$ is also lex-segment.*

Proof. Let b be the smallest monomial in L with respect to $>_{\text{lex}_W}$. It is clear that for any monomial $u \geq_{\text{lex}_W} b$ with $\deg u = \deg b$ and for any $m \in U$,

$$H(um) \geq_{\text{rev}} H(bt_{1,1} \cdots t_{r,1}).$$

Then what we must prove is that $\text{Shadow}^\uparrow(L)$ contains all monomials $\nu \in W$ of degree $\deg b + 1$ with $H(\nu) \geq_{\text{rev}} H(bt_{1,1} \cdots t_{r,1})$. Let $\nu = \nu_1 \cdots \nu_r$, where each $\nu_k \in K[t_{k,0}, t_{k,1}]$ for $k = 1, 2, \dots, r$, be such a monomial. Consider the monomial $\tilde{\nu} = \prod_{k=1}^r (\nu_k / t_{k, \max_k(\nu)})$. Then $\deg \tilde{\nu} = \deg b$ and $H(\tilde{\nu}) \geq_{\text{rev}} H(b)$. Thus $\tilde{\nu} \in L$. Since $\nu / \tilde{\nu} \in U$, we have $\nu \in \text{Shadow}^\uparrow(L)$ as desired. \square

Example 4.9. Suppose $r = 2$. Let $L = \{t_{1,0}t_{2,0}, t_{1,1}t_{2,0}, t_{1,0}t_{2,1}\}$. Then

$$V = \text{Shadow}^\uparrow(L) = \left\{ \begin{array}{l} t_{1,0}^2 t_{2,0}^2, t_{1,0} t_{1,1} t_{2,0}^2, t_{1,0}^2 t_{2,0} t_{2,1}, t_{1,0} t_{1,1} t_{2,0} t_{2,1}, \\ t_{1,1}^2 t_{2,0}^2, t_{1,1}^2 t_{2,0} t_{2,1}, t_{1,0}^2 t_{2,1}^2, t_{1,0} t_{1,1} t_{2,1}^2 \end{array} \right\}.$$

Note that L and V are lex-segment and

$$\{H(m) : m \in V\} = \{\{-1, 0\}, \{1, 0\}, \{-1, 2\}, \{1, 2\}, \{3, 0\}, \{3, 2\}, \{-1, 4\}, \{1, 4\}\}.$$

The next lemma shows that, to study Problem 1.1 for W , it is enough to consider colored Borel ideals.

Lemma 4.10. *Suppose $\text{char}(K) = 0$. For any graded ideal $J \subset W$, there exists a colored Borel ideal $M \subset W$ such that $H(M, t) = H(J, t)$ for all $t \in \mathbb{N}$ and $\beta_{i,j}^W(M) \geq \beta_{i,j}^W(J)$ for all i, j .*

Proof. Consider the change of coordinates ϕ on $T = K[t_{1,0}, t_{1,1}, t_{2,0}, t_{2,1}, \dots, t_{r,0}, t_{r,1}]$ defined by

$$\phi(t_{i,j}) = \alpha_{i,j} t_{i,0} + \gamma_{i,j} t_{i,1},$$

where we choose $\alpha_{i,j}, \gamma_{i,j} \in K$ generically. Then $\phi(W) = W$. Also, for any monomial order \succ on T satisfying $t_{k,0} \succ t_{k,1}$ for all k , it is easy to see that $\text{in}_\succ(\phi(J))$ is colored Borel (see e.g., [BN, Theorem 5.4]). Then the lemma follows from Corollary 2.9. \square

In the rest of this subsection, we study colored Borel ideals.

Lemma 4.11. *Let $B = \{u_1, \dots, u_\delta\} \subset W$ be a colored Borel set of monomials of the same degree. Then*

$$|\text{Shadow}^\uparrow(B)| = \sum_{k=1}^{\delta} 2^{r-\ell(u_k)}.$$

Proof. We order u_1, \dots, u_δ such that the set $B' = \{u_1, \dots, u_{\delta-1}\}$ is colored Borel. By Lemma 4.3,

$$\text{Shadow}^\uparrow(B) \setminus \text{Shadow}^\uparrow(B') = u_\delta \{t_{1,i_1} \cdots t_{r,i_r} : \max_k(u_\delta) \leq i_k \leq 1 \text{ for } k = 1, \dots, r\}.$$

Since the cardinality of the right-hand side of the above equation is $2^{r-\ell(u_\delta)}$, by arguing inductively, the desired formula follows. \square

For an integer $0 \leq p \leq r$, let $J_{[p]} \subset W$ be the ideal

$$J_{[p]} = \left(\{t_{1,i_1} \cdots t_{r,i_r} : i_k = 0 \text{ for some } k \in \{1, 2, \dots, p\}\} \right).$$

Lemma 4.12. *Let $M \subset W$ be a colored Borel ideal. Then*

$$\beta_{i,i+j}^W(M) = \sum_{u \in G(M), \deg u=j} \beta_i^W(W/J_{[\ell(u)]}) \quad \text{for all } i, j.$$

Proof. By Theorem 4.4, the graded Betti numbers of M are given by

$$\beta_{i,i+j}^W(M) = \sum_{u \in G(M), \deg u=j} \beta_i^W(W/(\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u) \text{ for some } k\})).$$

We show that the ideals $(\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u) \text{ for some } k\})$ and $J_{[\ell(u)]}$ have the same Betti numbers. Since these ideals are generated by a subset of U and since W is strongly Koszul, the ideals have a linear resolution. Thus it is enough to show that these ideals have the same Hilbert function. Fix $u \in G(M)$. Set $\ell = \ell(u)$. Then one has the isomorphism of rings

$$\begin{aligned} & W/(\{t_{1,i_1} \cdots t_{r,i_r} : i_k < \max_k(u) \text{ for some } k\}) \\ & \cong K[t_{1,i_1} \cdots t_{r,i_r} : \max_k(u) \leq i_k \leq 1 \text{ for } k = 1, \dots, r] \\ & \cong K[t_{1,1} \cdots t_{\ell,1} t_{\ell+1,i_{\ell+1}} \cdots t_{r,i_r} : i_k \in \{0, 1\} \text{ for } k = \ell + 1, \dots, r] \\ & \cong W/J_{[\ell]}. \end{aligned}$$

The above isomorphism guarantees the desired property. \square

For a colored Borel set $B \subset W$ of monomials of the same degree, let

$$\ell_i(B) = |\{m \in B : \ell(m) = i\}|$$

for $i = 0, 1, \dots, r$. The vector

$$\ell(B) = (\ell_0(B), \ell_1(B), \dots, \ell_r(B)) \in \mathbb{Z}^{r+1}$$

will be called the ℓ -vector of B . Lemmas 4.11 and 4.12 show that the ℓ -vector of B determine the size of its upper shadow as well as the Betti numbers of the ideal generated by B . We characterize the ℓ -vectors of colored Borel sets of monomials in terms of the face vectors of simplicial complexes.

Definition 4.13. A simplicial complex Δ on $[n] = \{1, 2, \dots, n\}$ is a collection of subsets of $[n]$ satisfying that if $F \in \Delta$ and $G \subset F$ then $G \in \Delta$ (we are not assuming that $\{i\} \in \Delta$ for any $i \in [n]$). For convenience, we assume $\{\emptyset\} \in \Delta$. The elements of Δ are called *faces*, and maximal faces (under inclusion) are called *facets*.

A simplicial complex Δ on $[n]$ is said to be *r-colored* if there exists a partition of $[n]$, $[n] = C_1 \cup \dots \cup C_r$, such that for every $F \in \Delta$ and every $1 \leq i \leq r$, $|F \cap C_i| \leq 1$. For an *r-colored* complex Δ on $[n]$, we write $f_{i-1}(\Delta) = |\{F \in \Delta : |F| = i\}|$ for $i = 0, 1, \dots, r$. The vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{r-1}(\Delta))$ is called the *f-vector* of Δ . (This is not an usual definition of *f-vectors* since $f_{r-1}(\Delta)$ might be 0.)

Let $\mathbb{Z}_i = \{i + rk : k \in \mathbb{Z}\}$ for $i = 1, 2, \dots, r$ and

$$\mathcal{C}_r = \{F \subset \mathbb{Z}_{>0} : |F \cap \mathbb{Z}_i| \leq 1 \text{ for } i = 1, 2, \dots, r\}.$$

An *r-colored shifted complex* Δ on $[n]$ is a simplicial complex on $[n]$ satisfying that $\Delta \subset \mathcal{C}_r$ and that, for every $F \in \Delta$ and $j \in F$ with $j > r$, $(F \setminus \{j\}) \cup \{j - r\} \in \Delta$. An *r-colored rev-lex complex* is a simplicial complex Δ satisfying that $\Delta \subset \mathcal{C}_r$ and that, for every $F \in \Delta$ and every $G \in \mathcal{C}_r$ with $|G| = |F|$ and with $G >_{\text{rev}} F$, one has $G \in \Delta$. Note that *r-colored rev-lex complexes* are *r-colored shifted*, and every *r-colored shifted complex* is indeed *r-colored*.

The next result is due to Frankl, Füredi and Kalai [FFK].

Theorem 4.14 (Frankl-Füredi-Kalai). *Let $f = (f_{-1}, f_0, \dots, f_{r-1}) \in \mathbb{Z}^{r+1}$. The following conditions are equivalent.*

- (i) *f is the f-vector of an r-colored simplicial complex on [n];*
- (ii) *f is the f-vector of an r-colored shifted complex on [n];*
- (iii) *f is the f-vector of an r-colored rev-lex complex on [n].*

An *r-colored rev-lex complex* is uniquely determined from its *f-vector*. Hence the above theorem characterizes *f-vectors* of *r-colored simplicial complexes* in terms of *r-colored rev-lex complexes*. A numerical characterization of *f-vectors* of colored simplicial complexes is also given in [FFK].

Theorem 4.15. *Let $\ell = (\ell_0, \ell_1, \dots, \ell_r) \in \mathbb{Z}^{r+1}$. The following conditions are equivalent.*

- (i) *ℓ is the ℓ -vector of a colored Borel set $B \subset W$ of monomials of degree d .*
- (ii) *ℓ is the f-vector of an r-colored simplicial complex on $[rd]$.*

Proof. (i) \Rightarrow (ii): Let $B \subset W$ be a colored Borel set of monomials of degree d . Define

$$(5) \quad \Delta(B) = \{H(m) \cap \mathbb{Z}_{>0} : m \in B\}.$$

We claim that $\Delta(B)$ is an *r-colored simplicial complex* on $[rd]$ with $f(\Delta(B)) = \ell(B)$.

Let $u = \prod_{k=1}^r (t_{k,0}^{p_k} t_{k,1}^{q_k}) \in B$. To prove that $\Delta(B)$ is a simplicial complex, what we must prove is that $(H(u) \cap \mathbb{Z}_{>0}) \setminus \{j\} \in \Delta(B)$ for any $j \in H(u) \cap \mathbb{Z}_{>0}$. By the

definition of $H(-)$, j must be an integer of the form $j = c + r(q_c - 1)$ with $1 \leq c \leq r$ and with $q_c \geq 1$. Then $u(t_{c,0}/t_{c,1})^{q_c} \in B$ since B is colored Borel and

$$H(u(t_{c,0}/t_{c,1})^{q_c}) = (H(u) \setminus \{j\}) \cup \{c - r\}.$$

Thus $(H(u) \cap \mathbb{Z}_{>0}) \setminus \{j\} = H(u(t_{c,0}/t_{c,1})^{q_c}) \cap \mathbb{Z}_{>0} \in \Delta(B)$ as desired. Hence $\Delta(B)$ is a simplicial complex.

The simplicial complex $\Delta(B)$ is r -colored since $|H(m) \cap \mathbb{Z}_i| \leq 1$ for any monomial $m \in W$ and for all i . Also, $\Delta(B)$ is a simplicial complex on $[rd]$ since $\max H(m) \leq rd$ for any monomial $m \in W$ of degree d . Finally we have $f(\Delta(B)) = \ell(B)$ since for any monomial $m \in W$ we have

$$(6) \quad |H(m) \cap \mathbb{Z}_{>0}| = |\{k : \max_k(m) \neq 0\}| = \ell(m).$$

(ii) \Rightarrow (i): Let Δ be an r -colored shifted complex on $[rd]$. Let $F \in \Delta$. Then F must be an element of the form

$$F = \{a_1 + rb_1, \dots, a_s + rb_s\},$$

where $1 \leq a_k \leq r$ and $b_k \in \mathbb{Z}_{\geq 0}$ for each k . Let

$$\mu(F) = \left(\prod_{k=1}^s (t_{a_k,0}^{d-b_k-1} t_{a_k,1}^{b_k+1}) \right) \left(\prod_{j \in [r] \setminus \{a_1, \dots, a_s\}} t_{j,0}^d \right).$$

Then $\mu(F)$ is a monomial in W of degree d such that $H(\mu(F)) \cap \mathbb{Z}_{>0} = F$. We claim that the set

$$B(\Delta) = \{\mu(F) : F \in \Delta\}$$

is colored Borel and $\ell(B(\Delta)) = f(\Delta)$. Since the equation (6) implies $\ell(\mu(F)) = |F|$, $\ell(B(\Delta)) = f(\Delta)$ is obvious. We show that $B(\Delta)$ is colored Borel.

What we must prove is $\mu(F) \left(\frac{t_{a_j,0}}{t_{a_j,1}} \right) \in B(\Delta)$ for $j = 1, 2, \dots, s$. If $b_j > 0$ then $(F \setminus \{a_j + rb_j\}) \cup \{a_j + r(b_j - 1)\} \in \Delta$ since Δ is colored shifted. Thus

$$\mu(F) \left(\frac{t_{a_j,0}}{t_{a_j,1}} \right) = \mu((F \setminus \{a_j + rb_j\}) \cup \{a_j + r(b_j - 1)\}) \in B(\Delta).$$

Suppose $b_j = 0$. Then $F \setminus \{a_j + rb_j\} \in \Delta$ since Δ is a simplicial complex. Thus

$$\mu(F) \left(\frac{t_{a_j,0}}{t_{a_j,1}} \right) = \mu(F \setminus \{a_j + rb_j\}) \in B(\Delta)$$

as desired. \square

Actually, the operations $\Delta(-)$ and $B(-)$ give a one-to-one correspondence between colored Borel sets of monomials of degree d and r -colored shifted complexes on $[rd]$.

Example 4.16. Consider the set $V = \text{Shadow}^\uparrow(L)$ given in Example 4.9. Then

$$\Delta(V) = \{\{\emptyset\}, \{1\}, \{2\}, \{1, 2\}, \{3\}, \{3, 2\}, \{4\}, \{1, 4\}\}$$

and

$$\ell(V) = (1, 4, 3).$$

Next, we study the structure of $\Delta(B)$ when B is lex-segment. For a simplicial complex Δ , let $\Delta^{(k-1)}$ denote the $(k-1)$ -skeleton of Δ , that is,

$$\Delta^{(k-1)} = \{F \in \Delta : |F| \leq k\}.$$

A simplicial complex Δ is said to be *pure* if all its facets have the same cardinality.

Lemma 4.17. *Let $L \subset W$ be a lex-segment set of monomials of the same degree and ν the smallest monomial in L with respect to $>_{\text{lex}_W}$. Let $G = H(\nu) \cap \mathbb{Z}_{>0}$ and $c = |G|$.*

- (i) $\Delta(L)$ is an r -colored rev-lex complex.
- (ii) $\Delta(L)^{(c-1)}$ is pure.

Proof. Any element $F \in \mathcal{C}_r$ can be written in the form

$$F = \{a_1 + rb_1, a_2 + rb_2, \dots, a_s + rb_s\},$$

where $1 \leq a_k \leq r$ and $b_k \in \mathbb{Z}_{\geq 0}$ for each k . Define

$$\rho(F) = F \cup \{i - r : i \in [r] \setminus \{a_1, \dots, a_s\}\}.$$

Thus, for any monomial $m \in W$,

$$\rho(H(m) \cap \mathbb{Z}_{>0}) = H(m)$$

and

$$\begin{aligned} (7) \quad & \{\rho(F) : F \in \Delta(L)\} \\ &= \{H(m) : m \in L\} \\ &= \{F \subset \mathbb{Z}_{\geq -r+1} : F \geq_{\text{rev}} \rho(G), |F \cap \mathbb{Z}_i| = 1 \text{ for } i = 1, \dots, r\}. \end{aligned}$$

Also, for all subsets $F, F' \in \mathcal{C}_r$ with $|F| = |F'|$, one has $\rho(F) >_{\text{rev}} \rho(F')$ if and only if $F >_{\text{rev}} F'$.

First we prove (i). Let $F \in \Delta(L)$ and $F' \in \mathcal{C}_r$ with $|F| = |F'|$ and $F' >_{\text{rev}} F$. We will show $F' \in \Delta(L)$. By the definition of $\rho(-)$, $\rho(F) \in \{H(m) : m \in L\}$ and $\rho(F') >_{\text{rev}} \rho(F) \geq_{\text{rev}} \rho(G)$. Then by (7) we have $\rho(F') \in \{H(m) : m \in L\}$. Thus $F' = \rho(F') \cap \mathbb{Z}_{>0} \in \Delta(L)$ as desired.

Next we prove (ii). Let $F \in \Delta^{(c-1)}$ with $|F| < |G|$. What we must prove is that there exists $F' \in \Delta^{(c-1)}$ such that $F' \supset F$ and $|F'| = |F| + 1$, in other words, there exists $F' \in \mathcal{C}_r$ such that $F' \supset F$, $|F'| = |F| + 1$ and $\rho(F') \geq_{\text{rev}} \rho(G)$.

Let

$$\rho(G) = \{i_1, i_2, \dots, i_r\},$$

where $i_1 < \dots < i_r$. Since $\rho(F) >_{\text{rev}} \rho(G)$, there exists a $1 \leq p \leq r$ such that $\rho(F)$ can be written in the form

$$\rho(F) = \{j_1, \dots, j_p, i_{p+1}, \dots, i_r\},$$

where $j_1 < \dots < j_p < i_p$. Note that $j_1 \leq 0$ since $|\rho(F) \cap \mathbb{Z}_{>0}| = |F| < |G| \leq r$.

Case 1: If $j_1 + r < i_p$ then $F' = F \cup \{j_1 + r\}$ satisfies the desired conditions. Indeed, $\rho(F') = (\rho(F) \setminus \{j_1\}) \cup \{j_1 + r\} >_{\text{rev}} \rho(G)$ since $\max\{j_2, \dots, j_p, j_1 + r\} < i_p$.

Case 2: Suppose $j_1 + r \geq i_p$. We will prove $F \cup \{j_1 + r\} = G$. For $k = 1, 2, \dots, p$, the integers i_k and j_k can be written in the form

$$i_k = c_k + r(d_k - 1)$$

and

$$j_k = c'_k + r(d'_k - 1)$$

where $1 \leq c_k, c'_k \leq r$, $d_k, d'_k \in \mathbb{Z}_{\geq 0}$ and $\{c_1, \dots, c_p\} = \{c'_1, \dots, c'_p\}$. Since $j_1 \leq 0$, we have $i_p \leq j_1 + r \leq r$. Then since $i_1 < \dots < i_p$ and $j_1 < \dots < j_p < i_p$, for $k = 1, 2, \dots, p$, we have

$$d_k, d'_k \in \{0, 1\}.$$

In particular, we have $d_p = 1$, and therefore $i_p = c_p$. Indeed, if $d_p = 0$ then d_k and d'_k are 0 for $k = 1, 2, \dots, p$ and one has $F = G$, a contradiction.

Since $j_1 < \dots < j_p < i_p = c_p$, we have

$$d'_k = 0 \quad \text{if } c'_k \geq c_p.$$

Also, since $c_p = i_p \leq j_1 + r < \dots < j_p + r$, we have

$$d'_k = 1 \quad \text{if } c'_k < c_p.$$

Then, since $\{c_1, \dots, c_p\} = \{c'_1, \dots, c'_p\}$, it follows that

$$(8) \quad \{j_1, \dots, j_p\} = \{c_k : c_k < c_p, k \in [p]\} \cup \{c_k - r : c_k \geq c_p, k \in [p]\}.$$

Similarly, since $i_1 < \dots < i_p = c_p$,

$$(9) \quad d_k = 0 \quad \text{if } c_k > c_p.$$

Then since $|\rho(G) \cap \mathbb{Z}_{>0}| = |G| > |F| = |\rho(F) \cap \mathbb{Z}_{>0}|$,

$$(10) \quad |\{i_1, \dots, i_p\} \cap \mathbb{Z}_{>0}| > |\{j_1, \dots, j_p\} \cap \mathbb{Z}_{>0}| = |\{k : c_k < c_p, k \in [p]\}|.$$

By (9) and (10), we have

$$d_k = 1 \quad \text{if } c_k \leq c_p.$$

Hence we have

$$\begin{aligned} \{i_1, \dots, i_p\} &= \{c_k : c_k \leq c_p, k \in [p]\} \cup \{c_k - r : c_k > c_p, k \in [p]\} \\ &= (\{j_1, \dots, j_p\} \setminus \{c_p - r\}) \cup \{c_p\}. \end{aligned}$$

(The second equality follows from (8).) In particular, $\rho(G) = (\rho(F) \setminus \{c_p - r\}) \cup \{c_p\}$.

Then since the equation (8) says $j_1 = c_p - r$, it follows that

$$F \cup \{j_1 + r\} = ((\rho(F) \setminus \{c_p - r\}) \cup \{c_p\}) \cap \mathbb{Z}_{>0} = \rho(G) \cap \mathbb{Z}_{>0} = G$$

as desired. \square

Let \mathcal{F} be a subset of $\{F \in \mathcal{C}_r : |F| = k\}$. The set

$$\text{Shadow}^\downarrow(\mathcal{F}) = \{G \in \mathcal{C}_r : |G| = k - 1, \text{ there exists } F \in \mathcal{F} \text{ such that } G \subset F\}$$

is called the *lower shadow* of \mathcal{F} . We say that \mathcal{F} is *rev-lex* if, for all $F \in \mathcal{F}$ and $G \in \mathcal{C}_r$ with $|G| = |F|$ and with $G \succ_{\text{rev}} F$, one has $G \in \mathcal{F}$. We need the next lemma which is essentially equivalent to Theorem 4.14.

Lemma 4.18 (Frankl-Füredi-Kalai). *Let \mathcal{F} and \mathcal{L} be subsets of $\{F \in \mathcal{C}_r : |F| = k\}$ such that $|\mathcal{F}| = |\mathcal{L}|$ and \mathcal{L} is rev-lex. Then $|\text{Shadow}^\downarrow(\mathcal{L})| \leq |\text{Shadow}^\downarrow(\mathcal{F})|$.*

Lemma 4.19. *Let B and L be colored Borel sets of monomials of degree d in W such that $|L| = |B|$ and L is lex-segment. Then*

$$\sum_{k=0}^j \ell_k(B) \geq \sum_{k=0}^j \ell_k(L) \quad \text{for } j = 0, 1, \dots, r.$$

Proof. The statement is obvious for $j = r$. Assume $\sum_{k=0}^{j+1} \ell_k(B) \geq \sum_{k=0}^{j+1} \ell_k(L)$. We will prove $\sum_{k=0}^j \ell_k(B) \geq \sum_{k=0}^j \ell_k(L)$.

If $\ell_{j+1}(B) \leq \ell_{j+1}(L)$ then $\sum_{k=0}^j \ell_k(B) \geq \sum_{k=0}^j \ell_k(L)$ is obvious. Thus we assume $\ell_{j+1}(B) > \ell_{j+1}(L)$. Then $f_j(\Delta(B)) > f_j(\Delta(L))$ since $f_j(\Delta(B)) = \ell_{j+1}(B)$ and $f_j(\Delta(L)) = \ell_{j+1}(L)$. Let $L' \supset L$ be the smallest lex-segment set of monomials of degree d such that $f_j(\Delta(L')) = f_j(\Delta(L)) + 1$. Then the smallest monomial ν in L' with respect to $>_{\text{lex}_W}$ must satisfy $|H(\nu) \cap \mathbb{Z}_{>0}| = j+1$. Hence, by Lemma 4.17, $\Delta(L')$ is rev-lex and the j -skeleton $\Delta(L')^{(j)}$ is pure. We claim that $f_k(\Delta(L')) \leq f_k(\Delta(B))$ for all $k \leq j$. Note that $f_j(\Delta(L')) = f_j(\Delta(L)) + 1 \leq f_j(\Delta(B))$. By arguing inductively, it follows that for all $k \leq j$

$$\begin{aligned} f_{k-1}(\Delta(L')) &= |\text{Shadow}^\downarrow(\{F \in \Delta(L') : |F| = k+1\})| \\ &\leq |\text{Shadow}^\downarrow(\{F \in \Delta(B) : |F| = k+1\})| \\ &\leq f_{k-1}(\Delta(B)). \end{aligned}$$

(The first equality follows from the fact that $\Delta(L')^{(j)}$ is pure. For the second line, we use Lemma 4.18 and the induction hypothesis $f_k(\Delta(L')) \leq f_k(\Delta(B))$.) Then since $L \subset L'$, we have

$$\sum_{k=0}^j \ell_k(L) \leq \sum_{k=0}^j \ell_k(L') = \sum_{k=0}^j f_{k-1}(\Delta(L')) \leq \sum_{k=0}^j f_{k-1}(\Delta(B)) = \sum_{k=0}^j \ell_k(B),$$

as desired. \square

We are ready to prove the main result of this section.

Theorem 4.20. *Suppose $\text{char}(K) = 0$. For any graded ideal $J \subset W$, there exists a lex-ideal $L \subset W$ such that $H(L, t) = H(J, t)$ for all $t \in \mathbb{N}$ and $\beta_{i,j}^W(L) \geq \beta_{i,j}^W(J)$ for all i, j .*

Proof. By Lemma 4.10, we may assume that J is colored Borel. Let B_d be the set of monomials of degree d in J and $C_d \subset W$ the lex-segment set of monomials of degree d with $|C_d| = |B_d|$. Let L be the K -vector space spanned by $\bigcup_{d \geq 0} C_d$. We claim that L is a lex-ideal having the same Hilbert function as J .

To prove that L is an ideal, by Lemma 4.8, it is enough to show that

$$|\text{Shadow}^\uparrow(C_d)| \leq |\text{Shadow}^\uparrow(B_d)|$$

for all $d \geq 0$. Fix an integer $d \geq 0$. Write $B_d = \{u_1, \dots, u_\delta\}$ and $C_d = \{v_1, \dots, v_\delta\}$ such that $\ell(u_1) \leq \dots \leq \ell(u_\delta)$ and $\ell(v_1) \leq \dots \leq \ell(v_\delta)$. Lemma 4.19 shows $\ell(u_k) \leq \ell(v_k)$ for all $k = 1, 2, \dots, \delta$. Then, by Lemma 4.11, we have

$$|\text{Shadow}^\uparrow(C_d)| = \sum_{k=1}^{\delta} 2^{r-\ell(v_k)} \leq \sum_{k=1}^{\delta} 2^{r-\ell(u_k)} = |\text{Shadow}^\uparrow(B_d)|$$

as desired. Thus L is an ideal of W . By the construction, it is clear that L is a lex-ideal and has the same Hilbert function as J .

It remains to show that $\beta_{i,j}^W(L) \geq \beta_{i,j}^W(J)$ for all i, j . Theorem 4.4 shows that J and L have strong linear quotients, and therefore are componentwise linear by Proposition 3.7. Thus, by Lemma 3.8, it is enough to prove that

$$\beta_i^W(L_{\langle d \rangle}) \geq \beta_i^W(J_{\langle d \rangle})$$

for all i and d .

We claim that, for integers $0 \leq p \leq q \leq r$, one has

$$\beta_i^W(J_{[p]}) \leq \beta_i^W(J_{[q]})$$

for all i ($J_{[p]}$ and $J_{[q]}$ are ideals considered in Lemma 4.12). Indeed, $J_{[p]}$ and $J_{[q]}$ are colored Borel ideals with $G(J_{[p]}) \subset G(J_{[q]})$. Thus Theorem 4.4 shows $\beta_i^W(J_{[p]}) \leq \beta_i^W(J_{[q]})$ for all i .

Recall that $L_{\langle d \rangle}$ and $J_{\langle d \rangle}$ are generated by C_d and B_d respectively. Then, since $\ell(u_k) \leq \ell(v_k)$ for $k = 1, 2, \dots, \delta$, it follows from Lemma 4.12 that

$$\beta_i^W(L_{\langle d \rangle}) = \sum_{k=1}^{\delta} \beta_i^W(W/J_{[\ell(v_k)]}) \geq \sum_{k=1}^{\delta} \beta_i^W(W/J_{[\ell(u_k)]}) = \beta_i^W(J_{\langle d \rangle})$$

as desired. \square

4.3. Remarks and Examples.

One may expect that Theorem 4.20 holds for any toric ring Q studied in subsection 4.1. The following example gives a negative answer for this.

Example 4.21. Let $H = \{s_i t_j : i = 0, 1, j = 0, 1, 2\}$. Consider the ring homomorphism

$$\begin{aligned} \psi : K[x_m : m \in H] &\rightarrow K[s_0, s_1, t_0, t_1, t_2] \\ x_m &\mapsto m \end{aligned}$$

where x_m with $m \in H$ are indeterminates. Let $R = K[x_m : m \in H]/\text{Ker}(\psi)$.

If every Hilbert function of a graded ideal in R is attained by a lex-ideal defined by the lex order $>_{\text{lex}_A}$ induced by the ordering $x_{u_1} > \dots > x_{u_6}$, where $\{u_1, \dots, u_6\} = H$, then for $j = 1, 2, \dots, 6$ and for any subset $\{u_{i_1}, \dots, u_{i_j}\} \subset H$ one has

$$H((u_1, \dots, u_j), 2) \leq H((u_{i_1}, \dots, u_{i_j}), 2).$$

We claim that there are no orderings u_1, \dots, u_6 of the elements of H satisfying the above property. Suppose contrary that there exists such an ordering u_1, \dots, u_6 . We may assume $u_1 = s_0t_0$. Then a routine computation shows

$$H((u_1, m), 2) = \begin{cases} 10, & \text{if } m \in H \setminus \{s_0t_0, s_1t_0\}, \\ 9, & \text{if } m = s_1t_0. \end{cases}$$

Hence u_2 must be s_1t_0 . However, for any $m \in H \setminus \{s_0t_0, s_1t_0\}$,

$$H((s_0t_0, s_1t_0, m), 2) = 13 > H((s_0t_0, s_0t_1, s_0t_2), 2) = 12.$$

This contradicts the assumption. Hence we cannot get the characterization of Hilbert functions of graded ideals in R by using lex-ideals.

The above argument is applicable to other toric rings. For example, by using a computer system, we verify that the Hilbert functions of graded ideals are not characterized by lex-ideals for the second squarefree Veronese ring with 5 variables $K[t_1t_2, t_1t_3, t_1t_4, t_1t_5, t_2t_3, t_2t_4, t_2t_5, t_3t_4, t_3t_5, t_4t_5]$.

Let $R = S/I$, where I is either a monomial or a projective toric ideal, such that every Hilbert function of a graded ideal in R is attained by a lex-ideal. It was asked in [MP, Conjecture 5.3] that the preimage of a lex-ideal in R to S has the greatest graded Betti numbers among all ideals in S containing I and having the same Hilbert function (we are considering the Betti numbers over S). It was proved in [MM] that there exists a monomial ideal I such that every Hilbert function of a graded ideal in $R = S/I$ is attained by a lex-ideal in R , however, the preimage of a lex-ideal does not always have the greatest graded Betti numbers among all ideals in S containing I and having the same Hilbert function. Here we show that the above question is also false when I is the defining ideal of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$.

Example 4.22. Consider the toric ring

$$K[x_1, \dots, x_8]/I_{\mathcal{A}} \cong K[s_0t_0u_0, s_1t_0u_0, s_0t_1u_0, s_1t_1u_0, s_0t_0u_1, s_1t_0u_1, s_0t_1u_1, s_1t_1u_1].$$

Set $S = K[x_1, \dots, x_8]$ and $R_{\mathcal{A}} = S/I_{\mathcal{A}}$. By Theorem 4.20, for any graded ideal in $R_{\mathcal{A}}$, there exists a lex-ideal having the same Hilbert function. Let

$$\tilde{J} = (x_1^2, x_1x_2, x_2^2) \subset R_{\mathcal{A}}.$$

Then the lex-ideal $\tilde{L} \subset R_{\mathcal{A}}$ having the same Hilbert function as \tilde{J} is

$$\tilde{L} = (x_1^2, x_1x_2, x_1x_3) \subset R_{\mathcal{A}}.$$

Let $J = (x_1^2, x_1x_2, x_2^2) + I_{\mathcal{A}} \subset S$ and $L = (x_1^2, x_1x_2, x_1x_3) + I_{\mathcal{A}} \subset S$ be the preimage of \tilde{J} and \tilde{L} to S . The following diagrams are the Betti diagrams of S/J and S/L computed by Macaulay 2 [GS]:

$$\begin{array}{r} \text{betti}(\text{res}(J)) = \text{total: } 1 \ 12 \ 29 \ 31 \ 20 \ 8 \ 1 \\ \quad \quad \quad 0: \ 1 \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \ \cdot \\ \quad \quad \quad 1: \ \cdot \ 12 \ 24 \ 15 \ \cdot \ \cdot \ \cdot \\ \quad \quad \quad 2: \ \cdot \ \cdot \ 5 \ 16 \ 20 \ 8 \ 1 \end{array}$$

```

betti(res(L)) = total: 1 12 26 28 20 8 1
                   0: 1  .  .  .  .  .  .
                   1: . 12 24 12  .  .  .
                   2: .  .  2 16 20 8 1

```

The above diagrams show that the preimage of lex-ideals in $R_{\mathcal{A}}$ to S do not always have the greatest graded Betti numbers over S .

Gasharov, Horwitz and Peeva [GHP, Problem 4.9] suggested to study the structure of the Hilbert scheme parameterizing all homogeneous ideals having the same Hilbert function over a projective toric ring. Studying this problem for a toric ring W would be interesting. We do not have any positive structural result even for $\mathbb{P}^1 \times \mathbb{P}^1$.

5. CONSECUTIVE CANCELLATIONS AND GOTZMANN'S PERSISTENCE THEOREM

In this section, we consider a ring $R = S/I$, where I is either a monomial or a projective toric ideal, satisfying the following properties:

- (A) For every graded ideal $J \subset R$, there exists a lex-ideal $L \subset R$ having the same Hilbert function as J .
- (B) Every lex-ideal $L \subset R$ has the greatest graded Betti numbers among all graded ideals in R having the same Hilbert function as L .
- (C) Every lex-ideal in R is componentwise linear.

Note that the condition (C) implies that R is Koszul. We refer the reader to [MP] for properties of lex-ideals over $R = S/I$ when I is a monomial ideal.

Suppose $\text{char}(K) = 0$. The polynomial ring S , the ring of the form $S/(x_1^2, \dots, x_m^2)$, where $m \leq n$, and the Veronese subring of a polynomial ring satisfy the conditions (A), (B) and (C). Clements-Lindström rings (that is, the rings of the form $S/(x_1^{a_1}, \dots, x_m^{a_m})$) satisfy conditions (A) and (B), but do not always satisfy the condition (C). We proved that a toric ring W in subsection 4.2 also satisfies the conditions (A), (B) and (C).

First of all, we study consecutive cancellations in Betti numbers. Given a sequence of numbers $\{c_{i,j}\}$, we obtain a new sequence by a *cancellation* as follows: fix a j , and choose i and i' so that one of the numbers is odd and the other is even; then replace $c_{i,j}$ by $c_{i,j} - 1$, and replace $c_{i',j}$ by $c_{i',j} - 1$. We have a *consecutive cancellation* when $i' = i + 1$. If we need to be specific, we call it a consecutive i, j -cancellation. The term “consecutive” is justified by the fact that we consider cancellations in Betti numbers of consecutive homological degrees.

Let J and J' be graded ideals in R . It is not hard to see that the graded Betti numbers of J are obtained from those of J' by a sequence of consecutive cancellations if and only if there exist integers $c_{i,j} \geq 0$ such that

$$\beta_{i,j}^R(J) = \beta_{i,j}^R(J') - c_{i,j} - c_{i+1,j} \quad \text{for all } i \text{ and } j,$$

where $c_{0,j} = 0$ for all j . The above equation says that a consecutive i, j -cancellation occurs $c_{i+1,j}$ times to obtain the graded Betti numbers of J from those of J' . It was proved by Peeva [Pe] that the graded Betti numbers of a graded ideal J in a

polynomial ring S are obtained from those of the lex-ideal having the same Hilbert function as J by a sequence of consecutive cancellations. Clearly this result of Peeva is stronger than the condition (B). However, the next result shows that these conditions are essentially equivalent over a Koszul ring R .

Theorem 5.1. *Suppose that R is Koszul and satisfies the conditions (A) and (B). Let $J \subset R$ be a graded ideal and $L \subset R$ the lex-ideal having the same Hilbert function as J . Then the graded Betti numbers of J are obtained from those of L by a sequence of consecutive cancellations.*

Proof. By Lemma 3.8 we have

$$\beta_{i,i+j}^R(J) = \beta_{i,i+j}^R(L) - (\beta_{i,i+j}^R(L_{\geq j}) - \beta_{i,i+j}^R(J_{\geq j})) - (\beta_{i+1,i+j}^R(L_{\geq j-1}) - \beta_{i+1,i+j}^R(J_{\geq j-1}))$$

for all i, j . Since $L_{\geq j}$ is a lex-ideal having the same Hilbert function as $J_{\geq j}$, by the condition (B), the number $\beta_{i,i+j}^R(L_{\geq j}) - \beta_{i,i+j}^R(J_{\geq j})$ is nonnegative. Then we get the desired property by setting $c_{i,i+j} = \beta_{i,i+j}^R(L_{\geq j}) - \beta_{i,i+j}^R(J_{\geq j})$. \square

Next we prove that the conditions (A), (B) and (C) imply an analogue of Gotzmann's persistence theorem. Our proof based on a simple idea given by Green [Gr]. Suppose that R satisfies the conditions (A), (B) and (C). Let $V \subset R_d$ be a K -vector space spanned by elements of degree d in R and let $L \subset R_d$ be the K -vector space spanned by the lex-segment set of monomials with $\dim_K V = \dim_K L$. Let $R_1V = \{fm \in R_{d+1} : f \in R_1, m \in V\}$. The condition (A) implies that

$$\dim_K R_1V \geq \dim_K R_1L.$$

We say that V is *Gotzmann* if $\dim_K R_1V = \dim_K R_1L$.

Theorem 5.2. *Suppose that $R = S/I$ satisfies the conditions (A), (B) and (C). If $V \subset R_d$ is Gotzmann then R_1V is also Gotzmann and the ideal $J \subset R$ generated by V has a d -linear resolution.*

Proof. Let $L \subset R$ be the lex-ideal having the same Hilbert function as J . Since V is Gotzmann, L has no generators of degree $d+1$. Since $L_{d+1} = R_1L_d$ is spanned by a lex-segment set of monomials, what we must prove is that $\dim_K R_1^2V = \dim_K R_1^2L_d$, that is, L has no generators of degree $d+2$.

We first claim that $\beta_{i,i+d+1}^R(L) = 0$ for all i . Let $L_{\leq d+1} \subset R$ be the ideal generated by all monomials in L of degree $\leq d+1$. Then $L_{\leq d+1}$ is a lex-ideal, and therefore is componentwise linear by (C). By Lemma 3.16, we have $\beta_{i,i+d+1}^R(L_{\leq d+1}) = 0$ for all i . Also, since all lex-ideals in R are componentwise linear and since $L_j = (L_{\leq d+1})_j$ for all $j \leq d+1$, Lemma 3.8 implies

$$\beta_{i,i+d+1}^R(L) = \beta_{i,i+d+1}^R(L_{\leq d+1}) = 0$$

for all i as desired.

Now, by Theorem 5.1, the graded Betti numbers of J are obtained from those of L by a sequence of consecutive cancellations. Thus there exist integers $c_{1,d+2}, c_{2,d+2}$ such that

$$\beta_{0,d+2}^R(J) = \beta_{0,d+2}^R(L) - c_{1,d+2}$$

and

$$\beta_{1,d+2}^R(J) = \beta_{1,d+2}^R(L) - c_{1,d+2} - c_{2,d+2}.$$

Since we proved $\beta_{1,d+2}^R(L) = 0$, we have $c_{1,d+2} = c_{2,d+2} = 0$. Then we have $\beta_{0,d+2}^R(L) = \beta_{0,d+2}^R(J) = 0$. Hence L has no generators of degree $d+2$.

It remains to show that J has a d -linear resolution. The first statement shows that

$$\dim_K L_{d+k} = \dim_K J_{d+k} = \dim_K R_1^k V = \dim_K R_1^k L_d$$

for all $k \geq 0$. This fact says that L is generated in degree d . Then since L is componentwise linear by (C), L has a d -linear resolution. Since $\beta_{i,j}^R(J) \leq \beta_{i,j}^R(L)$ for all i, j , it follows that J has a d -linear resolution. \square

When $R = S$, the above theorem was proved by Gotzmann [Go], and is known as Gotzmann's Persistence theorem. The persistence theorem was also proved for exterior algebras [AHH] and for Veronese subrings of a polynomial ring [GMP]. Theorem 5.2 covers all those cases, and moreover shows that the persistence theorem holds for a toric ring W studied in subsection 4.2. Note that Theorem 5.2 is false for Clements-Lindström rings [Ga, Example 1].

We say that a graded ideal $J \subset R$ is *Gotzmann* if each graded component J_d of degree d of J is Gotzmann for all integers $d \geq 0$. It is easy to see that $J \subset R$ is Gotzmann if and only if $\beta_0^R(J) = \beta_0^R(L)$, where $L \subset R$ is the lex-ideal having the same Hilbert function as J . An important corollary of Theorem 5.2 is the following.

Corollary 5.3. *Suppose that $R = S/I$ satisfies the conditions (A), (B) and (C). Every Gotzmann ideal in R is componentwise linear.*

Herzog and Hibi [HH] proved that the graded Betti numbers of a graded ideal J in a polynomial ring S are equal to those of the lex-ideal having the same Hilbert functions as J if and only if J is a Gotzmann ideal. We extend this result for a Koszul ring R satisfying the conditions (A), (B) and (C).

Theorem 5.4. *Suppose that $R = S/I$ satisfies the conditions (A), (B) and (C). Then a graded ideal $J \subset R$ is Gotzmann if and only if $\beta_{i,j}^R(J) = \beta_{i,j}^R(L)$ for all i, j , where $L \subset R$ is the lex-ideal having the same Hilbert function as J .*

Proof. The “if” part is obvious. We show that if J is Gotzmann then $\beta_{i,j}^R(J) = \beta_{i,j}^R(L)$ for all i, j . By Corollary 5.3, J and L are componentwise linear. Thus by Lemma 3.8 it is enough to show that $\beta_i^R(J_{\langle d \rangle}) = \beta_i^R(L_{\langle d \rangle})$ for all i and d . By Theorem 5.2, one has that

$$H(J_{\langle d \rangle}, d+k) = \dim_K R_1^k J_d = \dim_K R_1^k L_d = H(L_{\langle d \rangle}, d+k)$$

for all $k \geq 0$. Thus $J_{\langle d \rangle}$ and $L_{\langle d \rangle}$ have the same Hilbert function. Since J and L are componentwise linear, $J_{\langle d \rangle}$ and $L_{\langle d \rangle}$ have a d -linear resolution. Hence $J_{\langle d \rangle}$ and $L_{\langle d \rangle}$ have the same Betti numbers. \square

We conclude this paper with a few questions.

Question 5.5. Is it possible to prove Theorem 5.1 for non-Koszul rings?

It is known that if $R = S/I$ is Koszul and I is a monomial ideal then R satisfies the condition (C). Here we present the following question.

Question 5.6. Is there a Koszul toric ring R which satisfies the condition (A) but does not satisfy the condition (C)?

Suppose that $R = S/I$ satisfies the conditions (A), (B) and (C). Let $J \subset R$ be a graded ideal and $L \subset R$ the lex-ideal having the same Hilbert function as J . Theorem 5.4 says that $\beta_0^R(J) = \beta_0^R(L)$ implies $\beta_k^R(J) = \beta_k^R(L)$ for all $k \geq 0$. On the other hand, Conca, Herzog and Hibi [CHH] proved that if $R = S$ then $\beta_i^S(J) = \beta_i^S(L)$ implies $\beta_k^S(J) = \beta_k^S(L)$ for all $k \geq i$ in characteristic 0. The same property was proved for exterior algebras in [MS].

Question 5.7. With the same notations as above, is it true that if $\beta_i^R(J) = \beta_i^R(L)$ for some i then $\beta_k^R(J) = \beta_k^R(L)$ for all $k \geq i$?

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