ON FACE VECTORS OF BARYCENTRIC SUBDIVISIONS OF MANIFOLDS

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Abstract. We study face vectors of barycentric subdivisions of simplicial homology manifolds. Recently, Kubitzke and Nevo proved that the g-vector of the barycentric subdivision of a Cohen–Macaulay simplicial complex is an M-vector, which in particular proves the g-conjecture for barycentric subdivisions of simplicial homology spheres. In this paper, we prove an analogue of this result for Buchsbaum simplicial posets and simplicial homology manifolds.

1. Introduction

One of the most important open problems in the theory of face vectors of simplicial complexes is the g-conjecture for homology spheres, which states that the g-vector of a simplicial homology sphere is an M-vector (that is, the face vector of a multicomplex). In 2008, Brenti and Welker [BW] proved that the h-vector of the barycentric subdivision of a Cohen–Macaulay simplicial poset is unimodal. Later, Kubitzke and Nevo [KN] proved that the g-vector of the barycentric subdivision of a Cohen–Macaulay simplicial complex is an M-vector, and in particular they proved that the g-conjecture is true for barycentric subdivisions of simplicial homology spheres. On the other hand, recently, face vectors of simplicial homology manifolds became of great interest, and analogues of the g-conjecture for orientable homology manifolds were considered and studied in several papers (see [No, NS1, NS2]). The purpose of this paper is to extend the above result of Kubitzke and Nevo to homology manifolds.

We first recall the basics on simplicial complexes, the g-conjecture and simplicial posets. A simplicial complex Δ on the vertex set $V = \{v_1, v_2, \dots, v_n\}$ is a collection of subsets of V satisfying that $F \in \Delta$ and $G \subset F$ imply $G \in \Delta$. The elements of Δ are called *faces*, and maximal faces (under inclusion) are called *facets*. For convenience, we assume that Δ has the empty face $\emptyset$. For $i = 0, 1, 2, \dots$, let

$$f_{i-1}(\Delta) = \#\{F \in \Delta : \#F = i\},$$

where $f_{-1}(\Delta) = 1$. The *dimension* of Δ is $\dim \Delta = \max\{k : f_k(\Delta) \neq 0\}$. The vector $f(\Delta) = (f_{-1}(\Delta), f_0(\Delta), \dots, f_{\dim \Delta}(\Delta))$ is called the *f-vector* of Δ. Throughout this paper, we fix a field K. Let $\tilde{H}_i(\Delta; K)$ be the reduced homology groups of Δ over K. The numbers $\beta_i(\Delta) = \dim_K \tilde{H}_i(\Delta; K)$ are called the *Betti numbers* of Δ. A simplicial complex is said to be *pure* if all its facets have the same cardinality. A pure simplicial complex Δ is said to be *Cohen–Macaulay* if, for all faces $F \in \Delta$, $\beta_i(\mathrm{lk}_\Delta(F)) = 0$ for all $i \neq \dim \mathrm{lk}_\Delta(F)$, where $\mathrm{lk}_\Delta(F) = \{G \subset V \setminus F : F \cup G \in \Delta\}$ is

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the *link* of F in Δ. A *(simplicial) homology sphere* is a Cohen–Macaulay simplicial complex Δ such that $\beta_{\dim \mathrm{lk}_\Delta(F)}(\mathrm{lk}_\Delta(F)) = 1$ for all $F \in \Delta$.

The *h-vector* $h(\Delta) = (h_0(\Delta), h_1(\Delta), \dots, h_d(\Delta))$ of a $(d-1)$-dimensional simplicial complex Δ is defined by the relation

$$\sum_{i=0}^{d} h_i(\Delta)x^{d-i} = \sum_{i=0}^{d} f_{i-1}(\Delta)(x-1)^{d-i}.$$

The *g-vector* $g(\Delta) = (g_0(\Delta), g_1(\Delta), \dots, g_{\lfloor \frac{d}{2} \rfloor}(\Delta))$ of Δ is

$$g(\Delta) = \big(h_0(\Delta), h_1(\Delta) - h_0(\Delta), \dots, h_{\lfloor \frac{d}{2} \rfloor}(\Delta) - h_{\lfloor \frac{d}{2} \rfloor -1}(\Delta)\big)$$

where $\lfloor \frac{d}{2} \rfloor$ is the integer part of $\frac{d}{2}$. If Δ is a homology sphere then its h-vector is symmetric, that is, $h_i(\Delta) = h_{d-i}(\Delta)$ for all i. Thus, in this case, knowing $h(\Delta)$ is equivalent to knowing $g(\Delta)$. The g-conjecture states that

Conjecture 1.1 (The g-conjecture for homology spheres)**.** *If a simplicial complex Δ is a homology sphere then $g(\Delta)$ is an M-vector.*

The above conjecture is important since if true, it would yield the complete characterization of face vectors of simplicial homology spheres. We refer the readers to [BK2] for more backgrounds and details on the conjecture.

Next, we review simplicial posets. A *simplicial poset* (boolean complex in some literature) is a finite poset P with the minimal element $\hat{0}$ such that, for every $y \in P$, the interval $[\hat{0}, y]$ is a boolean algebra. Clearly, the face poset of a simplicial complex is a simplicial poset. Simplicial posets are special cases of CW-posets [Bj]. Thus a simplicial poset P is the face poset of a regular CW-complex $\Gamma(P)$. The *barycentric subdivision* $\mathrm{sd}(P)$ of P (or the order complex of $P \setminus \{\hat{0}\}$) is the simplicial complex whose faces are chains of $P \setminus \{\hat{0}\}$. Thus

$$\mathrm{sd}(P) = \big\{\{y_1, \dots, y_k\} \subset P \setminus \{\hat{0}\} : y_1 < \cdots < y_k\big\}.$$

The geometric realization of the barycentric subdivision $\mathrm{sd}(P)$ is homeomorphic to that of $\Gamma(P)$, and therefore taking barycentric subdivisions does not change the Betti numbers. For a simplicial complex Δ, we write $\mathrm{sd}(\Delta)$ for the barycentric subdivision of the face poset of Δ.

Kubitzke and Nevo [KN] proved that the g-vector of the barycentric subdivision of a Cohen–Macaulay simplicial complex is an M-vector. In the first part of this paper, we reprove this result in the following stronger form. Recall that the face vector of a simplicial complex is an M-vector.

Theorem 1.2. *Let P be a simplicial poset. If $\mathrm{sd}(P)$ is a $(d-1)$-dimensional Cohen–Macaulay complex then there exists a simplicial complex Δ such that*

$$g_i\big(\mathrm{sd}(P)\big) = f_{i-1}(\Delta) \quad \textit{for } i = 0, 1, \dots, \left\lfloor \frac{d}{2} \right\rfloor.$$

While Kubitzke and Nevo used an algebraic approach, we prove Theorem 1.2 using purely combinatorial methods. Our approach follows that of Brenti and Welker [BW] who study the h-vectors of barycentric subdivisions by using a concrete description of $h(\mathrm{sd}(P))$ in terms of the h-vector of P. In fact, most of the results of this paper are proved by analyzing the formula of Brenti and Welker.

Next, we consider homology manifolds. A *(simplicial) homology manifold* (without boundary) is a pure simplicial complex Δ such that, for every vertex v of Δ, $\mathrm{lk}_\Delta(v)$ is a homology sphere. A connected simplicial homology manifold is said to be *orientable* if its top Betti number is equal to 1. When we study face vectors of homology manifolds, h-vectors are not good invariants. They are not always non-negative and their behavior seems hard to understand. In 1998, Novik [No] introduced the h''-vectors of homology manifolds as "correct h-vectors for homology manifolds".

We consider a more general class of simplicial complexes, called Buchsbaum simplicial complexes. A pure simplicial complex Δ is said to be *Buchsbaum* if, for every vertex v of Δ, $\mathrm{lk}_\Delta(v)$ is Cohen–Macaulay. Clearly, simplicial homology manifolds are Buchsbaum. Let Δ be a $(d-1)$-dimensional Buchsbaum simplicial complex and $\beta_i = \beta_i(\Delta)$ for $i = 0, 1, \dots, d-1$. We define the *h''-vector* $h''(\Delta) = (h''_0(\Delta), h''_1(\Delta), \dots, h''_d(\Delta))$ of Δ by

$$(1) \qquad h''_k(\Delta) = \begin{cases} 1, & \text{if } k = 0,\\ h_k(\Delta) - \binom{d}{k} \left\{ \sum_{\ell=1}^{k} (-1)^{\ell-k} \beta_{\ell-1} \right\}, & \text{if } 1 \le k \le d-1,\\ h_d - \sum_{\ell=1}^{d-1} (-1)^{\ell-d} \beta_{\ell-1} = \beta_{d-1}(\Delta), & \text{if } k = d. \end{cases}$$

The following nice properties are known.

- (Novik-Swartz [NS1]) $h''(\Delta)$ is an M-vector;
- (Novik [No]) If Δ is a connected orientable simplicial homology manifold then $h''_i(\Delta) = h''_{d-i}(\Delta)$ for all i.

Actually, Novik and Swartz [NS1, NS2] proved stronger results. For example, it was proved in [NS2] that if Δ is a connected orientable simplicial homology manifold then its h''-vector is the Hilbert function of a Gorenstein graded algebra. From these nice results, h''-vectors seem essential in understanding the properties of face vectors of homology manifolds. In particular, the symmetry of h''-vectors of orientable homology manifolds leads to an extension of the g-conjecture for orientable homology manifolds. Define the *g''-vector* of Δ by

$$g''(\Delta) = \big(h''_0(\Delta), h''_1(\Delta) - h''_0(\Delta), \dots, h''_{\lfloor \frac{d}{2} \rfloor}(\Delta) - h''_{\lfloor \frac{d}{2} \rfloor - 1}(\Delta)\big).$$

Then it is natural to ask if $g''(\Delta)$ is an M-vector when Δ is a connected orientable homology manifold. Such a property was first conjectured by Kalai (see [No, Conjecture 7.5]) in a more algebraic form. The main result of this paper is the following.

Theorem 1.3. *Let P be a simplicial poset. If $\mathrm{sd}(P)$ is a $(d-1)$-dimensional Buchsbaum complex then there exists a simplicial complex Δ such that*

$$g''_i\big(\mathrm{sd}(P)\big) = f_{i-1}(\Delta) \quad \textit{for } i = 0, 1, \dots, \left\lfloor \frac{d}{2} \right\rfloor.$$

In particular, the above theorem shows that the g''-vector of the barycentric subdivision of an orientable simplicial homology manifold is an M-vector.

One may ask why g-vectors and g''-vectors of barycentric subdivisions become f-vectors of simplicial complexes (not just M-vectors). Barycentric subdivisions of simplicial posets are special cases of completely balanced complexes introduced by

Stanley [St1]. Stanley proved that the h-vector of a Cohen–Macaulay completely balanced complex is the f-vector of a simplicial complex, and, from the proof of this result, it seems plausible that if a simplicial homology sphere is completely balanced then its g-vector is the face vector of a simplicial complex.

This paper is organized as follows: In Section 2, we recall some known results on h-vectors of barycentric subdivisions of simplicial posets. In Section 3, we prove Theorem 1.2. In Section 4, we study h''-vectors of barycentric subdivisions and prove Theorem 1.3.

2. g-vectors of barycentric subdivisions

We first introduce some technical notations which will be used in this paper. Let $h=(h_0,h_1,\dots,h_n)\in\mathbb{Z}^{n+1}$, $h'=(h'_0,h'_1,\dots,h'_n)\in\mathbb{Z}^{n+1}$ and $a\in\mathbb{Z}$. We define

$$\begin{array}{rcl} (a,h) & = & (a,h_0,h_1,\dots,h_n)\in\mathbb{Z}^{n+2}\\ (h,a) & = & (h_0,h_1,\dots,h_n,a)\in\mathbb{Z}^{n+2}\\ h+h' & = & (h_0+h'_0,h_1+h'_1,\dots,h_n+h'_n)\in\mathbb{Z}^{n+1}\\ ah & = & (ah_0,ah_1,\dots,ah_n)\in\mathbb{Z}^{n+1}\\ h^\vee & = & (h_n,h_{n-1},\dots,h_0)\in\mathbb{Z}^{n+1}\\ \mathrm{del}(h) & = & (h_0,h_1,\dots,h_{n-1})\in\mathbb{Z}^{n}\\ \mathrm{last}(h) & = & h_n\\ h\le h' & \Leftrightarrow & h_i\le h'_i \text{ for } i=0,1,\dots,n. \end{array}$$

The vector h is said to be *unimodal* if there exists an integer $0\le k\le n$ such that $h_0\le h_1\le\cdots\le h_k\ge\cdots\ge h_{n-1}\ge h_n$, in which case we say that h has a *peak* at kth position.

In this section, we recall some known results on h-vectors of simplicial posets and barycentric subdivisions. Let P be a simplicial poset. For $i=-1,0,1,\dots$, let $f_i(P)$ be the number of elements $y\in P$ for which the interval $[\hat{0},y]$ is a boolean algebra of rank $i+1$. The *rank* of P is $\mathrm{rank}\,P=\max\{i : f_{i-1}(P)\neq 0\}$. We define the h-vector $h(P)=(h_0(P),h_1(P),\dots,h_d(P))$ of P, where $d=\mathrm{rank}\,P$, by the relation $\sum_{i=0}^d h_i(P)x^{d-i}=\sum_{i=0}^d f_{i-1}(P)(x-1)^{d-i}$. Thus, if P is the face poset of a simplicial complex Δ, $f_i(P)=f_i(\Delta)$, $h_i(P)=h_i(\Delta)$ and $\mathrm{rank}\,P=\dim\Delta+1$. We say that P is Cohen–Macaulay (respectively Buchsbaum) if $\mathrm{sd}(P)$ is Cohen–Macaulay (respectively Buchsbaum). The h-vectors of Cohen–Macaulay simplicial posets were characterized by Stanley [St2].

Theorem 2.1 (Stanley)**.** *Let $h=(h_0,h_1,\dots,h_d)\in\mathbb{Z}^{d+1}$. The following conditions are equivalent.*

(i) *There exits a Cohen–Macaulay simplicial poset P such that $h=h(P)$;*
(ii) *$h_0=1$ and $h_k\ge 0$ for $k=1,2,\dots,d$.*

Next, we recall how to obtain the h-vector of $\mathrm{sd}(P)$ from the h-vector of P. Let S_d be the symmetric group on $[d]=\{1,2,\dots,d\}$. For $\sigma\in S_d$, let $\mathrm{des}(\sigma)$ be the number of its *descents*, that is,

$$\mathrm{des}(\sigma)=\#\big\{i\in[d-1]:\sigma(i)>\sigma(i+1)\big\}.$$

For integers $1 \leq d$, $1 \leq j \leq d$ and $0 \leq i \leq d-1$, we write

$$A_d(i,j) = \#\big\{\sigma \in S_d : \sigma(1) = j \text{ and } \mathrm{des}(\sigma) = i\big\}.$$

For $1 \leq d$ and $0 \leq k \leq d$, define the vector $H_d(k) \in \mathbb{Z}^{d+1}$ by

$$H_d(k) = \big(A_{d+1}(0,k+1), A_{d+1}(1,k+1), \dots, A_{d+1}(d,k+1)\big) \in \mathbb{Z}^{d+1}.$$

The next formula was given in [BW, Theorem 1].

Theorem 2.2 (Brenti-Welker)**.** *Let P be a simplicial poset of rank d. Then*

$$h\big(\mathrm{sd}(P)\big) = \sum_{k=0}^{d} h_k(P) H_d(k).$$

Theorems 2.1 and 2.2 give a characterization of $h(\mathrm{sd}(P))$ for Cohen–Macaulay simplicial posets P. However, it is not clear from these results that $g(\mathrm{sd}(P))$ is an M-vector. Kubitzke and Nevo proved that $g(\mathrm{sd}(P))$ is an M-vector if P is the face poset of a Cohen–Macaulay simplicial complex by using an algebraic method. In the next section, we prove directly from Theorem 2.2 that $g(\mathrm{sd}(P))$ is an M-vector.

Definition 2.3. Let $1 \leq d$ and $0 \leq k \leq d$ be integers. Let $H_d(k) = (a_0, a_1, \dots, a_d)$. We define

$$G_d(k) = \big(a_0, a_1 - a_0, \dots, a_{\lfloor \frac{d}{2} \rfloor} - a_{\lfloor \frac{d}{2} \rfloor - 1}\big)$$

and

$$\hat{G}_d(k) = \big(a_0, a_1 - a_0, \dots, a_{\lfloor \frac{d}{2} \rfloor + 1} - a_{\lfloor \frac{d}{2} \rfloor}\big).$$

Thus $G_d(k)$ (respectively $\hat{G}_d(k)$) is the vector consisting of the first $\lfloor \frac{d}{2} \rfloor + 1$ (respectively $\lfloor \frac{d}{2} \rfloor + 2$) entries of the vector $(H_d(k), 0) - (0, H_d(k))$.

By Theorem 2.2,

Lemma 2.4. *Let P be a simplicial poset of rank d. Then*

$$g\big(\mathrm{sd}(P)\big) = \sum_{k=0}^{d} h_k(P) G_d(k).$$

To study properties of the h-vectors of barycentric subdivisions of simplicial posets, it is important to understand properties of $H_d(k)$ and $G_d(k)$. Basic techniques which we use in this paper are the following easy lemmas (see [BW, Lemma 2]).

Lemma 2.5. *Let $1 \leq d$, $0 \leq k \leq d$ and $0 \leq i \leq d$ be integers.*

(i) $A_{d+1}(i, k+1) = \sum_{\ell=1}^{k} A_d(i-1, \ell) + \sum_{\ell=k}^{d-1} A_d(i, \ell+1)$, *in other words,*

$$H_d(k) = \sum_{\ell=0}^{k-1} \big(0, H_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \big(H_{d-1}(\ell), 0\big),$$

where $H_0(0) = (1)$.

(ii) $A_{d+1}(j, k+1) = A_{d+1}(d-j, d-k+1)$, *in other words,*

$$H_d(k) = H_d(d-k)^{\vee}.$$

Below we list the vectors $H_d(k)$, $G_d(k)$ and $\hat{G}_d(k)$ for $d \leq 4$.

		$H_d(k)$	$G_d(k)$	$\hat{G}_d(k)$
$d=1$	$k=0$	$(1,0)$	(1)	$(1,-1)$
	$k=1$	$(0,1)$	(0)	$(0,1)$
$d=2$	$k=0$	$(1,1,0)$	$(1,0)$	$(1,0,-1)$
	$k=1$	$(0,2,0)$	$(0,2)$	$(0,2,-2)$
	$k=2$	$(0,1,1)$	$(0,1)$	$(0,1,0)$
$d=3$	$k=0$	$(1,4,1,0)$	$(1,3)$	$(1,3,-3)$
	$k=1$	$(0,4,2,0)$	$(0,4)$	$(0,4,-2)$
	$k=2$	$(0,2,4,0)$	$(0,2)$	$(0,2,2)$
	$k=3$	$(0,1,4,1)$	$(0,1)$	$(0,1,3)$
$d=4$	$k=0$	$(1,11,11,1,0)$	$(1,10,0)$	$(1,10,0,-10)$
	$k=1$	$(0,8,14,2,0)$	$(0,8,6)$	$(0,8,6,-12)$
	$k=2$	$(0,4,16,4,0)$	$(0,4,12)$	$(0,4,12,-12)$
	$k=3$	$(0,2,14,8,0)$	$(0,2,12)$	$(0,2,12,-6)$
	$k=4$	$(0,1,11,11,1)$	$(0,1,10)$	$(0,1,10,0)$

Table 1

Example 2.6. Let Δ be the boundary of a 3-dimensional simplex. Then $h(\Delta) = (1,1,1,1)$. Thus

$$h\big(\mathrm{sd}(\Delta)\big) = H_3(0) + H_3(1) + H_3(2) + H_3(3) = (1,11,11,1)$$

and

$$g\big(\mathrm{sd}(\Delta)\big) = G_3(0) + G_3(1) + G_3(2) + G_3(3) = (1,10).$$

The readers may verify Lemma 2.5 as well as the following properties of $H_d(k)$ for the examples given in Table 1:

- $H_d(k)$ is unimodal;
- If d is even then $H_d(k)$ has a peak at $\frac{d}{2}$th position;
- If d is odd then $H_d(k)$ has a peak at $\frac{d-1}{2}$th position when $k < \frac{d}{2}$ and has a peak at $\frac{d+1}{2}$th position when $k > \frac{d}{2}$.

The above properties were proved in [KN]. (Actually they proved a stronger result. See [KN, Corollary 4.10].) These properties relate to non-negativity of $G_d(k)$ and $\hat{G}_d(k)$. Indeed, the properties imply that $G_d(k)$ is non-negative for all d, k and $\hat{G}_d(k)$ is non-negative if d is odd and $k > \frac{d}{2}$.

Here we include a proof of non-negativity of $G_d(k)$ and $\hat{G}_d(k)$. Recall that $\mathrm{last}(h)$ denotes the rightmost entry of the vector h. We need the next lemma (check the statements for the values given in Table 1).

Lemma 2.7. *Let $1 \le d$ and $0 \le k \le d$ be integers.*

(i) *If d is odd then* $\mathrm{last}(\hat{G}_d(k)) + \mathrm{last}(\hat{G}_d(d-k)) = 0$.

(ii) *If d is even then* $\mathrm{last}(\hat{G}_d(k)) + \mathrm{last}(G_d(d-k)) = 0$.

Proof. Let $H_d(k) = (a_0, \dots, a_d)$ and $H_d(d-k) = (a'_0, \dots, a'_d)$.

(i) The statement is equivalent to $a_{\frac{d+1}{2}} + a'_{\frac{d+1}{2}} = a_{\frac{d-1}{2}} + a'_{\frac{d-1}{2}}$. This fact follows from Lemma 2.5(ii), which shows that $a_{\frac{d-1}{2}} = a'_{\frac{d+1}{2}}$ and $a_{\frac{d+1}{2}} = a'_{\frac{d-1}{2}}$.

(ii) The statement is equivalent to $a_{\frac{d}{2}+1} + a'_{\frac{d}{2}} = a_{\frac{d}{2}} + a'_{\frac{d}{2}-1}$. This fact follows from Lemma 2.5(ii), which shows that $a_{\frac{d}{2}} = a'_{\frac{d}{2}}$ and $a_{\frac{d}{2}+1} = a'_{\frac{d}{2}-1}$. □

Now, we prove non-negativity of $G_d(k)$ and $\hat{G}_d(k)$.

Lemma 2.8. *Let $1 \le d$ and $0 \le k \le d$ be integers.*

(i) *$G_d(k)$ is non-negative. Moreover, if $d \ge 2$ and $k > 0$ then $\mathrm{last}(G_d(k)) > 0$.*
(ii) *If d is odd then $\mathrm{last}(\hat{G}_d(k)) > 0$ for $k > \frac{d}{2}$ and $\mathrm{last}(\hat{G}_d(k)) < 0$ for $k < \frac{d}{2}$.*
(iii) *If d is even then $\mathrm{last}(\hat{G}_d(k)) \le 0$.*

Proof. We prove the statements simultaneously by using induction on d. For $d \le 2$, the statements follow from Table 1. Suppose $d > 2$.

Case 1: Suppose that d is even. By Lemma 2.5,

$$G_d(k) = \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \hat{G}_{d-1}(\ell).$$

Using the induction hypothesis, it only remains to prove that $\mathrm{last}(G_d(k)) > 0$ for $k > 0$ and $\mathrm{last}(G_d(0)) \ge 0$. We know that $\mathrm{last}(\hat{G}_{d-1}(\ell)) > 0$ for $\ell \ge \frac{d}{2}$ by the induction hypothesis. Thus the statement is obvious for $k \ge \frac{d}{2}$. Suppose $k < \frac{d}{2}$. Since Lemma 2.7(i) implies that the rightmost entry of $\sum_{\ell=k}^{d-1-k} \hat{G}_{d-1}(\ell)$ is 0,

$$\mathrm{last}\big(G_d(k)\big) \;=\; \sum_{\ell=0}^{k-1} \mathrm{last}\big(G_{d-1}(\ell)\big) + \sum_{\ell=d-k}^{d-1} \mathrm{last}\big(\hat{G}_{d-1}(\ell)\big). \tag{2}$$

Then statement (i) follows from the induction hypothesis. Also, statement (iii) follows from statement (i) and Lemma 2.7(ii).

Case 2: Suppose that d is odd. By Lemma 2.5,

$$\hat{G}_d(k) = \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \hat{G}_{d-1}(\ell).$$

Observe $(0, \mathrm{del}(G_{d-1}(0))) = G_{d-1}(d-1)$ by Lemma 2.5(i). Then statement (i) follows from the induction hypothesis. We prove statement (ii). Suppose $k > \frac{d}{2}$. Then

$$\mathrm{last}\big(\hat{G}_d(k)\big) = \sum_{\ell=0}^{d-1-k} \big\{\mathrm{last}\big(G_{d-1}(\ell)\big) + \mathrm{last}\big(\hat{G}_{d-1}(d-1-\ell)\big)\big\} + \sum_{\ell=d-k}^{k-1} \mathrm{last}\big(G_{d-1}(\ell)\big).$$

Then $\mathrm{last}(\hat{G}_d(k)) = \sum_{\ell=d-k}^{k-1} \mathrm{last}(G_{d-1}(\ell)) > 0$ by Lemma 2.7(ii) and the induction hypothesis. Finally, the fact that $\mathrm{last}(\hat{G}_d(k)) < 0$ for $k < \frac{d}{2}$ follows from Lemma 2.7(i). □

3. Barycentric subdivisions of Cohen–Macaulay simplicial posets

In this section, we prove Theorem 1.2. We first introduce a technique to prove that a given vector is the f-vector of a simplicial complex. We say that a vector $f = (1, f_0, f_1, \ldots, f_n) \in \mathbb{Z}^{n+2}$ is the f-vector of a simplicial complex if there exists a simplicial complex Δ such that $f_i = f_i(\Delta)$ for all $i \ge 0$ (we are not assuming $f_n \ne 0$). Let $f = (1, f_0, f_1, \ldots, f_n) \in \mathbb{Z}^{n+2}$ be the f-vector of a simplicial complex. We say that a vector $\alpha = (0, 1, \alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n+2}$ is a *basic admissible vector* of f if (i) $(1, \alpha_0, \ldots, \alpha_{n-1}) \in \mathbb{Z}^{n+1}$ is the f-vector of a simplicial complex and (ii) $f_i \ge \alpha_i$ for $i = 0, 1, \ldots, n-1$. Also, we say that a vector $\beta \in \mathbb{Z}^{n+2}$ is *admissible* to f if there exists a sequence of vectors $\beta_1, \dots, \beta_t \in \mathbb{Z}^{n+2}$ such that $\beta = \beta_1 + \cdots + \beta_t$ and each β_k is a basic admissible vector of $f + \beta_1 + \cdots + \beta_{k-1}$ for $k = 1, 2, \dots, t$.

Lemma 3.1. *Let $f = (1, f_0, f_1, \dots, f_n) \in \mathbb{Z}^{n+2}$ be the f-vector of a simplicial complex and $\alpha = (0, \alpha_{-1}, \alpha_0, \dots, \alpha_{n-1}) \in \mathbb{Z}^{n+2}$ a vector which is admissible to f. Then*

(i) *$f + \alpha$ is the f-vector of a simplicial complex;*
(ii) *If a vector $g \in \mathbb{Z}^{n+2}$ is the f-vector of a simplicial complex with $g \geq f$ then α is admissible to g;*
(iii) *If $\beta \in \mathbb{Z}^{n+2}$ is admissible to f then $\alpha + \beta$ is admissible to f;*
(iv) *For any integer $0 \leq b \leq \alpha_{n-1}$, $(0, \alpha_{-1}, \alpha_0, \dots, \alpha_{n-2}, \alpha_{n-1} - b) \in \mathbb{Z}^{n+2}$ is admissible to f.*

Proof. Statements (ii) – (iv) are obvious. We prove (i). Without loss of generality, we may assume that α is a basic admissible vector, say, $\alpha_{-1} = 1$. Then there exist simplicial complexes Δ and Γ such that $f_i(\Delta) = f_i$ and $f_i(\Gamma) = \alpha_i$ for $i \geq 0$. By the Kruskal–Katona Theorem (see [St3, p. 55]), we may take $\Gamma \subset \Delta$. Then

$$\Sigma = \Delta \cup \big\{\{v\} \cup F : F \in \Gamma\big\},$$

where v is a new vertex which is not in Δ, is a simplicial complex and $f_i(\Sigma) = f_i + \alpha_{i-1}$ for all $i \geq 0$. ☐

See [BK1] for another way to prove that a given vector is the f-vector of a simplicial complex.

Example 3.2. Let $f = G_4(0) = (1, 10, 0)$ and $\alpha = G_4(3) = (0, 2, 12)$. Then f is the f-vector of a simplicial complex and $\alpha = (0, 1, 9) + (0, 1, 3)$ is admissible to f. By Lemma 3.1, $f + c\alpha = (1, 10 + 2c, 12c)$ is the f-vector of a simplicial complex for all integers $c \geq 0$. By Lemma 2.4, this fact says that, for any simplicial poset P of rank 4 with $h(P) = (1, 0, 0, c, 0)$, $g(\mathrm{sd}(P)) = f + c\alpha$ is the f-vector of a simplicial complex. In Proposition 3.3 below, we prove a generalization of this example.

Now, we prove Theorem 1.2. By Theorem 2.1 and Lemmas 2.4 and 3.1, it is enough to prove the next proposition.

Proposition 3.3. *Let d be a positive integer.*

(A) *$G_d(0)$ is the f-vector of a simplicial complex;*
(B) *$G_d(k)$ is admissible to $G_d(0)$ for $k = 1, 2, \dots, d$;*
(C) *If d is odd and $k > \frac{d}{2}$ then $\hat{G}_d(k)$ is admissible to $(G_d(0), 0)$.*

Statement (C) is unnecessary to prove Theorem 1.2, but is required for proving (A) and (B) by induction on d.

Proof. We prove the statements simultaneously by using induction on d. For $d \leq 2$, the statements follow from Table 1. Suppose $d > 2$.

Case 1: Suppose that d is odd. By Lemma 2.5,

$$G_d(0) = G_{d-1}(0) + G_{d-1}(1) + \cdots + G_{d-1}(d-1). \tag{3}$$

By the induction hypothesis and Lemma 3.1(iii), $G_{d-1}(0)$ is the f-vector of a simplicial complex and $G_{d-1}(1) + \cdots + G_{d-1}(d-1)$ is admissible to $G_{d-1}(0)$. Hence $G_d(0)$ is the f-vector of a simplicial complex by Lemma 3.1(i). This proves (A).

We prove (B). Let $1 \le k \le d$. Then

$$G_d(k) = \sum_{\ell=0}^{k-1} \big(0, \mathrm{del}\big(G_{d-1}(\ell)\big)\big) + \sum_{\ell=k}^{d-1} G_{d-1}(\ell).$$

By the induction hypothesis, $G_{d-1}(\ell)$ is admissible to $G_{d-1}(0)$ for $\ell \ge 1$. Then, since $G_{d-1}(0) \le G_d(0)$ by (3), $\sum_{\ell=k}^{d-1} G_{d-1}(\ell)$ is admissible to $G_d(0)$. By Lemma 3.1(iii), it remains to prove that $\sum_{\ell=0}^{k-1}(0, \mathrm{del}(G_{d-1}(\ell)))$ is admissible to $G_d(0)$. By the induction hypothesis and Lemma 3.1, $\sum_{\ell=0}^{k-1} \mathrm{del}(G_{d-1}(\ell))$ is the f-vector of a simplicial complex. Also, (3) says that $\sum_{\ell=0}^{k-1} \mathrm{del}(G_{d-1}(\ell)) \le \mathrm{del}(G_d(0))$. This completes the proof of (B).

Finally we prove (C). Let $k > \frac{d}{2}$. Then

$$\hat{G}_d(k) = \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \hat{G}_{d-1}(\ell).$$

By Lemma 2.8(iii), $\mathrm{last}(\hat{G}_{d-1}(\ell)) \le 0$. Hence

$$\hat{G}_d(k) \le \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \big(G_{d-1}(\ell), 0\big).$$

The right-hand side of the above inequality is admissible to $(G_d(0), 0)$ as we saw in the proof of (B). Since $\hat{G}_d(k)$ is non-negative by Lemma 2.8(ii), Lemma 3.1(iv) implies that $\hat{G}_d(k)$ is admissible to $(G_d(0), 0)$ for $k > \frac{d}{2}$.

Case 2: Suppose that d is even. By Lemmas 2.5 and 2.7(i),

$$G_d(0) = \sum_{\ell=0}^{d-1} \hat{G}_{d-1}(\ell) = \sum_{\ell=0}^{d-1} \big(G_{d-1}(\ell), 0\big). \tag{4}$$

Then (A) follows from the induction hypothesis and Lemma 3.1.

We prove (B). Let $1 \le k \le d$. Then

$$\begin{aligned} G_d(k) &= \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \hat{G}_{d-1}(\ell) \\ &= \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{k \le \ell < \frac{d}{2}} \hat{G}_{d-1}(\ell) + \sum_{\frac{d}{2} \le \ell \le d-1} \hat{G}_{d-1}(\ell) \\ &\le \sum_{\ell=0}^{k-1} \big(0, G_{d-1}(\ell)\big) + \sum_{k \le \ell < \frac{d}{2}} \big(G_{d-1}(\ell), 0\big) + \sum_{\frac{d}{2} \le \ell \le d-1} \hat{G}_{d-1}(\ell). \qquad (5) \end{aligned}$$

(We use Lemma 2.8(ii) for the last step.) By Lemma 3.1(iv), it is enough to prove that the right-hand side of (5) is admissible to $G_d(0)$.

The vector $\sum_{\ell=0}^{k-1}(0, G_{d-1}(\ell))$ is admissible to $G_d(0)$ since $\sum_{\ell=0}^{k-1} G_{d-1}(\ell)$ is the f-vector of a simplicial complex by the induction hypothesis and since $\sum_{\ell=0}^{k-1} G_{d-1}(\ell) \le \mathrm{del}(G_d(0))$ by (4). Also, by the induction hypothesis, both $\sum_{k \le \ell < \frac{d}{2}}(G_{d-1}(\ell), 0)$

and $\sum_{\frac{d}{2}\leq \ell \leq d-1} \hat{G}_{d-1}(\ell)$ are admissible to $(G_{d-1}(0),0)$, and therefore to $G_d(0) \geq (G_{d-1}(0),0)$. Hence $G_d(k)$ is admissible to $G_d(0)$. $\square$

We need the following fact in the next section.

Corollary 3.4. *Let $1 \leq d$ and $1 \leq k \leq d$ be integers. Every entry of $G_d(k)$ is positive except for the first entry.*

Proof. Let $G_d(k)=(a_0,a_1,\dots,a_t)$. By Lemma 2.8, $a_t>0$. Since $G_d(k)$ is admissible to $G_d(0)$, there exists $\beta=(b_1,\dots,b_t)$ with $b_t>0$ and with $\beta \leq (a_1,\dots,a_t)$ such that β is the f-vector of a simplicial complex. Then every entry of β must be positive. $\square$

4. Face vectors of barycentric subdivisions of manifolds

In this section, we prove Theorem 1.3. Let P be a Buchsbaum simplicial poset of rank d and $\beta_i=\beta_i(\mathrm{sd}(P))$ for $i=0,1,\dots,d-1$. We define $h''(P)$ in the same way as for simplicial complexes (see (1) in the introduction).

Fix a positive integer d. Let $e_1,e_2,\dots,e_{d+1}\in \mathbb{Z}^{d+1}$ be the standard basis of $\mathbb{Z}^{d+1}$. Thus the ith entry of e_i is 1 and the other entries of e_i are 0. For $k=0,1,\dots,d$, we define

$$B_d(k)=\sum_{\ell=k}^{d}(-1)^{\ell-k}\binom{d}{\ell}\big(H_d(\ell)-e_{\ell+1}\big).$$

Also, if $B_d(k)=(a_0,a_1,\dots,a_d)$ then we write

$$C_d(k) \;=\; \Big(a_0,a_1-a_0,\dots,a_{\lfloor \frac{d}{2}\rfloor}-a_{\lfloor \frac{d}{2}\rfloor-1}\Big)$$

and

$$\hat{C}_d(k) \;=\; \Big(a_0,a_1-a_0,\dots,a_{\lfloor \frac{d}{2}\rfloor+1}-a_{\lfloor \frac{d}{2}\rfloor}\Big).$$

Thus $C_d(k)$ (respectively $\hat{C}_d(k)$) is the vector consisting of the first $\lfloor \frac{d}{2}\rfloor+1$ (respectively $\lfloor \frac{d}{2}\rfloor+2$) entries of the vector $(B_d(k),0)-(0,B_d(k))$.

Lemma 4.1. *Let P be a Buchsbaum simplicial poset of rank d and $\beta_i=\beta_i(\mathrm{sd}(P))$ for $i=0,1,\dots,d-1$. Then*

$$h''\big(\mathrm{sd}(P)\big)=\sum_{k=0}^{d}h''_k(P)H_d(k)+\sum_{k=1}^{d-1}\beta_{k-1}B_d(k)$$

and

$$g''\big(\mathrm{sd}(P)\big)=\sum_{k=0}^{d}h''_k(P)G_d(k)+\sum_{k=1}^{d-1}\beta_{k-1}C_d(k).$$

Proof. It is enough to prove the first equation. By the definition of h''-vectors,

$$h(P)=h''(P)+\sum_{k=1}^{d-1}\beta_{k-1}\left\{\sum_{\ell=k}^{d}(-1)^{\ell-k}\binom{d}{\ell}e_{\ell+1}\right\}.$$

By Theorem 2.2,

$$h\big(\mathrm{sd}(P)\big)=\sum_{k=0}^{d}h''_k(P)H_d(k)+\sum_{k=1}^{d-1}\beta_{k-1}\left\{\sum_{\ell=k}^{d}(-1)^{\ell-k}\binom{d}{\ell}H_d(\ell)\right\}.$$

Then

$$\begin{aligned} h''\big(\mathrm{sd}(P)\big) &= h\big(\mathrm{sd}(P)\big) - \sum_{k=1}^{d-1} \beta_{k-1} \left\{ \sum_{\ell=k}^{d} (-1)^{\ell-k} \binom{d}{\ell} e_{\ell+1} \right\} \\ &= \sum_{k=0}^{d} h''_k(P) H_d(k) + \sum_{k=1}^{d-1} \beta_{k-1} B_d(k) \end{aligned}$$

as desired. □

We study properties of $B_d(k)$. Below is the list of vectors $B_d(k)$, $C_d(k)$ and $\hat{C}_d(k)$ for $d \leq 4$.

		$B_d(k)$	$C_d(k)$	$\hat{C}_d(k)$
$d=1$	$k=0$	$(0,0)$	(0)	$(0,0)$
	$k=1$	$(0,0)$	(0)	$(0,0)$
$d=2$	$k=0$	$(0,0,0)$	$(0,0)$	$(0,0,0)$
	$k=1$	$(0,1,0)$	$(0,1)$	$(0,1,-1)$
	$k=2$	$(0,1,0)$	$(0,1)$	$(0,1,-1)$
$d=3$	$k=0$	$(0,0,0,0)$	$(0,0)$	$(0,0,0)$
	$k=1$	$(0,4,1,0)$	$(0,4)$	$(0,4,-3)$
	$k=2$	$(0,5,5,0)$	$(0,5)$	$(0,5,0)$
	$k=3$	$(0,1,4,0)$	$(0,1)$	$(0,1,3)$
$d=4$	$k=0$	$(0,0,0,0,0)$	$(0,0,0)$	$(0,0,0,0)$
	$k=1$	$(0,11,11,1,0)$	$(0,11,0)$	$(0,11,0,-10)$
	$k=2$	$(0,17,45,7,0)$	$(0,17,28)$	$(0,17,28,-38)$
	$k=3$	$(0,7,45,17,0)$	$(0,7,38)$	$(0,7,38,-28)$
	$k=4$	$(0,1,11,11,0)$	$(0,1,10)$	$(0,1,10,0)$

Table 2

Example 4.2. Let Δ be a standard triangulation of the 2-dimensional torus with 18 facets, 27 edges and 9 vertices. Then $h(\Delta) = (1,6,12,-1)$. Since $\beta_1(\Delta) = 2$, $h''(\Delta) = h(\Delta) - 2(0,0,3,-1) = (1,6,6,1)$. Then, by Lemma 4.1,

$$h''\big(\mathrm{sd}(\Delta)\big) = H_3(0) + 6H_3(1) + 6H_3(2) + H_3(3) + 2B_3(2) = (1,51,51,1).$$

It is not clear from the definition if the vectors $B_d(k)$ are non-negative. However, for the examples given in Table 2, we can see that all $B_d(k)$ are non-negative and have similar properties to $H_d(k)$: they are unimodal; they have the following symmetry $B_d(k) = B_d(d+1-k)^\vee$; $B_d(k)$ has a peak at $\frac{d}{2}$th position if d is even; $B_d(k)$ has a peak at $\frac{d-1}{2}$th position if d is odd and $k \leq \frac{d+1}{2}$; $B_d(k)$ has a peak at $\frac{d+1}{2}$th position if d is odd and $k \geq \frac{d+1}{2}$. We will prove all these properties. The non-negativity follows from the next recursion formula.

Lemma 4.3. *Let $2 \leq d$ and $0 \leq k \leq d$ be integers. Then*

$$\begin{aligned} B_d(k) &= \big(B_{d-1}(k),0\big) + \big(0,B_{d-1}(k-1)\big) \\ &\quad + \binom{d-1}{k-1} \left\{ \sum_{\ell=0}^{k-2} \big(0,H_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \big(H_{d-1}(\ell),0\big) \right\}, \end{aligned}$$

where $B_{d-1}(d)$ is the zero-vector and $B_{d-1}(-1) = -B_{d-1}(0)$.

Proof. The statement follows from the following computations.

$$\begin{aligned}
&B_d(k)\\
=\ &\sum_{i=k}^{d}(-1)^{i-k}\binom{d}{i}\left\{\sum_{\ell=0}^{i-1}\big(0,H_{d-1}(\ell)\big)+\sum_{\ell=i}^{d-1}\big(H_{d-1}(\ell),0\big)\right\}\\
&-\sum_{\ell=k}^{d}(-1)^{\ell-k}\left\{\binom{d-1}{\ell}+\binom{d-1}{\ell-1}\right\}e_{\ell+1}\\
=\ &\left[\sum_{\ell=k}^{d-1}\left\{\sum_{i=k}^{\ell}(-1)^{i-k}\binom{d}{i}\right\}\big(H_{d-1}(\ell),0\big)\right]-\sum_{\ell=k}^{d-1}(-1)^{\ell-k}\binom{d-1}{\ell}e_{\ell+1}\\
&+\left[\sum_{\ell=0}^{d-1}\left\{\sum_{i=\max\{k,\ell+1\}}^{d}(-1)^{i-k}\binom{d}{i}\right\}\big(0,H_{d-1}(\ell)\big)\right]-\sum_{\ell=k}^{d}(-1)^{\ell-k}\binom{d-1}{\ell-1}e_{\ell+1}\\
=\ &\left[\sum_{\ell=k}^{d-1}\left\{\sum_{i=0}^{\ell}(-1)^{i-k}\binom{d}{i}-\sum_{i=0}^{k-1}(-1)^{i-k}\binom{d}{i}\right\}\big(H_{d-1}(\ell),0\big)\right]\\
&-\sum_{\ell=k}^{d-1}(-1)^{\ell-k}\binom{d-1}{\ell}e_{\ell+1}\\
&+\left[\sum_{\ell=0}^{k-2}\binom{d-1}{k-1}\big(0,H_{d-1}(\ell)\big)\right]+\left[\sum_{\ell=k-1}^{d-1}(-1)^{\ell+1-k}\binom{d-1}{\ell}\big\{\big(0,H_{d-1}(\ell)\big)-e_{\ell+2}\big\}\right]\\
=\ &\left[\sum_{\ell=k}^{d-1}(-1)^{k-\ell}\binom{d-1}{\ell}\big\{\big(H_{d-1}(\ell),0\big)-e_{\ell+1}\big\}\right]+\left[\binom{d-1}{k-1}\sum_{\ell=k}^{d-1}\big(H_{d-1}(\ell),0\big)\right]\\
&+\left[\binom{d-1}{k-1}\sum_{\ell=0}^{k-2}\big(0,H_{d-1}(\ell)\big)\right]+\big(0,B_{d-1}(k-1)\big).
\end{aligned}$$

Note that we use Lemma 2.5(i) for the first step, and use $\sum_{i=\ell}^{d}(-1)^{i-\ell}\binom{d}{i}=\binom{d-1}{\ell-1}$ and $\sum_{i=0}^{\ell}(-1)^{i-\ell}\binom{d}{i}=\binom{d-1}{\ell}$ for the third and forth steps. □

Next, we prove the symmetry of $B_d(k)$.

Lemma 4.4. *Let $1\le d$ and $1\le k\le d$ be integers.*

(i) *$B_d(0)$ is the zero-vector;*
(ii) *$B_d(k)=B_d(d+1-k)^\vee$.*

Proof. Lemma 4.3 says $B_d(0)=(B_{d-1}(0),0)-(0,B_{d-1}(0))$. This proves (i) since $B_1(0)=(0,0)$. Let $k\ge 1$. Then

$$\begin{aligned}
B_d(0)\ &=\ (-1)^k\left\{B_d(k)+\sum_{\ell=0}^{k-1}(-1)^{\ell-k}\binom{d}{\ell}\big(H_d(\ell)-e_{\ell+1}\big)\right\}\\
&=\ (-1)^k\left\{B_d(k)+\sum_{\ell=0}^{k-1}(-1)^{\ell-k}\binom{d}{d-\ell}\big(H_d(d-\ell)-e_{d+1-\ell}\big)^\vee\right\}\\
&=\ (-1)^k\big\{B_d(k)-B_d(d+1-k)^\vee\big\}.
\end{aligned}$$

(We use Lemma 2.5(ii) for the second step.) Then (ii) follows from (i). □

Remark 4.5. It is possible to prove Lemma 4.4(i) in a geometric way. Consider the simplicial poset $P = \{\hat{0}\}$ which only has the minimal element and formally regard P as a simplicial poset of rank d. Then the h-vector of P is $\sum_{\ell=0}^{d}(-1)^\ell\binom{d}{\ell}e_{\ell+1}$ and by Theorem 2.2

$$\sum_{\ell=0}^{d}(-1)^\ell\binom{d}{\ell}H_d(\ell) = h\big(\mathrm{sd}(P)\big) = h(P) = \sum_{\ell=0}^{d}(-1)^\ell\binom{d}{\ell}e_{\ell+1}.$$

The above equation proves that $B_d(0)$ is the zero-vector.

Remark 4.6. If Δ is a $(d-1)$-dimensional connected orientable simplicial homology manifold then its h''-vector is symmetric. The barycentric subdivision of Δ must have the symmetric h''-vector again. This can be explained by using Lemma 4.4(ii). Recall that $\beta_i(\Delta) = \beta_{d-1-i}(\Delta)$ for $i = 1, 2, \ldots, d-2$ by the Poincaré duality. Lemmas 2.5(ii), 4.1 and 4.4(ii) show that if a connected Buchsbaum simplicial poset P of rank d has a symmetric h''-vector and has a symmetric Betti vector (say, $h''_i = h''_{d-i}$ for all i and $\beta_i = \beta_{d-1-i}$ for $i = 1, 2, \ldots, d-2$), then the h''-vector of $\mathrm{sd}(P)$ is again symmetric.

The next lemma follows from Lemma 4.4 in the same way as Lemma 2.7 follows from Lemma 2.5(ii).

Lemma 4.7. *Let $1 \le d$ and $1 \le k \le d$ be integers.*

(i) *If d is odd then* $\mathrm{last}(\hat{C}_d(k)) + \mathrm{last}(\hat{C}_d(d+1-k)) = 0$.

(ii) *If d is even then* $\mathrm{last}(\hat{C}_d(k)) + \mathrm{last}(C_d(d+1-k)) = 0$.

We now prove the non-negativity of $C_d(k)$.

Lemma 4.8. *Let d be a positive integer.*

(i) *$C_d(k)$ is non-negative for $k = 0, 1, \ldots, d$;*

(ii) *If d is odd then* $\mathrm{last}(\hat{C}_d(k)) \ge 0$ *for* $k \ge \frac{d+1}{2}$ *and* $\mathrm{last}(\hat{C}_d(k)) \le 0$ *for* $k \le \frac{d+1}{2}$. *In particular,* $\mathrm{last}(\hat{C}_d(\frac{d+1}{2})) = 0$.

(iii) *If d is even then* $\mathrm{last}(\hat{C}_d(k)) \le 0$ *for $k = 0, 1, \ldots, d$.*

Proof. We prove the statements simultaneously by using induction on d. By Table 2 we know that the statements are true for $d \le 2$. Suppose $d > 2$. Let $\tilde{e}_1, \ldots, \tilde{e}_{\lfloor \frac{d-1}{2}\rfloor+2} \in \mathbb{Z}^{\lfloor \frac{d-1}{2}\rfloor+2}$ be the standard basis of $\mathbb{Z}^{\lfloor \frac{d-1}{2}\rfloor+2}$, and let $\tilde{e}_i = 0$ for $i > \lfloor \frac{d-1}{2}\rfloor + 2$. By the definition of $B_d(k)$, for $k = 0, 1, \ldots, d-1$,

$$\hat{C}_{d-1}(k) \quad = \quad \binom{d-1}{k}\big\{\hat{G}_{d-1}(k) - \tilde{e}_{k+1} + \tilde{e}_{k+2}\big\} - \hat{C}_{d-1}(k+1). \tag{6}$$

Case 1: Suppose that d is even. We may assume $k \ne 0$ by Lemma 4.4(i). By Lemma 4.3,

$$\begin{aligned} C_d(k) \quad &= \quad \hat{C}_{d-1}(k) + \big(0, C_{d-1}(k-1)\big) \\ &\qquad + \binom{d-1}{k-1}\left\{\sum_{\ell=0}^{k-2}\big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1}\hat{G}_{d-1}(\ell)\right\}. \end{aligned} \tag{7}$$

Using the induction hypothesis it only remains to prove that $\mathrm{last}(C_d(k)) \geq 0$.

By Lemma 2.8(ii), $\mathrm{last}(\hat{G}_{d-1}(\ell)) \geq 0$ for $\ell \geq \frac{d}{2}$. Thus if $k \geq \frac{d}{2}$ then the statement follows from the induction hypothesis. Suppose $0 < k < \frac{d}{2}$. By Lemma 4.7(i),

$$\mathrm{last}\big(\hat{C}_{d-1}(k)\big) = -\mathrm{last}\big(\hat{C}_{d-1}(d-k)\big). \tag{8}$$

Since $\mathrm{last}(\hat{C}_{d-1}(d-k+1)) \geq 0$ by the induction hypothesis and since $\tilde{e}_{d-k+2} = 0$, (6) implies

$$\begin{aligned}(9)\quad & \mathrm{last}\big(\hat{C}_{d-1}(d-k)\big) \\ = & \binom{d-1}{d-k}\mathrm{last}\big(\hat{G}_{d-1}(d-k)\big) - \mathrm{last}(\tilde{e}_{d-k+1} - \tilde{e}_{d-k+2}) - \mathrm{last}\big(\hat{C}_{d-1}(d-k+1)\big) \\ \leq & \binom{d-1}{k-1}\mathrm{last}\big(\hat{G}_{d-1}(d-k)\big).\end{aligned}$$

Observe that Lemma 2.7(i) says that $\sum_{\ell=k}^{d-1-k} \mathrm{last}(\hat{G}_{d-1}(\ell)) = 0$. By applying last(-) to (7), we have

$$\begin{aligned}\mathrm{last}\big(C_d(k)\big) &= \mathrm{last}(\hat{C}_{d-1}(k)) + \mathrm{last}(C_{d-1}(k-1)) \\ &\quad + \binom{d-1}{k-1}\left\{\sum_{\ell=0}^{k-2}\mathrm{last}(G_{d-1}(\ell)) + \sum_{\ell=k}^{d-1}\mathrm{last}(\hat{G}_{d-1}(\ell))\right\} \\ &\geq -\mathrm{last}\big(\hat{C}_{d-1}(d-k)\big) + \binom{d-1}{k-1}\sum_{\ell=k}^{d-1}\mathrm{last}\big(\hat{G}_{d-1}(\ell)\big) \\ &\geq -\mathrm{last}\big(\hat{C}_{d-1}(d-k)\big) \\ &\quad + \binom{d-1}{k-1}\left[\sum_{\ell=k}^{d-1-k}\mathrm{last}\big(\hat{G}_{d-1}(\ell)\big)\right] + \binom{d-1}{k-1}\left[\sum_{\ell=d-k}^{d-1}\mathrm{last}\big(\hat{G}_{d-1}(\ell)\big)\right] \\ &\geq 0,\end{aligned}$$

where we use (8) and the non-negativity of $C_{d-1}(k-1)$ and $G_d(\ell)$ for the second step, and use (9), $\sum_{\ell=k}^{d-1-k} \mathrm{last}(\hat{G}_{d-1}(\ell)) = 0$ and the induction hypothesis $\mathrm{last}(\hat{G}_{d-1}(\ell)) \geq 0$ for $\ell \geq d-k$ for the last step. This proves statement (i).

Statement (iii) follows from statement (i) and Lemma 4.7(ii).

Case 2: Suppose that d is odd. We may assume $k \neq 0$. By Lemma 4.3,

$$\begin{aligned}C_d(k) &= C_{d-1}(k) + \big(0, \mathrm{del}\big(C_{d-1}(k-1)\big)\big) \\ &\quad + \binom{d-1}{k-1}\left\{\sum_{\ell=0}^{k-2}\big(0, \mathrm{del}\big(G_{d-1}(\ell)\big)\big) + \sum_{\ell=k}^{d-1} G_{d-1}(\ell)\right\},\end{aligned}$$

and this expression is non-negative by Lemma 2.8 and the induction hypothesis.

We prove statement (ii). By Lemma 4.7(i), $\mathrm{last}(\hat{C}_d(\frac{d+1}{2})) = 0$. Suppose $k > \frac{d+1}{2}$. Then

$$\hat{C}_d(k) = \hat{C}_{d-1}(k) + \big(0, C_{d-1}(k-1)\big) + \binom{d-1}{k-1}\left\{\sum_{\ell=0}^{k-2}\big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1}\hat{G}_{d-1}(\ell)\right\}$$

and

$$
\begin{aligned}
(10)\qquad & \mathrm{last}\big(\hat{C}_d(k)\big) \\
\geq\ & \mathrm{last}\big(\hat{C}_{d-1}(k)\big) + \binom{d-1}{k-1}\sum_{\ell=k}^{d-1}\big\{\mathrm{last}\big(\hat{G}_{d-1}(\ell)\big) + \mathrm{last}\big(G_{d-1}(d-1-\ell)\big)\big\} \\
& + \binom{d-1}{k-1}\sum_{\ell=d-k}^{k-2}\big(\mathrm{last}\big(G_{d-1}(\ell)\big)\big) \\
\geq\ & -\mathrm{last}\big(C_{d-1}(d-k)\big) + \binom{d-1}{k-1}\sum_{\ell=d-k}^{k-2}\big(\mathrm{last}\big(G_{d-1}(\ell)\big)\big) \\
\geq\ & -\mathrm{last}\big(C_{d-1}(d-k)\big) + \binom{d-1}{k-1}\big\{\mathrm{last}\big(G_{d-1}(d-k)\big) + \mathrm{last}\big(G_{d-1}(k-2)\big)\big\}.
\end{aligned}
$$

(We use Lemmas 2.7(ii) and 4.7(ii) for the second step, and use the fact that $k-2 > d-k$ for the third step.) Since (6) says

$$\mathrm{last}\big(C_{d-1}(d-k)\big) \leq \binom{d-1}{k-1}\big\{\mathrm{last}\big(G_{d-1}(d-k)\big) + 1\big\},$$

(10) and Corollary 3.4 guarantee that $\mathrm{last}(\hat{C}_d(k)) \geq 0$.

Finally, the fact that $\mathrm{last}(\hat{C}_d(k)) \leq 0$ for $k < \frac{d+1}{2}$ follows from Lemma 4.7(i). ☐

To prove the main result, we need a few more technical lemmas.

Lemma 4.9. *Let $1 \leq d$ and $1 \leq k < d$ be integers. For every $3 \leq m \leq \lfloor \frac{d}{2} \rfloor + 1$,*

$$G_d(k) + \tilde{e}_m \leq \big(0, \mathrm{del}\big(G_d(0)\big)\big) + \sum_{\ell=k+1}^{d} G_d(\ell),$$

where $\tilde{e}_1, \ldots, \tilde{e}_{\lfloor \frac{d}{2} \rfloor+1} \in \mathbb{Z}^{\lfloor \frac{d}{2} \rfloor+1}$ is the standard basis of $\mathbb{Z}^{\lfloor \frac{d}{2} \rfloor+1}$.

Proof. We induct on d. For $d \leq 4$, the statement follows from Table 1. Suppose $d > 4$. We often use the next equation which follows from Lemma 2.5.

$$(11)\qquad \big(0, \mathrm{del}\big(G_d(0)\big)\big) = G_d(d) \quad \text{and} \quad \big(0, G_d(0)\big) = \hat{G}_d(d).$$

Suppose that d is odd. By the induction hypothesis,

$$G_d(i) = \sum_{\ell=0}^{i-1}(0, \mathrm{del}(G_{d-1}(\ell))) + \sum_{\ell=i}^{d-1} G_{d-1}(\ell) \geq G_{d-1}(i-1)$$

for $i = 2, \ldots, d-1$. Also, $G_d(d) \geq (0, \mathrm{del}(G_{d-1}(0))) = G_{d-1}(d-1)$ by (11). Then, these facts and Lemma 2.5(i) say

$$
\begin{aligned}
& \big(0, \mathrm{del}\big(G_d(0)\big)\big) + \sum_{\ell=k+1}^{d} G_d(\ell) \\
\geq\ & \left[\sum_{\ell=0}^{k-1}\big(0, \mathrm{del}\big(G_{d-1}(\ell)\big)\big)\right] + \big(0, \mathrm{del}\big(G_{d-1}(d-1)\big)\big) + \sum_{\ell=k+1}^{d} G_{d-1}(\ell-1) \\
=\ & \big(0, \mathrm{del}\big(G_{d-1}(d-1)\big)\big) + G_d(k).
\end{aligned}
$$

Then the desired inequality follows from Corollary 3.4.

Suppose that d is even. The desired inequality follows in the same way as in the case when d is odd except for the rightmost entry. Thus what we must prove is

$$\mathrm{last}\big(G_d(k)\big)+1 \le \mathrm{last}\big(\mathrm{del}\big(G_d(0)\big)\big)+\sum_{\ell=k+1}^{d}\mathrm{last}\big(G_d(\ell)\big).$$

Since $\hat{G}_{d-1}(k)=\sum_{\ell=0}^{k-1}(0,G_{d-2}(\ell))+\sum_{\ell=k}^{d-2}\hat{G}_{d-2}(\ell)$ by Lemma 2.5 and $\mathrm{last}(\hat{G}_{d-2}(\ell))\le 0$ for all ℓ by Lemma 2.8(iii),

$$\mathrm{last}\big(\hat{G}_{d-1}(0)\big)\le \mathrm{last}\big(\hat{G}_{d-1}(1)\big)\le\cdots\le \mathrm{last}\big(\hat{G}_{d-1}(d-1)\big)=\mathrm{last}\big(G_{d-1}(0)\big). \tag{12}$$

On the other hand, by (2) (in the proof of Lemma 2.8) we have $\mathrm{last}(G_d(\ell))\ge \mathrm{last}(G_{d-1}(0))$ for $\ell=1,2,\dots,d$. Then

$$\begin{aligned}
&\mathrm{last}\big(\mathrm{del}\big(G_d(0)\big)\big)+\sum_{\ell=k+1}^{d}\mathrm{last}\big(G_d(\ell)\big)\\
\ge\ & \mathrm{last}\big(G_d(d)\big)+\sum_{\ell=k+1}^{d}\mathrm{last}\big(G_{d-1}(0)\big)\\
=\ & \sum_{\ell=0}^{d-1}\mathrm{last}\big(G_{d-1}(\ell)\big)+\sum_{\ell=k}^{d-1}\mathrm{last}\big(G_{d-1}(0)\big)\\
\ge\ & \left[\left\{\sum_{\ell=0}^{k-1}\mathrm{last}\big(G_{d-1}(\ell)\big)\right\}+\mathrm{last}\big(G_{d-1}(d-1)\big)\right]+\sum_{\ell=k}^{d-1}\mathrm{last}\big(\hat{G}_{d-1}(\ell)\big)\\
\ge\ & \mathrm{last}\big(G_d(k)\big)+1,
\end{aligned}$$

as desired. (We use (11) for the first step, use Lemma 2.5 for the second step and use (12) for the third step.) □

Corollary 4.10. *Let $1\le d$ and $1\le k\le d$ be integers. Then*

$$C_d(k)\le\binom{d}{k}\left\{\big(0,\mathrm{del}\big(G_d(0)\big)\big)+\sum_{\ell=k+1}^{d}G_d(\ell)\right\}.$$

Proof. If $k=d$ then $C_d(d)=G_d(d)=(0,\mathrm{del}(G_d(0)))$ by the definition. Suppose $1\le k<d$. By Lemma 4.9 and (6),

$$C_d(k)=\binom{d}{k}\big\{G_d(k)-\tilde{e}_{k+1}+\tilde{e}_{k+2}\big\}-C_d(k+1)\le\binom{d}{k}\left\{\big(0,\mathrm{del}\big(G_d(0)\big)\big)+\sum_{\ell=k+1}^{d}G_d(\ell)\right\},$$

as desired, where $\tilde{e}_i=0$ for $i>\lfloor\frac{d}{2}\rfloor+1$. □

Lemma 4.11. *Let $1\le d$ be an odd integer. Then $\sum_{\ell=k}^{d}\hat{G}_d(\ell)$ is admissible to $(G_d(0),0)$ for $k=1,2,\dots,d$.*

Proof. The statement follows from Proposition 3.3 if $k>\frac{d}{2}$. Suppose $k<\frac{d}{2}$. By Lemma 2.7,

$$\sum_{\ell=k}^{d}\hat{G}_d(\ell)=\sum_{\ell=k}^{d-k}\hat{G}_d(\ell)+\sum_{\ell=d-k+1}^{d}\hat{G}_d(\ell)=\sum_{\ell=k}^{d-k}\big(G_d(\ell),0\big)+\sum_{\ell=d-k+1}^{d}\hat{G}_d(\ell).$$

The above vector is admissible to $(G_d(0),0)$ by Proposition 3.3. □

Proposition 4.12. *Let $1 \le d$ and $b_1, b_2, \dots, b_d$ be non-negative integers.*

(A) *The vector $X = G_d(0)+\sum_{k=1}^{d} b_k C_d(k)$ is the f-vector of a simplicial complex.*
(B) *If d is odd then the vector*

$$Y = \big(G_d(0),0\big) + \sum_{1\le k\le \frac{d}{2}} b_k\big(C_d(k),0\big) + \sum_{\frac{d}{2}\le k\le d} b_k\hat{C}_d(k)$$

is the f-vector of a simplicial complex.

Proof. We induct on d. For $d \le 2$ the statements follow from Table 2. Suppose $d > 2$.

Case 1: Suppose that d is even. For $k = 1, 2, \dots, d$, let

$$D(k) = \binom{d-1}{k-1}\left\{\sum_{\ell=0}^{k-2}\big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1}\hat{G}_{d-1}(\ell)\right\}.$$

By Lemma 4.3

$$C_d(k) = \hat{C}_{d-1}(k) + \big(0, C_{d-1}(k-1)\big) + D(k).$$

Let

$$Z = G_d(0)+\sum_{k=1}^{d} b_k\big\{\big(0, C_{d-1}(k-1)\big)+D(k)\big\} + \sum_{1\le k<\frac{d}{2}} b_k\big(C_{d-1}(k),0\big) + \sum_{\frac{d}{2}\le k\le d} b_k\hat{C}_{d-1}(k).$$

Then $\mathrm{del}(X) = \mathrm{del}(Z)$ and $X \le Z$ by Lemma 4.8(ii). Since X is non-negative by Lemma 4.8, it is enough to prove that Z is the f-vector of a simplicial complex by Lemma 3.1(iv).

By the induction hypothesis,

$$N_1 = \big(G_{d-1}(0),0\big) + \sum_{1\le k<\frac{d}{2}} b_k\big(C_{d-1}(k),0\big) + \sum_{\frac{d}{2}\le k\le d-1} b_k\hat{C}_{d-1}(k)$$

is the f-vector of a simplicial complex. By Proposition 3.3, $(G_{d-1}(\ell),0)$ is admissible to $N_1 \ge (G_{d-1}(0),0)$ for $\ell = 1, 2, \dots, d-1$. Then the vector

$$N_2 = N_1 + \sum_{\ell=1}^{d-2}\big(G_{d-1}(\ell),0\big)$$

is the f-vector of a simplicial complex. By Lemma 4.11, $\sum_{\ell=k}^{d-1}\hat{G}_{d-1}(\ell)$ is admissible to $N_2 \ge (G_{d-1}(0),0)$ for $k = 1, 2, \dots, d-1$. Also, for $k \le d$, since $\sum_{\ell=0}^{k-2} G_{d-1}(\ell)$ is the f-vector of a simplicial complex by Proposition 3.3, $\sum_{\ell=0}^{k-2}(0, G_{d-1}(\ell))$ is admissible to $N_2 \ge \sum_{\ell=0}^{d-2}(G_{d-1}(\ell),0)$. Thus $D(k)$ is admissible to N_2 for $k = 1, 2, \dots, d$. Hence

$$N_3 = N_1 + \sum_{\ell=1}^{d-2}\big(G_{d-1}(\ell),0\big) + \sum_{k=1}^{d} b_k D(k)$$

is the f-vector of a simplicial complex. On the other hand, by (4) (in the proof of Proposition 3.3) and (11), we have

$$\begin{aligned} Z &= N_3 + \big(G_{d-1}(d-1),0\big) + \sum_{k=1}^{d} b_k\big(0, C_{d-1}(k-1)\big) \\ &= N_3 + \left\{ \big(0, \mathrm{del}\big(G_{d-1}(0)\big),0\big) + \sum_{k=1}^{d} b_k\big(0, C_{d-1}(k-1)\big) \right\}. \end{aligned}$$

Set

$$E = \big(\mathrm{del}\big(G_{d-1}(0)\big),0\big) + \sum_{k=1}^{d} b_k C_{d-1}(k-1).$$

We claim that $(0,E)$ is admissible to N_3. By the induction hypothesis, $G_{d-1}(0) + \sum_{k=1}^{d} b_k C_{d-1}(k-1)$ is the f-vector of a simplicial complex. Then E is also the f-vector of a simplicial complex. On the other hand, since $\mathrm{del}(D(k)) \geq C_{d-1}(k-1)$ by Corollary 4.10, we have

$$\mathrm{del}(N_3) \geq G_{d-1}(0) + \sum_{k=1}^{d} b_k C_{d-1}(k-1) \geq E.$$

Thus $(0,E)$ is admissible to N_3, and $Z = N_3 + (0,E)$ is the f-vector of a simplicial complex.

Case 2: Suppose that d is odd. It is enough to prove (B). For $k = 1,2,\dots,d$, let

$$D(k) = \binom{d-1}{k-1} \left\{ \sum_{\ell=0}^{k-2} \big(0, G_{d-1}(\ell)\big) + \sum_{\ell=k}^{d-1} \big(G_{d-1}(\ell),0\big) \right\}.$$

By Lemmas 2.8, 4.3 and 4.8

$$\begin{aligned} \hat{C}_d(k) &\leq \big(C_{d-1}(k),0\big) + \big(0, C_{d-1}(k-1)\big) + D(k), \\ \big(C_d(k),0\big) &\leq \big(C_{d-1}(k),0\big) + \big(0, C_{d-1}(k-1)\big) + D(k) \end{aligned}$$

and

$$C_d(k) = \mathrm{del}\big(\hat{C}_d(k)\big) = \mathrm{del}\big\{ \big(C_{d-1}(k),0\big) + \big(0, C_{d-1}(k-1)\big) + D(k) \big\}.$$

Let

$$Z = \big(G_d(0),0\big) + \sum_{k=1}^{d} b_k \big\{ \big(C_{d-1}(k),0\big) + \big(0, C_{d-1}(k-1)\big) + D(k) \big\}.$$

Then $Y \leq Z$ and $\mathrm{del}(Y) = \mathrm{del}(Z)$. Since Y is non-negative, it is enough to prove that Z is the f-vector of a simplicial complex by Lemma 3.1(iv). This can be proved in the same way as in Case 1.

First, by the induction hypothesis,

$$N_1 = \big(G_{d-1}(0),0\big) + \sum_{k=1}^{d} b_k \big(C_{d-1}(k),0\big)$$

is the f-vector of a simplicial complex. Second, since $(G_{d-1}(\ell),0)$ is admissible to N_1 for $\ell=1,\dots,d-1$ and since $\sum_{\ell=0}^{k-2}(0,G_{d-1}(\ell))$ is admissible to $N_1+\sum_{\ell=1}^{d-2}(G_{d-1}(\ell),0)$,

$$N_2 = N_1 + \sum_{\ell=1}^{d-2}\big(G_{d-1}(\ell),0\big) + \sum_{k=1}^{d} b_k D(k)$$

is the f-vector of a simplicial complex. Observe that

$$Z = N_2 + \left\{\big(0,\mathrm{del}\big(G_{d-1}(0)\big),0\big) + \sum_{k=1}^{d} b_k\big(0,C_{d-1}(k-1)\big)\right\}.$$

Let $E=(\mathrm{del}(G_{d-1}(0)),0)+\sum_{k=1}^{d} b_k C_{d-1}(k-1)$. Then $(0,E)$ is admissible to N_2. Indeed, E is the f-vector of a simplicial complex by the induction hypothesis and is smaller than or equal to $\mathrm{del}(N_2)$ by Corollary 4.10. □

Proof of Theorem 1.3. Since P is Buchsbaum, it follows from [NS1, Theorem 6.4] that $h''_0(P)=1$ and $h''_i(P)\geq 0$ for $i\geq 1$. Let $\beta_i=\beta_i(\mathrm{sd}(P))$ for $i=0,1,\dots,d-1$. By Lemma 4.1,

$$g''\big(\mathrm{sd}(P)\big) = G_d(0) + \sum_{k=1}^{d} h''_k(P)G_d(k) + \sum_{k=1}^{d-1}\beta_{k-1}C_d(k).$$

By Proposition 4.12,

$$X = G_d(0) + \sum_{k=1}^{d-1}\beta_{k-1}C_d(k)$$

is the f-vector of a simplicial complex. Since $G_d(\ell)$ is admissible to $X\geq G_d(0)$ for $\ell=1,\dots,d$ by Proposition 3.3, $g''(\mathrm{sd}(P))$ is the f-vector of a simplicial complex. □

Corollary 4.13. *Let P be a Buchsbaum simplicial poset. Then $h''(\mathrm{sd}(P))$ is unimodal.*

Proof. Lemmas 2.5(ii) and 2.8(i) imply that $H_d(k)$ is a unimodal vector whose peak lies in the middle for $k=0,1,\dots,d$. Also, the same property holds for $B_d(k)$ by Lemmas 4.4(ii) and 4.8. Then, since $h''(P)$ is non-negative, $h''(\mathrm{sd}(P))$ must be unimodal by Lemma 4.1. □

We conclude this paper with a few questions about h''-vectors.

Let P be a Cohen–Macaulay simplicial poset and let $(h_0,\dots,h_d)=h(\mathrm{sd}(P))$. Brenti and Welker [BW, Theorem 2] proved that the zeros of the polynomial $f=\sum_{i=0}^{d} h_i t^{d-i}$ are all real numbers. When f is symmetric, this result is of particular interest in connection with the Charney-Davis conjecture (see [BW, p. 857]). It is natural to ask if the same property holds for h''-vectors of orientable homology manifolds (or more generally Buchsbaum simplicial posets), that is,

Question 4.14. Let P be a Buchsbaum simplicial poset of rank d (such that $\mathrm{sd}(P)$ is an orientable homology manifold) and $(h_0,\dots,h_d)=h''(\mathrm{sd}(P))$. Are all zeros of the polynomial $f=\sum_{i=0}^{d} h_i t^{d-i}$ real numbers?

Let Δ be a simplicial complex with the vertex set V. A subset $F\subset V$ is called a *missing face* of Δ if $F\notin\Delta$ and all proper subsets of F are contained in Δ. A *flag* simplicial complex is a simplicial complex Δ such that all missing faces of Δ

have cardinality 2. It is easy to see that the order complex of a finite poset is a flag complex. Kalai conjectured that if Δ is a $(d-1)$-dimensional Cohen–Macaulay flag simplicial complex then $h(\Delta)$ is the f-vector of a simplicial complex. (Actually, Kalai proposed a stronger conjecture. He conjectured that $h(\Delta)$ is the f-vector of a d-colored simplicial complex. See [St3, p. 100].) In view of Kalai's conjecture and the results of this paper, we ask the following question which implies a stronger version of the g-conjecture for flag homology spheres.

Question 4.15. Let Δ be a flag simplicial complex which is a connected orientable homology manifold. Is $g''(\Delta)$ the f-vector of a simplicial complex?

Even the non-negativity of $g''(\Delta)$ is not known. The new part of the above question is the upper bound conditions of Kruskal–Katona theorem. We mention that Nevo, Petersen and Tenner [NPT] recently proposed a stronger conjecture for flag simplicial spheres.

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