

FACE VECTORS OF SIMPLICIAL CELL DECOMPOSITIONS OF MANIFOLDS

SATOSHI MURAI

ABSTRACT. In this paper, we study face vectors of simplicial posets that are the face posets of cell decompositions of topological manifolds without boundary. We characterize all possible face vectors of simplicial posets whose geometric realizations are homeomorphic to the product of spheres. As a corollary, we obtain the characterization of face vectors of simplicial posets whose geometric realizations are odd dimensional manifolds without boundary.

1. INTRODUCTION

The study of face numbers is one of the central topics in combinatorics. A goal of the study is to obtain characterizations of face vectors of various classes of combinatorial objects. In this paper, we study face vectors of simplicial posets, particularly those whose geometric realizations are manifolds.

A *simplicial poset* is a finite poset P with a minimal element $\hat{0}$ such that every interval $[\hat{0}, \sigma]$ for $\sigma \in P$ is a Boolean algebra. It is known that any simplicial poset is the face poset of a regular CW-complex $\Gamma(P)$ [Bj]. A CW-complex whose face poset is a simplicial poset is called a *simplicial cell complex* (also called a Boolean cell complex or a pseudocomplex).

Let P be a simplicial poset. We say that an element $\sigma \in P$ has *rank* i , denoted $\text{rank } \sigma = i$, if $[\hat{0}, \sigma]$ is a Boolean algebra of rank i . Thus those elements correspond to $(i - 1)$ -dimensional cells of $\Gamma(P)$. The *dimension* of P is

$$\dim P = \max\{\text{rank } \sigma : \sigma \in P\} - 1.$$

Let $f_i = f_i(P)$ be the number of elements $\sigma \in P$ having rank $i + 1$ and $d = \dim P + 1$. The vector $f(P) = (f_{-1}, f_0, \dots, f_{d-1})$ is called the *f-vector* of P . To study *f-vectors*, it is often convenient to consider the *h-vector* $h(P) = (h_0, h_1, \dots, h_d)$ of P defined by

$$\sum_{i=0}^d f_{i-1} t^i (1-t)^{d-i} = \sum_{i=0}^d h_i t^i.$$

It is easy to see that knowing $f(P)$ is equivalent to knowing $h(P)$.

A *simplicial cell sphere* is a simplicial poset P such that $\Gamma(P)$ is homeomorphic to a sphere. One of the most important results on face vectors of simplicial posets is the next result due to Stanley [St] and Masuda [Ma], which characterize all possible *h-vectors* of simplicial cell spheres.

This work was supported by KAKENHI 22740018.

Theorem 1.1 (Stanley, Masuda). *Let $h = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$. Then h is the h -vector of a $(d-1)$ -dimensional simplicial cell sphere if and only if it satisfies the following conditions:*

- (1) $h_0 = h_d = 1$ and $h_i = h_{d-i}$ for all i .
- (2) $h_i \geq 0$ for all i .
- (3) if $h_i = 0$ for some $1 \leq i \leq d-1$ then $h_0 + h_1 + \dots + h_d$ is even.

Note that Theorem 1.1 characterizes the face vectors of simplicial cell spheres. We say that a poset P is a *simplicial cell decomposition of a topological space X* if P is a simplicial poset such that $\Gamma(P)$ is homeomorphic to X . From topological and combinatorial viewpoints, it is natural to ask a characterization of face vectors of simplicial cell decompositions of a given topological manifold. In this paper, we give such a characterization for the product of spheres.

Before stating the result, we define h'' -vectors introduced by Novik [No]. From now on, we fix a field K . For a simplicial poset P , let

$$\beta_i = \beta_i(P) = \dim_K \tilde{H}_i(P; K)$$

be the i th Betti number of P , where $\tilde{H}_i(P; K)$ is the i th reduced homology group of P (or $\Gamma(P)$) over K . The h'' -vector $h''(P) = (h''_0, h''_1, \dots, h''_d)$ of P (over K) is defined by

$$h''_k(P) = \begin{cases} 1, & \text{if } k = 0, \\ h_k - \binom{d}{k} \left\{ \sum_{\ell=1}^k (-1)^{\ell-k} \beta_{\ell-1} \right\}, & \text{if } 1 \leq k \leq d-1, \\ h_d - \sum_{\ell=1}^{d-1} (-1)^{\ell-d} \beta_{\ell-1} = \beta_{d-1}, & \text{if } k = d. \end{cases}$$

If one knows Betti numbers, then knowing $h(P)$ is equivalent to knowing $h''(P)$. (Since Betti numbers depend on the characteristic, h'' -vectors depend on the characteristic of the base field K .) It was proved by Novik [No] and Novik-Swartz [NS] that the h'' -vector of a simplicial cell decomposition of an orientable manifold is symmetric and non-negative (see section 2). The main result of this paper is the next result, which characterizes face vectors of simplicial cell decompositions of the product of spheres $S^n \times S^m$.

Theorem 1.2. *Fix integers $n, m \geq 1$. Let $d = n + m + 1$ and $h = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$. There exists a simplicial cell decomposition P of $S^n \times S^m$ with $h''(P) = h$ if and only if h satisfies the conditions (1), (2) and (3) in Theorem 1.1.*

The technique used in the proof of the above theorem is also applicable to other classes of simplicial posets. We characterize all possible face vectors of simplicial cell decompositions of real projective spaces (Theorem 7.1) and face vectors of simplicial cell complexes that are odd dimensional topological manifolds without boundary (Theorem 7.2).

This paper is organized as follows: In section 2, we recall known conditions on h'' -vectors and prove the necessity of Theorem 1.2. In section 3–6, we prove the sufficiency of Theorem 1.2. The key idea to prove the sufficiency is a graph theoretical construction of simplicial cell decompositions of manifolds called crystallizations

[FGG]. In section 7, we discuss face vectors of simplicial cell decompositions of real projective spaces and odd dimensional manifolds.

2. NECESSITY CONDITIONS OF h'' -VECTORS

In this section, we recall some known necessity conditions of h'' -vectors.

Let P be a $(d-1)$ -dimensional simplicial poset. For an element $\sigma \in P$, the *link* of σ in P , denoted by $P_{\geq\sigma}$, is a poset

$$P_{\geq\sigma} = \{\tau \in P : \tau \geq \sigma\}.$$

It is easy to see that $P_{\geq\sigma}$ is again a simplicial poset with the minimal element σ . For $k = 0, 1, \dots, d$ we write

$$P_k = \{\sigma \in P : \text{rank } \sigma = k\}.$$

An element in P_1 is called a *vertex* of P and a maximal element in P is called a *facet* of P . A simplicial poset P is said to be *pure* if all its facets have the same rank.

A $(d-1)$ -dimensional simplicial poset P is said to be a *homology sphere (over K)* if, for all $\sigma \in P$, $\beta_i(P_{\geq\sigma}) = 0$ for all $i \neq d-1-\text{rank } \sigma$ and $\beta_{d-1-\text{rank } \sigma}(P_{\geq\sigma}) = 1$. Also, a pure simplicial poset P is said to be a *homology manifold (without boundary)* if $P_{\geq v}$ is a homology sphere for all vertices $v \in P_1$. A *simplicial cell homology manifold* is a simplicial poset which is a homology manifold. From now on, we assume that all homology manifolds are connected. A simplicial cell homology manifold P is said to be *orientable* if $\beta_{d-1}(P) = 1$.

The next result is crucial for the necessity of Theorem 1.2.

Theorem 2.1. *Let P be a $(d-1)$ -dimensional simplicial cell homology manifold. Then*

- (i) (Novik-Swartz) $h''_i(P) \geq 0$ for all i .
- (ii) (Novik) if P is orientable then $h''_i(P) = h''_{d-i}(P)$ for all i .

The first condition was recently proved by Novik and Swartz [NS, Proposition 6.3 and Theorem 6.4]. The second condition was proved by Novik in [No, Lemma 7.3] for simplicial complexes. However, this condition essentially follows from the Dehn–Sommerville equations for homology manifolds and the Poincaré duality. Since both the Dehn–Sommerville equations and the Poincaré duality hold for simplicial posets, (ii) holds for simplicial posets. See [MMP, section 8] for Dehn–Sommerville equations for simplicial posets.

Another necessary condition of h'' -vectors which we need is the following condition.

Lemma 2.2. *Let P be a $(d-1)$ -dimensional orientable simplicial cell homology manifold. If $h''_i(P) = 0$ for some $1 \leq i \leq d-1$ then the number of facets of P is even.*

For homology spheres, the above statement was conjectured by Stanley [St] and proved by Masuda [Ma].

To prove Lemma 2.2, we need an algebraic tool, called face rings. Since the proof is essentially the same as the simple proof of Masuda’s result given by Miller

and Reiner [MR], we just sketch the proof. We refer the readers to [St] and [Du] for basic algebraic properties of face rings and basic algebraic notations on commutative algebra.

Let P be a simplicial poset, $R = K[x_\sigma : \sigma \in P \setminus \{\hat{0}\}]$ the polynomial ring over a field K in indeterminates indexed by the elements in $P \setminus \{\hat{0}\}$ and $S = K[x_v : v \in P_1]$. The *face ring of P* is the quotient ring $K[P] = R/I_P$, where I_P is the ideal generated by the following elements

- $x_\sigma x_\tau$ for all pairs $\sigma, \tau \in P$ that have no common upper bounds in P .
- $x_\sigma x_\tau - x_{\sigma \wedge \tau} \sum_\rho x_\rho$, where the summation runs over the minimal elements among all upper bounds of σ and τ , and where $\sigma \wedge \tau$ is the meet (largest lower bounds) of σ and τ . (We consider $x_{\sigma \wedge \tau} = 1$ if $\sigma \wedge \tau = \hat{0}$.)

It is known that, by setting $\deg x_\sigma = \text{rank } \sigma$, $K[P]$ is a d -dimensional finitely generated S -module whose Hilbert series determines the f -vector of P , where $d = \dim P + 1$. See [St, Proposition 3.8 and Lemma 3.9].

Proof of Lemma 2.2. (Sketch). Throughout the proof we regard $K[P]$ as an S -module. Let $\theta_1, \dots, \theta_d \in S$ be an l.s.o.p. of $K[P]$ (it exists by assuming that K is infinite if necessary) and $A_P = K[P]/(\theta_1, \dots, \theta_d)K[P]$. By Schenzel's results (see [NS, Proposition 6.3]), we have

(NS1) $\dim_K(A_P)_d = h''_d = 1$ and $\dim_K(A_P)_k = h''_k + \binom{d}{k} \beta_{k-1}(P)$ for $k = 1, \dots, d-1$, where $(A_P)_k$ is the homogeneous component of A_P of degree k . Since $h''_i(P) = 0$, it follows from [NS, Theorem 6.4] that all elements in $(A_P)_i$ are socle elements, that is, for any $f \in (A_P)_i$ and for any homogeneous polynomial $h \in S$ with $h \notin K$, we have $fh = 0$ in A_P . In particular, for distinct vertices v_1, v_2, \dots, v_d of P , we have

(NS2) $x_{v_1} x_{v_2} \cdots x_{v_d} = 0$ in A_P .

For an element $\sigma \in P$, let $V(\sigma) = \{v \in P_1 : v \leq \sigma\}$ be the set of vertices of σ . Since P is pure, by the definition of the ideal I_P , $x_{v_1} x_{v_2} \cdots x_{v_d} = \sum_{\sigma \in P_d, V(\sigma) = \{v_1, \dots, v_d\}} x_\sigma$ in $K[P]$. Since P is a pseudomanifold (see section 4 for the definition of pseudomanifolds) it follows from [MR, Propositions 5 and 6] that

(MR1) if $(A_P)_d \neq \{0\}$ then, for any facet $\sigma \in P$, $x_\sigma \neq 0$ in A_P .

(MR2) for all facets σ and τ of P with $V(\sigma) = V(\tau)$, $x_\sigma = \pm x_\tau$.

(NS1) shows that the assumption of (MR1) is satisfied. Then, for distinct vertices v_1, \dots, v_d of P , since (NS2) says $\sum_{\sigma \in P_d, V(\sigma) = \{v_1, \dots, v_d\}} x_\sigma = 0$ in A_P , by (MR1) and (MR2) it follows that the number of faces σ of P with $V(\sigma) = \{v_1, \dots, v_d\}$ is even. Hence the number of facets of P is even. \square

Corollary 2.3. *Let P be a $(d-1)$ -dimensional orientable simplicial cell homology manifold. If $h''_i(P) = 0$ for some $1 \leq i \leq d-1$, then $\sum_{i=0}^d h''_i(P)$ is even.*

Proof. By the symmetry of h'' -vectors, we may assume that d is even. Since $f_{d-1}(P) = \sum_{i=0}^d h_i(P)$ is even by Lemma 2.2, it is enough to prove that $\sum_{i=0}^d h_i(P) \equiv \sum_{i=0}^d h''_i(P)$

mod 2. By the definition of h'' -vectors,

$$\begin{aligned} \sum_{i=0}^d h''_i(P) &= \sum_{i=0}^d h_i(P) + \left[\sum_{i=1}^{d-2} \beta_i(P) \left\{ \sum_{l=i+1}^d (-1)^{l-i-1} \binom{d}{l} \right\} \right] \\ &= \sum_{i=0}^d h_i(P) + \sum_{i=1}^{d-2} \beta_i(P) \binom{d-1}{i} \\ &= \sum_{i=0}^d h_i(P) + 2 \left\{ \sum_{i=1}^{\frac{d-2}{2}} \beta_i(P) \binom{d-1}{i} \right\} \end{aligned}$$

as desired, where we use the Poincaré duality $\beta_i = \beta_{d-1-i}$ for the last equality. \square

Theorem 2.1 and Corollary 2.3 prove the necessity of Theorem 1.2. More precisely,

Theorem 2.4. *If P is a $(d-1)$ -dimensional orientable simplicial cell homology manifold, then $h''(P)$ satisfies the conditions (1), (2) and (3) in Theorem 1.1.*

3. HOW TO CHARACTERIZE h'' -VECTORS

In this section, we show that, to characterize h'' -vectors of simplicial cell decompositions of a manifold M , it is enough to find simplicial cell decompositions of M with minimal h'' -vectors. From now on, all manifolds are connected, compact and without boundary. Also, all constructions in this paper are done in the piecewise linear (PL) category (see [Hu, RS]).

Let P and Q be $(d-1)$ -dimensional simplicial posets, $\sigma \in P_d$ and $\tau \in Q_d$. The connected sum of P and Q with respect to σ and τ is the simplicial poset, denoted $P\#_{\sigma,\tau}Q$ (or $P\#Q$ for short), obtained from P and Q by removing σ and τ from P and Q and by identifying $[\hat{0}, \sigma] \setminus \{\sigma\}$ and $[\hat{0}, \tau] \setminus \{\tau\}$. Thus, topologically, $P\#Q$ is obtained by removing $(d-1)$ -cells σ and τ from P and Q and gluing them along the boundaries of σ and τ . (The combinatorial type and the topological type of $P\#_{\sigma,\tau}Q$ may depend on the specific identification of the vertices.)

Lemma 3.1. *Let P be a $(d-1)$ -dimensional orientable simplicial cell homology manifold and Q a $(d-1)$ -dimensional simplicial cell homology manifold. Then $P\#Q$ is a homology manifold satisfying the following conditions*

- (i) $\beta_i(P\#Q) = \beta_i(P) + \beta_i(Q)$ for $i \neq d-1$ and $\beta_{d-1}(P\#Q) = \beta_{d-1}(Q)$.
- (ii) $h''_i(P\#Q) = h''_i(P) + h''_i(Q)$ for $i \neq 0, d-1$ and $h''_d(P\#Q) = h''_d(Q)$.

Proof. It is straightforward that $P\#Q$ is a homology manifold. (i) follows from a simple Mayer–Vietoris argument. Observe $f_{i-1}(P\#Q) = f_{i-1}(P) + f_{i-1}(Q) - \binom{d}{i}$ for $i \neq d$ and $f_{d-1}(P\#Q) = f_{d-1}(P) + f_{d-1}(Q) - 2$. Straightforward computations show $h_i(P\#Q) = h_i(P) + h_i(Q)$ for $i \neq d$ and $h_d(P\#Q) = h_d(P) + h_d(Q) - 1$. Then (ii) follows from (i) and the definition of h'' -vectors. \square

Let M be a $(d-1)$ -dimensional manifold. We write $\mathcal{H}(M)$ for the set of all h'' -vectors of simplicial cell decompositions of M , where we consider h'' -vectors over a field of characteristic 2 if M is non-orientable. For example, if $M = S^{d-1}$ the

$(d-1)$ -dimensional sphere, then $\mathcal{H}(M)$ is the set of all vectors $h \in \mathbb{Z}^{d+1}$ satisfying the conditions (1), (2) and (3) in Theorem 1.1.

Corollary 3.2. *With the same notation as above, if there is a simplicial cell decomposition P of M with $h''(P) = (1, 0, \dots, 0, 1)$, then $\mathcal{H}(M) = \mathcal{H}(S^{d-1})$.*

Proof. Since any manifold is an orientable homology manifold over a field of characteristic 2, Theorem 2.4 shows $\mathcal{H}(M) \subset \mathcal{H}(S^{d-1})$. We prove the reverse inclusion. Let $h \in \mathcal{H}(S^{d-1})$. There exists a $(d-1)$ -dimensional simplicial cell sphere Q with $h''(Q) = h(Q) = h$ by Theorem 1.1. Then $P \# Q$ is a simplicial cell decomposition of M with the desired h'' -vector by Lemma 3.1. \square

Corollary 3.2 shows that the existence of a simplicial cell decomposition P of M with $h''(P) = (1, 0, \dots, 0, 1)$ induces a characterization of face vectors of simplicial cell decompositions of M .

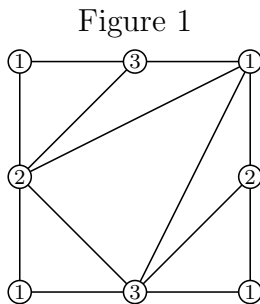
We define a partial order $>_{\mathcal{P}}$ on $\mathcal{H}(S^{d-1})$ by, for $h, h' \in \mathcal{H}(S^{d-1})$, $h >_{\mathcal{P}} h'$ if $h - h' + (1, 0, \dots, 0, 1) \in \mathcal{H}(S^{d-1})$. The proof of Corollary 3.2 says that, to characterize h'' -vectors of simplicial cell decompositions of M , it is enough to find all minimal elements of $\mathcal{H}(M)$ with respect to $>_{\mathcal{P}}$. This fact suggests the following problems.

Problem 3.3. For a given manifold M , find all minimal elements in $\mathcal{H}(M)$.

Problem 3.4. For which manifold M , $\mathcal{H}(M)$ possess the unique minimal element? In particular, for which M , one has $(1, 0, \dots, 0, 1) \in \mathcal{H}(M)$?

For present, we do not even have an example of a manifold M such that $\mathcal{H}(M)$ has more than two minimal elements. On the other hand, there are many manifolds M with $(1, 0, \dots, 0, 1) \notin \mathcal{H}(M)$. Indeed, a recent result of Klee [Kl] says that any manifold M whose fundamental group requires more generators than its first Betti number, such as any non-trivial homology sphere, satisfies $(1, 0, \dots, 0, 1) \notin \mathcal{H}(M)$.

Example 3.5. Figure 1 is a simplicial cell complex that presents $S^1 \times S^1$. (Identify parallel edges of the square.)



Its f -vector is $f = (1, 3, 9, 6)$ and its h -vector is $h = (1, 0, 6, -1)$. Since $\beta_1(S^1 \times S^1) = 2$, the h'' -vector is $h'' = h - 2(0, 0, 3, -1) = (1, 0, 0, 1)$.

4. GRAPHICAL SIMPLICIAL POSETS

To study problems given in the previous section, it is important to have a good construction of simplicial cell homology manifolds. We use graph theoretic approach

called *crystallizations*. In this section, we briefly introduce crystallization theory. Most statements of this section are not new, but we rewrite it to adapt the theory to simplicial posets. A good survey of crystallization theory is [FGG].

Let $G = (V, E, \phi)$ be a (finite) multi-graph (without loops), where V is a finite set of vertices, E is a finite set of edges and ϕ is a function that assigns to each edge $e \in E$ a 2-element set of vertices $\phi(e) \subset V$. For an integer $d \geq 1$, a pair $\Lambda = (G, \gamma)$ of a graph $G = (V, E, \phi)$ and a map $\gamma : E \rightarrow [d] = \{1, 2, \dots, d\}$ is called a *d-colored multi-graph*. For a *d*-colored multi-graph $\Lambda = (G, \gamma)$ and $S \subset [d]$, let

$$E_S = \{e \in E : \gamma(e) \in S\}$$

and

$$G_S = (V, E_S, \phi_S),$$

where ϕ_S is the restriction of ϕ to E_S . Thus G_S is the multi-graph whose edges are the edges in G having color $i \in S$. We write $\Lambda_S = (G_S, \gamma_S)$, where $\gamma_S : E_S \rightarrow [d]$ is the restriction of γ to E_S . We say that a *d*-colored multi-graph $\Lambda = (G, \gamma)$ is *admissible* if it satisfies the following conditions:

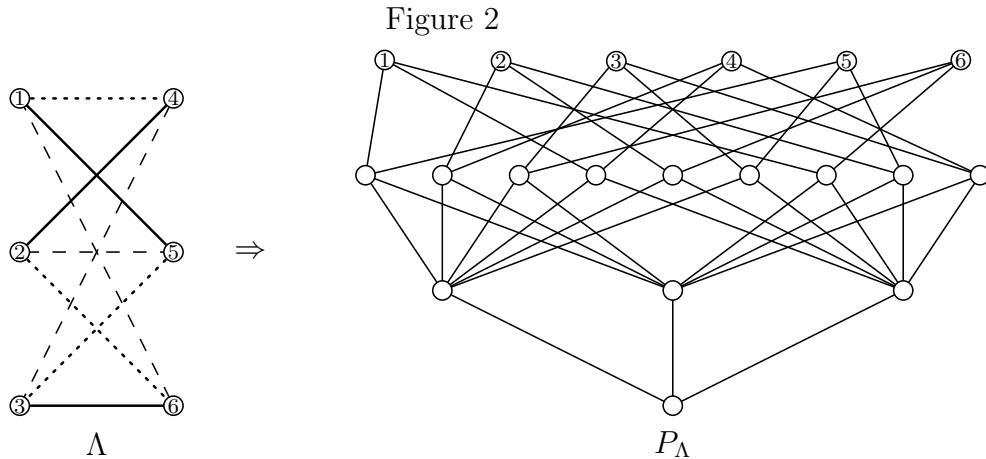
- (a) G is connected.
- (b) for each $i \in [d]$, $G_{\{i\}}$ is a complete matching on V . In other words, all edges in $G_{\{i\}}$ are vertex-disjoint and every vertex in V is a vertex of an edge of $G_{\{i\}}$.

Note that the number of the vertices of G must be even by (b).

For an admissible *d*-colored multi-graph Λ , we define a poset P_Λ such that its elements are the pairs (H, S) of a connected component H of G_S and a subset $S \subset [d]$ and the order on P_Λ is defined by

$$(H, S) \geq (H', S') \Leftrightarrow S \subset S' \text{ and } H \text{ is a subgraph of } H'.$$

Thus H consists of a single vertex of G if $S = \emptyset$ (since $G_\emptyset = (V, \emptyset, \phi_\emptyset)$) and H consists of a single edge if $S = \{i\}$ since $G_{\{i\}}$ is a matching. Figure 2 is an example of an admissible 3-colored multi-graph Λ and the poset P_Λ . (This P_Λ is the simplicial cell decomposition of $S^1 \times S^1$ given in Figure 1.) Many examples of admissible colored graphs that present manifolds can be found in [FGG].



It is not hard to see that P_Λ is simplicial. Indeed, for $(H, S) \in P_\Lambda$ and for every $S' \supset S$, there is the unique connected component H' of $G_{S'}$ which contains H . Also, it is clear that $(G, [d])$ is the unique minimal element of P_Λ . These facts show that the interval $[(G, [d]), (H, S)]$ is isomorphic to the poset of the set of all subsets of $[d] \setminus S$ ordered by inclusion. Hence P_Λ is simplicial. Moreover, $\text{rank}(H, S) = d - \#S$ and $\dim P_\Lambda = d - 1$, where $\#X$ is the cardinality of a finite set X .

We say that a simplicial poset P is *graphical* if there exists an admissible colored multi-graph Λ such that P is isomorphic to P_Λ as posets. In the rest of this section, we study which simplicial posets are graphical.

A $(d - 1)$ -dimensional simplicial poset is said to be a *pseudomanifold (without boundary)* if P satisfies the following conditions:

- (i) P is *pure*.
- (ii) every element $\sigma \in P_{d-1}$ is covered by exactly two elements in P_d .
- (iii) P is *strongly connected*. In other words, for all $\sigma, \tau \in P_d$, there is a sequence $\sigma = \sigma_1, \sigma_2, \dots, \sigma_p = \tau$ of elements of P_d such that σ_i and σ_{i+1} has a common lower bound having rank $d - 1$ for all i .

Moreover, a pseudomanifold P is said to be *normal* if for every $\sigma \in P$ with $\text{rank } \sigma \leq d - 2$, its link $P_{\geq \sigma}$ is connected (as a CW-complex). It is not hard to see that a link of a normal pseudomanifold is again a normal pseudomanifold (see [BD, p. 331]).

Observe that if P_Λ is a graphical simplicial poset and $(H, S) \in P_\Lambda$ then the link $(P_\Lambda)_{\geq (H, S)}$ is the graphical simplicial poset $P_{\Lambda'}$ of the graph $\Lambda' = (H, \gamma_H)$, where γ_H is the restriction of the coloring map of Λ to H . Then it is straightforward that any graphical simplicial poset is a normal pseudomanifold.

Another combinatorial property of graphical simplicial posets is the fact that it has a nice coloring on their vertices. Let P be a simplicial poset. Recall that $V(\sigma) = \{v \in P_1 : v \leq \sigma\}$, where $\sigma \in P$, is the set of vertices of σ . We say that P is *d-colored* if there exists a map $\psi : P_1 \rightarrow [d]$ such that, for every $\sigma \in P$, $\psi(u) \neq \psi(v)$ for all $u, v \in V(\sigma)$ with $u \neq v$. Recall that any vertex of a $(d - 1)$ -dimensional graphical simplicial poset P_Λ is an element of the form $(H, [d] \setminus \{i\})$. Then P_Λ is *d-colored* by defining $\psi((H, [d] \setminus \{i\})) = i$.

Proposition 4.1. *A $(d - 1)$ -dimensional simplicial poset P is graphical if and only if P is a d -colored normal pseudomanifold.*

Proof. We already proved the “only if” part. We prove that if P is a d -colored normal pseudomanifold then there exists an admissible d -colored multi-graph $\Lambda = (G, \gamma)$ such that P is isomorphic to P_Λ .

Let $G = (V, E, \phi)$ be the multi-graph such that $V = P_d$, $E = P_{d-1}$ and, for any $\sigma \in E$, $\phi(\sigma)$ is the set of the elements in P_d which cover σ (this is well-defined by the condition (ii) of pseudomanifolds). Let $\psi : P_1 \rightarrow [d]$ be a coloring map of P . Define a d -colored multi-graph $\Lambda = (G, \gamma)$ by setting $\gamma(\sigma)$ to be the integer $i \in [d]$ such that $i \notin \{\psi(v) : v \in V(\sigma)\}$, where $\sigma \in E = P_{d-1}$.

Since P is strongly connected, the graph G is connected. Also, each $G_{\{i\}}$ is a complete matching since for every vertex $\sigma \in V = P_d$ and $i \in [d]$, there is the unique $\tau \in P_{d-1}$ with $\sigma \geq \tau$ and $\{\psi(v) : v \in V(\tau)\} = [d] \setminus \{i\}$. Hence Λ is admissible.

We claim that P is isomorphic to P_Λ as posets. Let $\sigma \in P$ and $S = \{\psi(v) : v \in V(\sigma)\}$. Choose a facet $\sigma' \in P_d$ with $\sigma' \geq \sigma$. Then there exists the unique connected component H of $\Lambda_{[d] \setminus S}$ which contains the vertex $\sigma' \in V = P_d$ of Λ . We define

$$\Phi(\sigma) = (H, [d] \setminus S).$$

This $\Phi(\sigma)$ do not depend on the choice of $\sigma' \in P_d$ with $\sigma' \geq \sigma$. Indeed, for $\sigma'' \in P_d$ with $\sigma'' \geq \sigma$, since $P_{\geq \sigma}$ is strongly connected there exist edges $\tau_1, \dots, \tau_m \in (P_{\geq \sigma})_{d-1-\text{rank } \sigma} \subset E$ which connect σ'' and σ' . Since $\tau_i \in P_{\geq \sigma}$, $\gamma(\tau_i) \notin S$ for all i . Hence τ_1, \dots, τ_m are edges in $\Lambda_{[d] \setminus S}$ and therefore σ' and σ'' are in the same connected component of $\Lambda_{[d] \setminus S}$.

We claim that Φ is an order-preserving bijection. It is clear that $\Phi : P \rightarrow P_\Lambda$ is order preserving. It remains to prove that Φ is a bijection.

Let $(H, S) \in P_\Lambda$. Choose a facet τ which is a vertex of H . Since P is d -colored, there is $\sigma \in P$ with $\sigma \leq \tau$ and with $\{\psi(v) : v \in V(\sigma)\} = [d] \setminus S$. By the definition of Φ , we have $\Phi(\sigma) = (H, S)$. Hence Φ is surjective.

Let $\sigma, \tau \in P$ such that $\Phi(\sigma) = \Phi(\tau) = (H, [d] \setminus S)$. We prove $\sigma = \tau$. Since P is simplicial, for any facet $\rho \in P_d$ and $T \subset [d]$, there exists the unique element $\rho' \leq \rho$ in P with $\{\psi(v) : v \in V(\rho')\} = T$. This fact shows that, for any facet $\rho \geq \sigma$ and an edge $e \in P_{d-1}$ in $\Lambda_{[d] \setminus S}$ with $\rho \in \phi(e)$, one has $e \geq \sigma$. Hence the vertices of H are the facets of $P_{\geq \sigma}$, and therefore there is a facet $\rho \in P_{d-1}$ satisfying $\rho \geq \sigma$ and $\rho \geq \tau$. Since $\Phi(\sigma) = \Phi(\tau)$, $\{\psi(v) : v \in V(\sigma)\} = \{\psi(v) : v \in V(\tau)\}$. Then σ and τ have a common upper bound ρ and has the same color. This fact implies $\sigma = \tau$. Hence Φ is injective. \square

Since homology manifolds are normal pseudomanifolds, we have the following result.

Corollary 4.2. *A $(d-1)$ -dimensional simplicial cell homology manifold is graphical if and only if it is d -colored.*

Finally, we explain what crystallizations are. Given a $(d-1)$ -dimensional manifold M , an admissible d -colored multi-graph Λ is called a *crystallization of M* if (the barycentric subdivision of) the simplicial cell complex $\Gamma(P_\Lambda)$ is PL-homeomorphic to M and, for every $i \in [d]$, $\Lambda_{[d] \setminus \{i\}}$ is connected. Since the latter condition is equivalent to $f_0(P_\Lambda) = d$, which is also equivalent to $h_1''(P_\Lambda) = h_1(P_\Lambda) = 0$, and since any $(d-1)$ -dimensional simplicial poset with d vertices are d -colored, considering crystallizations of M is almost equivalent to considering simplicial cell decompositions of M with d vertices.

5. CONSTRUCTION OF A GRAPH THAT PRESENTS $S^n \times S^m$.

Let Λ_1 and Λ_2 be admissible colored multi-graphs such that P_{Λ_1} and P_{Λ_2} are simplicial cell decompositions of manifolds M_1 and M_2 . Gagliardi and Grasselli [GG] gave a way to construct an admissible colored multi-graph Λ that gives a simplicial cell decomposition of $M_1 \times M_2$ from Λ_1 and Λ_2 . Cristofori [Cr] studied their construction for the products of spheres. In this section, we recall this construction of a graph that presents $S^n \times S^m$.

We first recall a standard triangulation of the product of simplexes. We just list the known facts and do not give a proof. See [GG] for the details.

Let σ be an n -dimensional (geometric) simplex in \mathbb{R}^n with vertices v_0, v_1, \dots, v_n and τ an m -dimensional (geometric) simplex in \mathbb{R}^m with vertices u_0, u_1, \dots, u_m . The product of σ and τ is the polytope

$$\sigma \times \tau = \{(x, y) : x \in \sigma \text{ and } y \in \tau\} \subset \mathbb{R}^{n+m}.$$

Then the set of the vertices of $\sigma \times \tau$ is

$$W = \{(v_i, u_j) : 0 \leq i \leq n, 0 \leq j \leq m\}.$$

To simplify the notation, we write $w_{ij} = (v_i, u_j)$.

Let $\mathbf{pd}(\sigma \times \tau)$ be the abstract simplicial complex on the vertex set W (that is, a family of subsets of W closed under inclusion) defined by

$$\mathbf{pd}(\sigma \times \tau) = \left\{ \{w_{i_0 j_0}, w_{i_1 j_1}, \dots, w_{i_p j_p}\} : i_0 \leq i_1 \leq \dots \leq i_p, j_0 \leq j_1 \leq \dots \leq j_p \right\}.$$

In particular, the facets of $\mathbf{pd}(\sigma \times \tau)$ are the sets of the form

$$F(i_0, i_1, \dots, i_{n+m}; j_0, j_1, \dots, j_{n+m}) = \{w_{i_0 j_0}, w_{i_1 j_1}, \dots, w_{i_{n+m} j_{n+m}}\},$$

where $(i_0, j_0) = (0, 0)$ and where (i_{l+1}, j_{l+1}) is either $(i_l + 1, j_l)$ or $(i_l, j_l + 1)$ for all l . Then, by taking a convex hull of each face, $\mathbf{pd}(\sigma \times \tau)$ gives a triangulation of $\sigma \times \tau$. Also, $\mathbf{pd}(\sigma \times \tau)$ satisfies the following conditions.

- $\mathbf{pd}(\sigma \times \tau)$ is $(n + m + 1)$ -colored by the coloring map

$$(1) \quad \begin{array}{ll} \psi : W & \rightarrow [n + m + 1], \\ w_{ij} & \rightarrow \psi(w_{ij}) = i + j + 1. \end{array}$$

- The boundary of $\mathbf{pd}(\sigma \times \tau)$ is generated by the following faces

$$(2) F(i_0, \dots, i_{n+m}; j_0, \dots, j_{n+m}) \setminus \{w_{i_l j_l}\} \text{ such that } \{i_0, \dots, i_{n+m}\} \setminus \{i_l\} \neq \{0, \dots, n\},$$

$$(3) F(i_0, \dots, i_{n+m}; j_0, \dots, j_{n+m}) \setminus \{w_{i_l j_l}\} \text{ such that } \{j_0, \dots, j_{n+m}\} \setminus \{j_l\} \neq \{0, \dots, m\}.$$

Moreover, the convex hull of a face (2) belongs to $\partial\sigma \times \tau$ and that of a face (3) belongs to $\sigma \times \partial\tau$.

Definition 5.1. For any n -subset $S \subset [n+m]$, we associate a facet $F(S)$ of $\mathbf{pd}(\sigma \times \tau)$ as follows: We define $(0, 0) = (i_0, j_0), (i_1, j_1), \dots, (i_{n+m}, j_{n+m}) = (n, m)$ by

$$(i_l, j_l) = \begin{cases} (i_{l-1}, j_{l-1}) + (1, 0), & \text{if } l \in S, \\ (i_{l-1}, j_{l-1}) + (0, 1), & \text{if } l \notin S, \end{cases}$$

for $l = 1, 2, \dots, n + m$ and let

$$F(S) = F(i_0, \dots, i_{n+m}; j_0, \dots, j_{n+m}).$$

Then $\{F(S) : S \subset [n + m], \#S = n\}$ is the set of facets of $\mathbf{pd}(\sigma \times \tau)$.

Now we consider the product of spheres. Let σ_1 and σ_2 be n -dimensional simplexes and τ_1 and τ_2 m -dimensional simplexes. Let

$$\begin{aligned} A &= \mathbf{pd}(\sigma_1 \times \tau_1), & B &= \mathbf{pd}(\sigma_2 \times \tau_1), \\ C &= \mathbf{pd}(\sigma_1 \times \tau_2), & D &= \mathbf{pd}(\sigma_2 \times \tau_2). \end{aligned}$$

Then we obtain a simplicial cell decomposition of $S^n \times S^m$ by identifying $(\partial\sigma_1) \times \tau_1$ in A and $(\partial\sigma_2) \times \tau_1$ in B , $(\partial\sigma_1) \times \tau_2$ in C and $(\partial\sigma_2) \times \tau_2$ in D , $\sigma_1 \times (\partial\tau_1)$ in A and $\sigma_1 \times (\partial\tau_2)$ in C , $\sigma_2 \times (\partial\tau_1)$ in B and $\sigma_2 \times (\partial\tau_2)$ in D . In particular, by identifying the same types of faces described in (2) and (3), we can construct such a simplicial cell decomposition in a unique way so that it is $(n + m + 1)$ -colored by the coloring map (1).

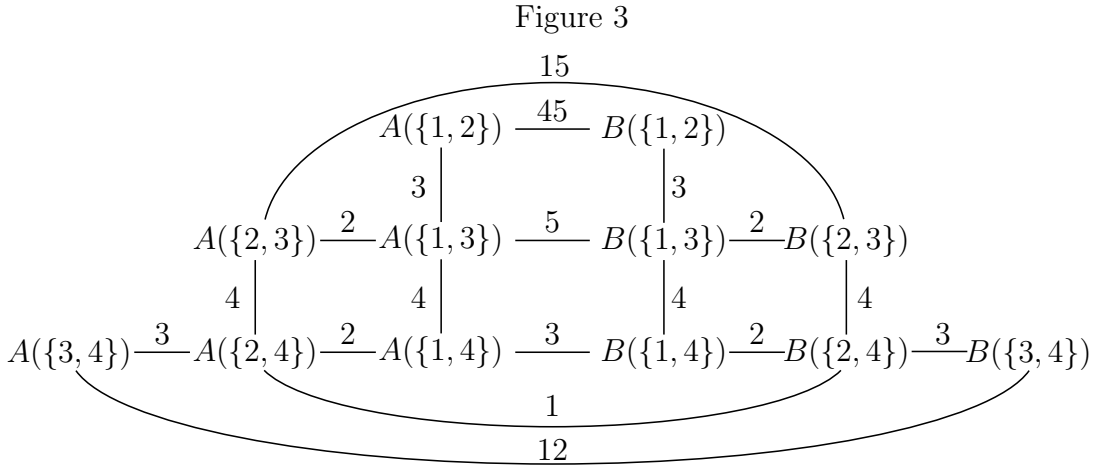
Let $P(n, m)$ be the simplicial cell decomposition of $S^n \times S^m$ obtained by the above construction. Since $P(n, m)$ is $(n + m + 1)$ -colored, it is graphical. Let $\Lambda(n, m)$ be the admissible $(n + m + 1)$ -colored graph with $P(n, m) \cong P_{\Lambda(n, m)}$.

For an n -subset $S \subset [n + m]$, let $A(S)$ be the facet of $\mathbf{pd}(\sigma_1 \times \tau_1)$ defined in the same way as in Definition 5.1. Also, we define $B(S)$, $C(S)$ and $D(S)$ similarly. We may consider that these $A(S)$, $B(S)$, $C(S)$ and $D(S)$ are the vertices of $\Lambda(n, m)$. By (2) and (3), any edge of $\Lambda(n, m)$ is one of the following edges:

- (E1) an edge of color $k \in [n + m + 1]$ whose vertices are $A(S)$ and $B(S)$ (or $C(S)$ and $D(S)$) such that $\{k - 1, k\} \cap S = \emptyset$;
- (E2) an edge of color $k \in [n + m + 1]$ whose vertices are $A(S)$ and $C(S)$ (or $B(S)$ and $D(S)$) such that $\{k - 1, k\} \subset S$;
- (E3) an edge of color $2 \leq k \leq n + m$ whose vertices are $A(S)$ and $A((S \setminus \{k\}) \cup \{k - 1\})$ such that $k \in S$ and $k - 1 \notin S$ (and the same type of edges for $B(-)$, $C(-)$ and $D(-)$);

where we consider $\{k, k - 1\} = \{1\}$ if $k = 1$ and $\{k, k - 1\} = \{n + m\}$ if $k = n + m + 1$.

Example 5.2. The following Figure 3 is a part of the graph $\Lambda(2, 2)$ (the whole graph can be found in [GG, p. 567]).



The numbers on edges are colors of edges. For example, there are two edges between $A(\{1, 2\})$ and $B(\{1, 2\})$ such that one edge has color 4 and the other edge has color 5. In Figure 3, we omit edges between A and C (and B and D), but they are edges between $A(S)$ and $C(S)$ whose colors are the colors which do not appear in the above figure. For example, there are two edges between $A(\{1, 2\})$ and $C(\{1, 2\})$ whose color is 1 or 2.

6. PROOF OF THEOREM 1.2

In this section, we prove the sufficiency of Theorem 1.2.

6.1. Cancellations of dipoles.

Let $\Lambda = (G, \gamma)$ with $G = (V, E, \phi)$ be an admissible d -colored multi-graph. Let $x, y \in V$ be vertices of G . We define a new admissible d -colored multi-graph $\Lambda' = \text{del}_{\{x,y\}}\Lambda = (G', \gamma')$ with $G' = (V', E', \phi')$ as follows: Let $C = \{\gamma(e) : e \in E, \phi(e) = \{x, y\}\}$. Thus C is the set of colors of edges between x and y . Then, for each $i \in [d] \setminus C$, there is the unique pair (a_i, b_i) of vertices in G such that there are edges e and e' in E of color i with $\phi(e) = \{a_i, x\}$ and with $\phi(e') = \{y, b_i\}$. Then we define the graph $G' = (V', E', \phi')$ by

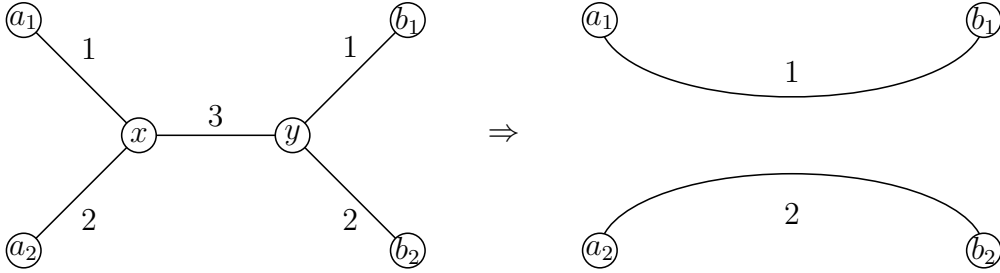
$$\begin{aligned} V' &= V \setminus \{x, y\} \\ E' &= \{e \in E : \phi(e) \cap \{x, y\} = \emptyset\} \cup \{\sigma_i : i \in [d] \setminus C\}, \\ \phi'(e) &= \begin{cases} \phi(e), & \text{if } e \in E, \\ \{a_i, b_i\}, & \text{if } e = \sigma_i \text{ for some } i \in [d] \setminus C. \end{cases} \end{aligned}$$

Also, we define the coloring γ' of G' by

$$\gamma'(e) = \begin{cases} \gamma(e), & \text{if } e \in E, \\ i, & \text{if } e = \sigma_i \text{ for some } i \in [d] \setminus C. \end{cases}$$

Thus Λ' is the graph obtained from Λ by removing the vertices x and y and by adding, for each color $i \in [d] \setminus C$, a new edge σ_i of color i between the vertices a_i and b_i (see Figure 4). By the construction, it is easy to see that Λ' is an admissible d -colored multi-graph. We call the operation $\Lambda \rightarrow \Lambda'$ a *cancellation (of x and y)*.

Figure 4



We say that two vertices u and v of a multi-graph G are *connected on G* if there exists a sequence of edges, called a *path*, e_1, e_2, \dots, e_l of G such that $u \in \phi(e_1) \setminus \phi(e_2)$, $v \in \phi(e_l) \setminus \phi(e_{l-1})$ and $\phi(e_i) \cap \phi(e_{i+1}) \neq \emptyset$ for $i = 1, 2, \dots, l-1$. If two vertices u and v are not connected on G , then we say that they are *disconnected on G* .

Let $C = \{\gamma(e) : e \in E, \phi(e) = \{x, y\}\}$. The cancelling $\Lambda \rightarrow \Lambda' = \text{del}_{\{x,y\}}\Lambda$ is said to be a *cancellation of a dipole (of type C)* if $C \neq \emptyset$ and the vertices x and y are disconnected on $\Lambda_{[d] \setminus C}$. The following result is known in crystallization theory ([FG, Lemma 1]).

Lemma 6.1 (Ferri-Gagliardi). *Let Λ be an admissible d -colored multi-graph such that (the barycentric subdivision of) the simplicial cell complex $\Gamma(P_\Lambda)$ is a PL-manifold. If $\Lambda \rightarrow \Lambda'$ is a cancellation of a dipole then $\Gamma(P_{\Lambda'})$ is PL-homeomorphic to $\Gamma(P_\Lambda)$.*

6.2. Construction.

Recall that by Corollary 3.2, to prove Theorem 1.2, it is enough to construct a simplicial cell decomposition P of $S^n \times S^m$ with $h''(P) = (1, 0, \dots, 0, 1)$.

Lemma 6.2. *Let Λ be an admissible $(n + m + 1)$ -colored multi-graph such that P_Λ is a simplicial cell decomposition of $S^n \times S^m$. If the number of vertices of Λ is equal to $2 + 2\binom{n+m}{n}$ then $h''(P_\Lambda) = (1, 0, \dots, 0, 1)$.*

Proof. Observe $\beta_i(P_\Lambda) = 0$ if $i \neq n, m, n + m$, $\beta_n(P_\Lambda) = \beta_m(P_\Lambda) = 1$ if $n \neq m$ and $\beta_n(P_\Lambda) = 2$ if $n = m$. In the proof of Corollary 2.3, we show

$$f_{n+m}(P_\Lambda) = \sum_{i=0}^{n+m+1} h_i(P_\Lambda) = \sum_{i=0}^{n+m+1} h''_i(P_\Lambda) + \binom{n+m}{n} + \binom{n+m}{m}.$$

Since the number of vertices of Λ is equal to $f_{n+m}(P_\Lambda)$, $\sum_{i=0}^{n+m+1} h''_i(P_\Lambda) = 2$. Then the statement follows since $h''(P_\Lambda)$ is non-negative. \square

By the above lemma, to prove Theorem 1.2, what we must prove is the existence of a crystallization of $S^n \times S^m$ with $2 + 2\binom{n+m}{n}$ vertices. Unfortunately, the graph $\Lambda(n, m)$ given in the previous section has $4\binom{n+m}{n}$ vertices. We make a desired crystallization by repeating cancellations of dipoles to $\Lambda(n, m)$.

From now on we fix positive integers n and m . For integers i, j , we write $[i, j] = \{i, i + 1, \dots, j\}$ where $[i, j] = \emptyset$ if $j < i$. For $j = 1, 2, \dots, n$, let

$$X_j = \{S \subset [j + 1, n + m] : \#S = n + 1 - j\}$$

and

$$X = \bigcup_{j=1}^n X_j.$$

Remark 6.3. There is a natural bijection $\Phi : X \rightarrow \{S \subset [n + m] : \#S = n\} \setminus \{[n]\}$ defined by $\Phi(S) = [n - \#S] \cup S$. In particular $\#X = \binom{n+m}{n} - 1$.

In the rest of this section, for a set $\{i_1, \dots, i_k\}$ of integers, we always assume $i_1 < \dots < i_k$.

Definition 6.4. Recall that the vertices of $\Lambda(n, m)$ are denoted by $A(S), B(S), C(S)$ and $D(S)$ where $S \subset [n + m]$ and $\#S = n$. For $S = \{i_1, i_2, \dots, i_{n+1-j}\} \in X_j$ we define the pair $\mathcal{D}_j(S)$ of vertices of $\Lambda(n, m)$ as follows: Let $S' = S \setminus \{i_1\}$. If j is odd then

$$\mathcal{D}_j(S) = \begin{cases} \{A([j - 1] \cup S), A([j] \cup S')\}, & \text{if } i_1 = j + 1, \\ \{A([j - 1] \cup S), B([i_1 - j - 1, i_1 - 2] \cup S')\}, & \text{if } i_1 > j + 1, \end{cases}$$

and if j is even then

$$\mathcal{D}_j(S) = \{B([i_1 - j, i_1 - 2] \cup S), B([i_1 - j, i_1 - 1] \cup S')\}.$$

Since $j = n + 1 - \#S$, we simply write $\mathcal{D}_j(S) = \mathcal{D}(S)$.

Lemma 6.5.

- (1) *Suppose that n is even. Then*
 - (a) *for any $F \subset [n + m]$ with $\#F = n$ and $F \neq [n]$, there is the unique $S \in X$ such that $A(F) \in \mathcal{D}(S)$.*
 - (b) *for any $F \subset [n + m]$ with $\#F = n$ and $F \neq [m + 1, m + n]$, there is the unique $S \in X$ such that $B(F) \in \mathcal{D}(S)$.*
- (2) *Suppose that n is odd. Then*
 - (a) *for any $F \subset [n + m]$ with $\#F = n$, there is the unique $S \in X$ such that $A(F) \in \mathcal{D}(S)$.*
 - (b) *for any $F \subset [n + m]$ with $\#F = n$, $F \neq [m, m + n - 1]$ and $F \neq [m + 1, m + n]$, there is the unique $S \in X$ such that $B(F) \in \mathcal{D}(S)$.*

Proof. The uniqueness follows from the existence. Indeed, the number of vertices $A(F)$ and $B(F)$ appearing in (a) and (b) is $2^{\binom{n+m}{n}} - 2$. On the other hand, since $\#X = \binom{n+m}{n} - 1$, the number of vertices which appears in $\mathcal{D}(S)$ for some $S \in X$ is at most $2^{\binom{n+m}{n}} - 2$. Thus if it is $2^{\binom{n+m}{n}} - 2$ then each vertex cannot appear twice. We will prove the existence. Let $F = \{i_1, i_2, \dots, i_n\} \subset [n + m]$.

We first consider $A(F)$. Let k be the smallest positive integer which is not in F . Thus $F \supset [k - 1]$ and $k \notin F$. Let $F = [k - 1] \cup F'$ where $\min F' > k$. If k is odd and $F' \neq \emptyset$ then $A(F)$ is the first vertex of $\mathcal{D}_k(F')$. If k is even then $A(F)$ is the second vertex of $\mathcal{D}_{k-1}(\{k\} \cup F')$. These prove (1)-(a) and (2)-(a).

Next, we consider $B(F)$. Let l be the smallest integer such that $i_{l+1} \neq i_l + 1$. Thus $F \supset \{i_1, i_1 + 1, \dots, i_1 + l - 1\} = \{i_1, \dots, i_l\}$ and $i_1 + l \notin F$. Let $F = [i_1, i_1 + l - 1] \cup F'$ with $\min F' > i_1 + l$.

Suppose $F' \neq \emptyset$. Then $l < n$. If l is even then $B(F)$ is the second vertex of $\mathcal{D}_l(\{i_1 + l\} \cup F')$. If l is odd and $i_{l+1} > i_1 + l + 1$ then $B(F)$ is the second vertex of $\mathcal{D}_l(\{i_1 + l + 1\} \cup F')$. If l is odd and $i_{l+1} = i_1 + l + 1$ then $B(F)$ is the first vertex of $\mathcal{D}_{l+1}(F')$.

Suppose $F' = \emptyset$, that is, $F = [i_1, i_1 + n - 1]$ with $1 \leq i_1 \leq m + 1$. If n is even and $i_1 \neq m + 1$ then $B(F)$ is the second vertex of $\mathcal{D}_n(\{i_1 + n\})$. If n is odd and $i_1 < m$ then $B(F)$ is the second vertex of $\mathcal{D}_n(\{i_1 + n + 1\})$. \square

We define the total order \succ on $\{\mathcal{D}(S) : S \in X\}$ by $\mathcal{D}_j(S) \succ \mathcal{D}_{j'}(S')$ if (i) $j < j'$ or (ii) $j = j'$ and $S >_{\text{rev}} S'$, where $>_{\text{rev}}$ is the reverse lexicographic order. Thus $S >_{\text{rev}} S'$ if the largest integer in the symmetric difference $(S \setminus S') \cup (S' \setminus S)$ is contained in S' . From the proof of Lemma 6.5, we obtain the next corollary.

Corollary 6.6. *Let $F = [i_1, i_1 + k - 1] \cup F' \subset [n + m]$ with $\min F' > i_1 + k$ and with $\#F = n$. If $B(F)$ appears in $\mathcal{D}_l(T)$ then $l \geq k$ and $\mathcal{D}_l(T) \preceq \mathcal{D}_k(\{i_1 + k\} \cup F')$.*

Let

$$\{\mathcal{D}(S) : S \in X\} = \left\{ \mathcal{D}_1 \succ \mathcal{D}_2 \succ \dots \succ \mathcal{D}_{\binom{n+m}{n}-1} \right\}.$$

We define the admissible $(n + m + 1)$ -colored multi-graph $\Lambda^{(k)}$ recursively by $\Lambda^{(1)} = \Lambda(n, m)$ and

$$\Lambda^{(k+1)} = \text{del}_{\mathcal{D}_k} \Lambda^{(k)}$$

for $k = 1, 2, \dots, \binom{n+m}{n} - 1$ (these graphs are well defined by Lemma 6.5). If $\mathcal{D}_k = \mathcal{D}_j(S)$, we write

$$\Lambda(S) = \Lambda_j(S) = \Lambda^{(k)} \text{ and } \Lambda'(S) = \Lambda^{(k+1)} = \text{del}_{\mathcal{D}(S)}\Lambda^{(k)}.$$

Clearly the number of vertices of $\Lambda^{(k)}$ is $4\binom{n+m}{n} - 2(k-1)$. Then by Lemma 6.1 the next statement completes the proof of Theorem 1.2.

Lemma 6.7. *For $k = 1, 2, \dots, \binom{n+m}{n} - 1$, the cancelling $\Lambda^{(k)} \rightarrow \Lambda^{(k+1)}$ is a cancelling of a dipole.*

We prove the above lemma in subsections 6.3 and 6.4 in a series of lemmas.

Example 6.8. Suppose $n = m = 2$. Then

$$\begin{aligned} \mathcal{D}_1 &= \mathcal{D}_1(\{2, 3\}) = \{A(\{2, 3\}), A(\{1, 3\})\}, \\ \mathcal{D}_2 &= \mathcal{D}_1(\{2, 4\}) = \{A(\{2, 4\}), A(\{1, 4\})\}, \\ \mathcal{D}_3 &= \mathcal{D}_1(\{3, 4\}) = \{A(\{3, 4\}), B(\{1, 4\})\}, \\ \mathcal{D}_4 &= \mathcal{D}_2(\{3\}) = \{B(\{1, 3\}), B(\{1, 2\})\}, \\ \mathcal{D}_5 &= \mathcal{D}_2(\{4\}) = \{B(\{2, 4\}), B(\{2, 3\})\}. \end{aligned}$$

6.3. Proof of Lemma 6.7: disconnectivity of $\Lambda(S)$.

Let $\Lambda^{(k)} = \Lambda_j(S)$ and let

$$\text{color}(S) = \{l \in S : l - 1 \notin S\}.$$

The next lemma gives a part of a proof of Lemma 6.7.

Lemma 6.9. *With the same notation as above, two vertices in $\mathcal{D}(S)$ are disconnected on $(\Lambda(S))_{[n+m+1] \setminus \text{color}(S)}$.*

We need the following technical but obvious lemma.

Lemma 6.10. *Let Λ be an admissible d -colored multi-graph on the vertex set V and $T \subset [d]$. Let $X \cup (V \setminus X)$ be a partition of V such that, for all $x \in X$ and $y \in V \setminus X$, x and y are disconnected on Λ_T . If u and v are vertices in X , then, for all $x \in X \setminus \{u, v\}$ and $y \in V \setminus X$, x and y are disconnected on $(\text{del}_{\{u, v\}}\Lambda)_T$.*

Proof of Lemma 6.9. Let

$$S = \{i_1, i_2, \dots, i_{n+1-j}\}.$$

For $\{p_1, p_2, \dots, p_{n+1-j}\} \subset [n+m]$, we write $\{p_1, p_2, \dots, p_{n+1-j}\} \supseteq S$ if $p_l \geq i_l$ for all l . Let

$$\begin{aligned} X &= \{A(\{p_1, \dots, p_n\}) : \{p_j, \dots, p_n\} \supseteq S\} \\ &\quad \cup \{B(\{p_1, \dots, p_n\}) : \{p_j, \dots, p_n\} \supseteq S\} \\ &\quad \cup \{C(\{p_1, \dots, p_n\}) : \{p_j, \dots, p_n\} \supseteq S\} \\ &\quad \cup \{D(\{p_1, \dots, p_n\}) : \{p_j, \dots, p_n\} \supseteq S\}. \end{aligned}$$

Let V be the set of vertices of $\Lambda(n, m)$. We claim that the partition $X \cup (V \setminus X)$ and the set of colors $T = [n + m + 1] \setminus \text{color}(S)$ satisfy the assumption of Lemma 6.10 for $\Lambda(n, m)$.

We use the description (E1), (E2) and (E3) of edges of $\Lambda(n, m)$. By the description, if e_1, \dots, e_ℓ is a path on $\Lambda(n, m)$ from $x \in X$ to $y \in V \setminus X$, then there is an edge e_q whose vertices are of the form

$$x' = \diamond(\{p_1, \dots, p_{j-1}, i_1, i_2, \dots, i_{n+1-j}\})$$

and

$$y' = \diamond(\{p_1, \dots, p_{j-1}, i_1, \dots, i_{\ell-1}, i_\ell - 1, i_{\ell+1}, \dots, i_{n+1-j}\})$$

with $i_\ell - 1 \notin S$, where \diamond is A, B, C or D . Also such an edge e_q has color i_ℓ by (E3). Since $i_\ell \in S$, we have $e_q \notin \Lambda(n, m)_T$. Thus e_1, \dots, e_ℓ is not a path on $\Lambda(n, m)_T$. This fact shows that $x \in X$ and $y \in V \setminus S$ are disconnected on $\Lambda(n, m)_T$.

Now by lemma 6.10 what we must prove is that, for any integer $k' < k$, $\mathcal{D}_{k'}$ is contained in either X or $V \setminus X$. Let $\mathcal{D}_{k'} = \mathcal{D}_{j'}(T) = \{\square(\{p_1, \dots, p_n\}), \square'(\{q_1, \dots, q_n\})\}$ where \square and \square' are either A or B . Since $\mathcal{D}_{k'} \succ \mathcal{D}_k$, we have $j' < j$ or $j' = j$ and $T \succ_{\text{rev}} S$. If $j' < j$ then $\{p_j, \dots, p_n\} = \{q_j, \dots, q_n\}$ by Definition 6.4, which guarantees $\mathcal{D}_{k'} \subset X$ or $\mathcal{D}_{k'} \subset V \setminus X$. Suppose $j' = j$. Then $T \succ_{\text{rev}} S$. By Definition 6.4, $\{p_j, \dots, p_n\} \geq_{\text{rev}} T$ and $\{q_j, \dots, q_n\} \geq_{\text{rev}} T$. Hence we have $\mathcal{D}_{k'} \subset V \setminus X$, as desired. \square

6.4. Proof of Lemma 6.7: existence of edges with desired colors.

We say that two vertices u and v in $\Lambda^{(k)}$ are *directly connected on $\Lambda^{(k)}$ by colors $H \subset [n + m + 1]$* if, for each $i \in H$, there is an edge e of $\Lambda^{(k)}$ whose vertices are u and v and whose color is i . The next lemma and Lemma 6.9 prove Lemma 6.7.

Lemma 6.11. *For every $S \in X$, the vertices in $\mathcal{D}(S)$ are directly connected on $\Lambda(S)$ by colors $\text{color}(S)$.*

We need two technical lemmas.

Lemma 6.12. *Let $S = \{i_1, \dots, i_{n+1-j}\} \in X_j$ and $S' = S \setminus \{i_1\}$.*

- (i) *Suppose j is odd. Then $B([i_1 - j, i_1 - 1] \cup S')$ and $B([i_1 - j + 1, i_1] \cup S')$ are vertices of $\Lambda(S)$. Moreover, if $i_1 + 1 \notin S$ and $i_1 + 1 \leq n + m$, then $A([j - 1] \cup \{i_1 + 1\} \cup S')$ is a vertex of $\Lambda(S)$.*
- (ii) *Suppose j is even. Then $A([j] \cup S')$ and $B([i_1 - j + 1, i_1] \cup S')$ are vertices of $\Lambda(S)$.*

Proof. By Lemma 6.5, to prove that $A(F)$ (or $B(F)$) is a vertex of $\Lambda(S)$, what we must prove is that it appears in some $\mathcal{D}(T)$ with $\mathcal{D}(T) \preceq \mathcal{D}(S)$ or it does not appear in any $\mathcal{D}(T)$.

(i) If $B([i_1 - j, i_1 - 1] \cup S')$ or $B([i_1 - j + 1, i_1] \cup S')$ appears in some $\mathcal{D}(T)$, then Corollary 6.6 says $\mathcal{D}(T) \preceq \mathcal{D}_j(\{i_1\} \cup S') = \mathcal{D}_j(S)$. If $i_1 + 1 \notin S$ then $A([j - 1] \cup \{i_1 + 1\} \cup S')$ appears in $\mathcal{D}_j(\{i_1 + 1\} \cup S') \prec \mathcal{D}_j(S)$ by Definition 6.4.

(ii) If $A([j] \cup S')$ appears in some $\mathcal{D}_l(T)$, then by Definition 6.4 $A([j] \cup S') \in \mathcal{D}_{j+1}(S') \prec \mathcal{D}_j(S)$. Also, if $B([i_1 - j + 1, i_1] \cup S')$ appears in some $\mathcal{D}(T)$ then Corollary 6.6 says $\mathcal{D}(T) \preceq \mathcal{D}(\{i_1 + 1\} \cup S') \prec \mathcal{D}_j(S)$. \square

Lemma 6.13. *Let $S = \{i_1, \dots, i_{n+1-j}\} \in X_j$ and $S' = S \setminus \{i_1\}$.*

(i) *If j is odd then $B([i_1 - j, i_1 - 1] \cup S')$ and $B([i_1 - j + 1, i_1] \cup S')$ are directly connected on $\Lambda'(S)$ by colors*

$$H = \{r \in [i_1 + 2, n + m + 1] : \{r - 1, r\} \cap S' = \emptyset\}.$$

(ii) *If j is odd and $i_1 + 1 \notin S$, where $i_1 + 1 \leq n + m$, then $A([j - 1] \cup \{i_1 + 1\} \cup S')$ and $B([i_1 - j, i_1 - 1] \cup S')$ are directly connected on $\Lambda'(S)$ by color $i_1 + 1$.*

(iii) *If j is even then $A([j] \cup S')$ and $B([i_1 - j + 1, i_1] \cup S')$ are directly connected on $\Lambda'(S)$ by colors*

$$H' = \{r \in [i_2 + 2, n + m + 1] : \{r - 1, r\} \cap S' = \emptyset\}.$$

Proof. We prove the statement by induction on the total order \succ on $\{\mathcal{D}(T) : T \in X\}$. Note that all vertices appearing in the statements are vertices of $\Lambda(S)$ by Lemma 6.12.

We often use the following fact: if two vertices are directly connected on $\Lambda^{(k)}$ by colors C and if they are still vertices of $\Lambda^{(l)}$ with $l > k$ then they are directly connected on $\Lambda^{(l)}$ by colors C .

Case 1. Suppose j is odd and $i_1 = j + 1$. Then

$$\mathcal{D}(S) = \{A([j - 1] \cup S), A([j] \cup S')\}.$$

By the description (E1) of edges in $\Lambda(n, m)$,

(6.1) $A([j] \cup S')$ and $B([i_1 - j, i_1 - 1] \cup S') = B([j] \cup S')$ are directly connected on $\Lambda(n, m)$ by colors H (and $j + 2$ if $j + 2 \notin S$).

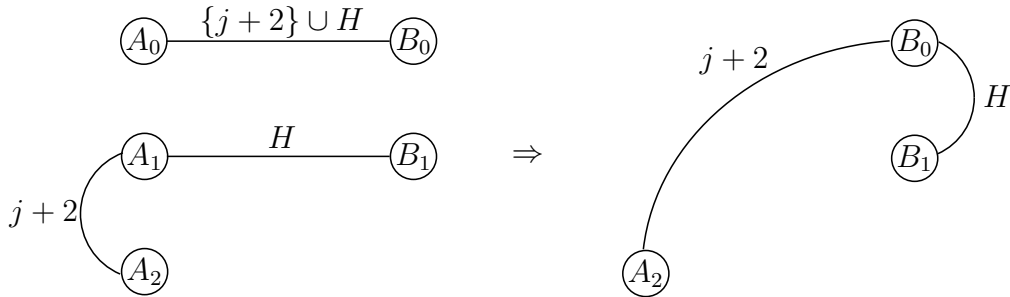
By applying the induction hypothesis to $\tilde{S} = \{j\} \cup S \in X_{j-1}$,

(6.2) $A([j - 1] \cup S)$ and $B([2, j] \cup S)$ are directly connected on $\Lambda(S)$ by colors H , where the above statement follows from (E1) when $j = 1$. Also, by (E3),

(6.3) if $i_1 + 1 \notin S$ then $A([j - 1] \cup S)$ and $A([j - 1] \cup \{i_1 + 1\} \cup S')$ are directly connected on $\Lambda(n, m)$ by color $i_1 + 1 = j + 2$.

Then it is straightforward that (i) and (ii) follow from (6.1), (6.2), (6.3) and the definition of cancellations. See Figure 5.

Figure 5



$$\left(\begin{array}{l} A_0 = A([j] \cup S'), A_1 = A([j - 1] \cup S), A_2 = A([j - 1] \cup \{i_1 + 1\} \cup S'), \\ B_0 = B([j] \cup S'), B_1 = B([2, j] \cup S), \mathcal{D}(S) = (A_0, A_1). \end{array} \right)$$

Note that Case 1 contains a proof of Lemma 6.13 for $\mathcal{D}_1 = \mathcal{D}_1([2, n + 1])$ which is the starting point of the induction.

Case 2. Suppose j is odd and $i_1 > j + 1$. Then

$$\mathcal{D}(S) = \{A([j-1] \cup S), B([i_1 - j - 1, i_1 - 2] \cup S')\}.$$

By applying the induction hypothesis to $\hat{S} = \{i_1 - 1\} \cup S' \in X_j$,

$$(6.4) \quad B([i_1 - j - 1, i_1 - 2] \cup S') \text{ and } B([i_1 - j, i_1 - 1] \cup S') \text{ are directly connected on } \Lambda(S) \text{ by colors } H \text{ (and } i_1 + 1 \text{ if } i_1 + 1 \notin S).$$

By applying the induction hypothesis to $\tilde{S} = \{i_1 - 1\} \cup S \in X_{j-1}$,

$$(6.5) \quad A([j-1] \cup S) \text{ and } B([i_1 - j + 1, i_1 - 1] \cup S) = B([i_1 - j + 1, i_1] \cup S') \text{ are directly connected on } \Lambda(S) \text{ by colors } H,$$

where the above statement follows from (E1) when $j = 1$. Also, by (E3),

$$(6.6) \quad \text{if } i_1 + 1 \notin S \text{ then } A([j-1] \cup S) \text{ and } A([j-1] \cup \{i_1 + 1\} \cup S') \text{ are directly connected on } \Lambda(n, m) \text{ by color } i_1 + 1.$$

Then it is straightforward that statement (i) and (ii) follow from (6.4), (6.5), (6.6) and the definition of cancellations.

Case 3. Suppose j is even. Then

$$\mathcal{D}(S) = \{B([i_1 - j, i_1 - 2] \cup S), B([i_1 - j, i_1 - 1] \cup S')\}.$$

We claim

$$(6.7) \quad A([j] \cup S') \text{ and } B([i_1 - j, i_1 - 1] \cup S') \text{ are directly connected on } \Lambda(S) \text{ by colors } H'.$$

If $i_1 = j + 1$ then (6.7) follows from (E1). If $i_1 > j + 1$ then apply the induction hypothesis to $\hat{S} = \{i_1 - 1\} \cup S' \in X_j$. Hence (6.7) holds.

Also, by applying the induction hypothesis to $\tilde{S} = \{i_1 - 1\} \cup S \in X_{j-1}$,

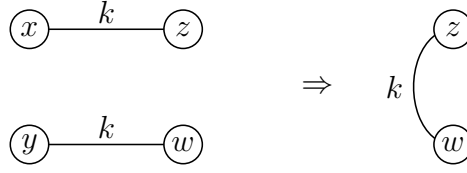
$$(6.8) \quad B([i_1 - j, i_1 - 2] \cup S) \text{ and } B([i_1 - j + 1, i_1 - 1] \cup S) = B([i_1 - j + 1, i_1] \cup S') \text{ are directly connected on } \Lambda(S) \text{ by colors } H'.$$

It is straightforward that statement (iii) follows from (6.7), (6.8) and the definition of cancellations. \square

Proof of Lemma 6.11. Let $S = \{i_1, \dots, i_{n+1-j}\}$. We first prove that the vertices in $\mathcal{D}(S)$ are directly connected on $\Lambda(S)$ by color i_1 . If j is even, then this is obvious since by (E3) they are directly connected on $\Lambda(n, m)$ by color i_1 . Suppose j is odd. If $i_1 = j + 1$ then by (E3) they are directly connected on $\Lambda(n, m)$ by color i_1 . If $i_1 > j + 1$ then the claim follows by applying Lemma 6.13(ii) to $\{i_1 - 1\} \cup S'$.

It remains to prove that, for any $k \in \text{color}(S) \setminus \{i_1\}$, vertices in $\mathcal{D}(S)$ are directly connected on $\Lambda(S)$ by color k . Let $k \in \text{color}(S) \setminus \{i_1\}$ and $T = (S \setminus \{k\}) \cup \{k-1\}$. Let $\mathcal{D}(T) = \{x, y\}$ and $\mathcal{D}(S) = \{z, w\}$. Since $\mathcal{D}(T) \succ \mathcal{D}(S)$, x, y, z, w are vertices of $\Lambda(T)$. By (E1), x and z are directly connected on $\Lambda(T)$ by color k . Similarly y and w are directly connected on $\Lambda(T)$ by color k . These facts say that z and w are directly connected on $\Lambda'(T) = \text{del}_{\mathcal{D}(T)}\Lambda(T)$ by color k (see Figure 6), and therefore they are directly connected on $\Lambda(S)$ by color k . \square

Figure 6



7. REAL PROJECTIVE SPACES AND ODD DIMENSIONAL MANIFOLDS

In this section, we characterize face vectors of a few more classes of simplicial posets by using Theorems 1.2 and 2.4.

7.1. Simplicial cell decompositions of $\mathbb{R}P^n$.

For integers $1 \leq i < n$, let $r(n, i) = \binom{n}{i}$ if i is even and $r(n, i) = 0$ if i is odd, and let $r(n, n) = -1$ if n is odd and $r(n, n) = 0$ if n is even. The next result was proved by Masuda (unpublished) except for the necessity of (3).

Theorem 7.1. *A vector $h = (h_0, h_1, \dots, h_n) \in \mathbb{Z}^{n+1}$ is the h -vector of a simplicial cell decomposition of $\mathbb{R}P^{n-1}$ if and only if it satisfies the following conditions:*

- (1) $h_0 = h_n - r(n, n) = 1$ and $h_i - r(n, i) = h_{n-i} - r(n, n-i)$ for $i = 1, 2, \dots, n-1$.
- (2) $h_i - r(n, i) \geq 0$ for $i = 1, 2, \dots, n-1$.
- (3) if $h_i - r(n, i) = 0$ for some $1 \leq i \leq n-1$ then $h_0 + h_1 + \dots + h_n$ is even.

Proof. We work over a field K of characteristic 2.

(Necessity.) Let P be a simplicial cell decomposition of $\mathbb{R}P^{n-1}$. Then P is an orientable homology manifold with $\beta_i(P) = 1$ for all $i \geq 1$. Thus $h_0''(P) = h_0$, $h_i''(P) = h_i(P) - r(n, i)$ for $i = 1, 2, \dots, n$. Then the desired conditions follow from Theorem 2.4.

(Sufficiency.) By Corollary 3.2, what we must prove is the existence of a simplicial cell decomposition P of $\mathbb{R}P^{n-1}$ with $h''(P) = (1, 0, \dots, 0, 1)$. In the same way as in the proof of Lemma 6.2, it is enough to find a simplicial cell decomposition P of $\mathbb{R}P^{n-1}$ with

$$f_{n-1}(P) = 2 + \sum_{i=1}^{n-2} \beta_i(P) \binom{n-1}{i} = 2^{n-1}.$$

Consider the boundary complex $\partial \diamond^n$ of the n -dimensional cross polytope $\diamond^n \subset \mathbb{R}^n$. Thus \diamond^n is the convex hull of $\{\pm \mathbf{e}_i : i = 1, 2, \dots, n\}$, where \mathbf{e}_i is the i th unit vector of \mathbb{R}^n . Since the cross polytope \diamond^n is simplicial and centrally symmetric (say, if $F \subset \mathbb{R}^n$ is a face of \diamond^n then $-F$ is also a face of \diamond^n), by identifying F and $-F$ for all faces F of $\partial \diamond^n$, we obtain a simplicial cell decomposition of $\mathbb{R}P^{n-1}$. Since the number of the facets of \diamond^n is 2^n , the number of the facets of such a simplicial cell decomposition is 2^{n-1} . \square

7.2. Odd dimensional manifolds.

For a $(d-1)$ -dimensional simplicial poset P , the vector

$$\beta(P) = (1, \beta_1(P), \dots, \beta_{d-1}(P)) \in \mathbb{Z}_{\geq 0}^d$$

is called the *Betti vector of P* . If P is an orientable homology manifold then the Poincaré duality guarantees the symmetry $\beta_i(P) = \beta_{d-1-i}(P)$ for $i = 1, 2, \dots, d-2$.

For any vector $\beta = (1, \beta_1, \dots, \beta_{d-1}) \in \mathbb{Z}_{\geq 0}^d$ and a vector $h = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$, we define $h^\beta = (h_0^\beta, h_1^\beta, \dots, h_d^\beta)$ by $h_0^\beta = h_0$, $h_k^\beta = h_k - \binom{d}{k} \{\sum_{\ell=2}^k (-1)^{\ell-k} \beta_{\ell-1}\}$ for $k = 1, 2, \dots, d-1$ and $h_d^\beta = h_d - \sum_{\ell=2}^{d-1} (-1)^{\ell-d} \beta_{\ell-1}$. Thus, for a connected simplicial poset P , if $h = h(P)$ and $\beta = \beta(P)$, then $h^\beta = h''(P)$.

Theorem 7.2. *Let d be an even number. The vector $h = (h_0, h_1, \dots, h_d) \in \mathbb{Z}^{d+1}$ is the h -vector of a simplicial cell decomposition of a $(d-1)$ -dimensional topological manifold without boundary if and only if there exists a symmetric vector $\beta = (1, \beta_1, \dots, \beta_{d-1}) \in \mathbb{Z}_{\geq 0}^d$ such that h^β satisfies the conditions (1), (2) and (3) in Theorem 1.1.*

Proof. By considering a field of characteristic 2, any topological manifold is an orientable homology manifold. Then the necessity follows from Theorem 2.4.

We prove the sufficiency. By Lemma 3.1 and Corollary 3.2, for $(d-1)$ -dimensional orientable manifolds M_1 and M_2 , if $\mathcal{H}(M_1) = \mathcal{H}(S^{d-1})$ and $\mathcal{H}(M_2) = \mathcal{H}(S^{d-1})$, then $\mathcal{H}(M_1 \# M_2) = \mathcal{H}(S^{d-1})$. Since, for any symmetric vector $\beta = (1, \beta_1, \dots, \beta_{d-1}) \in \mathbb{Z}_{\geq 0}^d$, we can make a $(d-1)$ -dimensional manifold whose Betti vector is equal to β from a sphere by taking a connected sum with the product of spheres repeatedly, the statement follows from Theorem 1.1. \square

Remark 7.3. The same argument characterizes all possible h -vectors of $(d-1)$ -dimensional orientable simplicial cell (homology) manifolds in characteristic 0 when $d \not\equiv 3 \pmod{4}$ since any Betti vector is attained by the same construction (see [CJS]).

We also note that since $\mathcal{H}(\mathbb{R}P^2) = \mathcal{H}(S^2)$ in characteristic 2 by Theorem 7.1, and since $\mathcal{H}(\mathbb{C}P^2) = \mathcal{H}(S^4)$ by the result of Gagliardi [Ga] and Corollary 3.2, the same argument characterizes all possible face vectors of simplicial cell decompositions of d -dimensional topological manifolds without boundary for $d \leq 5$.

As we suggested in section 3, it would be interesting to find characterizations of face vectors of simplicial cell decompositions of several types of manifolds. For 3-manifolds M , it seems to be plausible that $\mathcal{H}(M)$ has the unique minimal element. (Indeed, this is true if we restrict the problem to graphical simplicial posets since h'' -vector decreases by cancelling a dipole.) Also, while we only consider manifolds without boundary in this paper, it is of interest to consider manifolds with boundary. The h -vectors of simplicial cell decompositions of balls were recently characterized by Kolins [Ko] (up to dimension 6) and the author [Mu] (in any dimension). These results may help to study face vectors of simplicial cell decompositions of manifolds with boundary.

REFERENCES

- [BD] B. Bagchi and B. Datta, Lower bound theorem for normal pseudomanifolds, *Expo. Math.* **26** (2008), 327–351.
- [Bj] A. Björner, Posets, regular CW complexes and Bruhat order, *European J. Combin.* **5** (1984), 7–16.
- [Cr] P. Cristofori, On the genus of $\mathbb{S}^m \times \mathbb{S}^n$, *J. Korean Math. Soc.* **41** (2004), 407–421.

- [CJS] C. Chan, D. Jungreis and R. Stong, Buchsbaum and Eulerian complexes, *J. Pure Appl. Algebra* **98** (1995), 7–13.
- [Du] A. Duval, Free resolutions of simplicial posets, *J. Algebra* **188** (1997), 363–399.
- [FG] M. Ferri and C. Gagliardi, Crystallization moves, *Pacific J. Math.* **100** (1982), 85–103.
- [FGG] M. Ferri, C. Gagliardi and L. Grasselli, A graph-theoretical representation of PL-manifolds—a survey on crystallizations, *Aequationes Math.* **31** (1986), 121–141.
- [Ga] C. Gagliardi, On the genus of the complex projective plane, *Aequationes Math.* **37** (1989), 130–140.
- [GG] C. Gagliardi and L. Grasselli, Representing products of polyhedra by products of edge-colored graphs, *J. Graph Theory* **17** (1993), 549–579.
- [Hu] J.F.P. Hudson, Piecewise linear topology, Benjamin Inc., New York, 1969.
- [Kl] S. Klee, The fundamental group of balanced simplicial complexes and posets, *Electron. J. Combin.* **16** (2009), Research Paper 7, 12 pp.
- [Ko] S. Kolins, f -vectors of Simplicial Posets that are Balls, *J. Algebraic Combin.* **34** (2011), 587–605.
- [Ma] M. Masuda, h -vectors of Gorenstein* simplicial posets, *Adv. Math.* **194** (2005), 332–344.
- [MMP] H. Maeda, M. Masuda and T. Panov, Torus graphs and simplicial posets, *Adv. Math.* **212** (2007), 458–483.
- [MR] E. Miller and V. Reiner, Stanley’s simplicial poset conjecture, after M. Masuda, *Comm. Algebra* **34** (2006), 1049–1053.
- [Mu] S. Murai, h -vectors of simplicial cell balls, *Trans. Amer. Math. Soc.*, to appear.
- [No] I. Novik, Upper bound theorems for homology manifolds, *Israel J. Math.* **108** (1998), 45–82.
- [NS] I. Novik and E. Swartz, Socles of Buchsbaum modules, complexes and posets, *Adv. Math.* **222** (2009), 2059–2084.
- [RS] C.P. Rourke and B.J. Sanderson, Introduction to piecewise-linear topology, Springer-Verlag, 1982.
- [St] R.P. Stanley, f -vectors and h -vectors of simplicial posets, *J. Pure Appl. Algebra* **71** (1991), 319–331.

SATOSHI MURAI, DEPARTMENT OF MATHEMATICAL SCIENCE, FACULTY OF SCIENCE, YAMAGUCHI UNIVERSITY, 1677-1 YOSHIDA, YAMAGUCHI 753-8512, JAPAN.