Geometry of weighted Lorentz–Finsler manifolds I: Singularity theorems

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Abstract

We develop the theory of weighted Ricci curvature in a weighted Lorentz–Finsler framework and extend the classical singularity theorems of general relativity. In order to reach this result, we generalize the Jacobi, Riccati and Raychaudhuri equations to weighted Finsler spacetimes and study their implications for the existence of conjugate points along causal geodesics. We also show a weighted Lorentz–Finsler version of the Bonnet–Myers theorem based on a generalized Bishop inequality.

1 Introduction

The aim of this work is to develop the theory of weighted Ricci curvature on *weighted Lorentz–Finsler manifolds* and show that the classical singularity theorems of general relativity [HE] can be generalized to this setting. It is known that singularity theorems can be generalized to Finsler spacetimes [AJ, Min4], and at least some of them have been generalized to the weighted Lorentzian framework [Ca, GW, WW1, WW2] (we refer to [GW, WW1] for some physical motivations in connection with the Brans–Dicke theory). We will generalize many singularity theorems, including the classical ones by Penrose, Hawking, and Hawking–Penrose, to the weighted Lorentz–Finsler setting.

By a *weighted Lorentzian manifold* we mean a pair of a Lorentzian manifold (M, g) and a weight function ψ on M. This is equivalent to considering a pair of (M, g) and a measure $\mathfrak{m}$ on M via the relation $\mathfrak{m} = \mathrm{e}^{-\psi} \mathrm{vol}_g$, where vol_g is the canonical volume measure of g. The latter formulation was studied also in the Finsler framework [Oh1], where the weight function associated with a measure needs to be a function on the tangent bundle $TM \setminus \{0\}$ (since we do not have a unique canonical measure like vol_g). Motivated by these investigations, we work with an even more general structure, namely a pair given by a Lorentz–Finsler spacetime (M, L) and a (positively 0-homogeneous) function ψ on

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the set of causal vectors. In this framework we can include the unweighted case as well (as the case $\psi = 0$), while in general a constant function may not be associated with any measure. See Section 4 for a more detailed discussion.

Our results will be formulated with the *weighted Ricci curvature* Ric_N, which is defined for (M, L, ψ) in a similar way to the Finsler case [Oh1] (see Definition 4.1). The real parameter N is called the *effective dimension* or the *synthetic dimension* (in connection with the synthetic theory of curvature-dimension condition, see below). Our work not only unifies previous results, but also improves previous findings already in the non-Finsler case, particularly in dealing with the weight. A weighted generalization of the Bishop inequality leads us to a weighted Lorentz–Finsler version of the Bonnet–Myers theorem (Theorem 5.17). For what concerns singularity theorems, we obtain not only the weighted Raychaudhuri equation, but also the weighted Jacobi and Riccati equations (Section 5). Moreover, we show that the genericity condition can be used in its classical formulation (we need to introduce a weighted version as in [Ca, WW2] only in the extremal case of $N = 0$, see Remarks 7.2, 7.5). This fact simplifies the statements of some theorems.

Our results apply to every effective dimension, $N \in (-\infty, 0] \cup [n, +\infty]$ in the timelike case and $N \in (-\infty, 1] \cup [n, +\infty]$ in the null case. The idea of including negative values of N is recent, see [WW1, WW2] for the Lorentzian case (for us the spacetime dimension is $n+1$, which means that the formulas in previous references have to undergo the replacements $n \mapsto n + 1$ and $N \mapsto N + 1$ to be compared with our owns, see Remark 4.2). Our formulation of ϵ-*completeness* (Definitions 5.10, 6.4), which is a key concept in singularity theorems, generalizes that in [WW1, WW2] and is very accurate: We are able to identify a family of time parameters, depending on a real variable ϵ belonging to an ϵ-range dependent on N, for which the incompleteness holds (see Propositions 5.8, 6.3). For $\epsilon = 1$ one recovers the ordinary concept of completeness, while for $\epsilon = 0$ one recovers the ψ-completeness studied in [WW1, WW2]. Our N-dependent ϵ-range explains why for $N \in [n, \infty)$ one can infer both (unweighted and weighted) forms of incompleteness, while for negative N one can infer only the ψ-incompleteness.

The investigation of weighted Lorentz–Finsler manifolds is meaningful also from the view of synthetic studies of Lorentzian geometry. This is motivated by the important breakthrough in the positive-definite case, a characterization of the lower (weighted) Ricci curvature bound by the convexity of an entropy in terms of optimal transport theory, called the *curvature-dimension condition* $\mathrm{CD}(K, N)$ (roughly speaking, $\mathrm{CD}(K, N)$ is equivalent to $\mathrm{Ric}_N \geq K$). We refer to [CMS, LV, vRS, St1, St2, Vi] for the Riemannian case and to [Oh1] for the Finsler case. The curvature-dimension condition can be formulated in metric measure spaces without differentiable structures. Then one can successfully develop comparison geometry and geometric analysis on such metric measure spaces. Lorentzian counterparts of such a synthetic theory attracted growing interest recently, see for instance [AB, KuSa] for triangle comparison theorems, [BP, Br, EM, KeSu, Su] for optimal transport theory, [Mc] for a direct analogue to the curvature-dimension condition, and [MS] for an optimal transport interpretation of the Einstein equation. We also refer to [GKS, Min6], the proceedings [CGKM] and the references therein for related investigations of less regular Lorentzian spaces. Since the curvature-dimension condition is available both in Riemannian and Finsler manifolds, it is important to know what kind of comparison geometric results can be generalized to the Finsler setting. Thus the results

in this article will give some insights in the synthetic study of Lorentzian geometry. We will continue the study of weighted Lorentz–Finsler manifolds in a forthcoming paper on splitting theorems.

1.1   The structure of singularity theorems

Although our work will contain results whose scope exceeds that of singularity theorems, it will be convenient to mention how singularity theorems are typically structured, for that will clarify the focus of this work.

Singularity theorems are composed of the following three steps (see [Min7, Section 6.6] for further discussions):

I. A non-causal statement assuming some form of geodesic completeness plus some genericity and convergence conditions, and implying the existence of conjugate points along geodesics or focal points for certain (hyper)surfaces with special convergence properties, e.g., our Corollaries 5.11 and 6.5. This step typically makes use of the Raychaudhuri equation.

II. A non-causal statement to the effect that the presence of conjugate or focal points spoils some length maximization property (achronal property in the null case), for instance [Min4, Proposition 5.1] will be used to show Proposition 8.2.

III. A statement to the effect that under some causality conditions as well as in presence of some special set (trapped set, Cauchy hypersurface) the spacetime necessarily has a *causal line* (a maximizing inextendible causal geodesic) or a causal S-ray.

The first two results go in contradiction with the last one, so from here one infers the geodesic incompleteness.

Interestingly, the first two steps basically coincide for all the singularity theorems. For instance, Penrose's and Gannon's singularity theorems [Ga, Pen], but also the topological censorship theorem [FSW], use the same versions of Steps I, II. Similarly, Hawking–Penrose's and Borde's singularity theorems [Bo, HP] use the same versions of Steps I, II. Most singularity theorems really differ just for the causality statement in Step III. For this reason, it is often convenient to identify the singularity theorem with its *causality core statement*, namely Step III. It turns out that this causality core statement in most cases involves just the cone distribution, thereby it is fairly robust.

For instance, we shall work with a Lorentz–Finsler space of Beem's type which is a special case of a more general object called a *locally Lipschitz proper Lorentz–Finsler space*, see [Min6, Theorem 2.52], which is basically a distribution of closed cones $x \mapsto \overline{\Omega}_x$ plus a function $F : \overline{\Omega} \longrightarrow \mathbb{R}$ satisfying certain regularity properties. For this structure and hence for our setting, one can prove the following causality statement [Min6, Theorem 2.67] (this result actually holds for more general *closed cone structures*): *In a Finsler spacetime admitting a non-compact Cauchy hypersurface every nonempty compact set S admits a future lightlike S-ray.* (The various terms will be clarified in what follows.)

There is also a simpler approach by which one can understand the validity of this type of causality core statements. The local causality theory makes use of the existence

of convex neighborhoods, but does not make use of the curvature tensor. The curvature tensor really makes its appearance only in Steps I and II above. Thus all the proofs of these causality core statements, being of topological nature, pass through word-for-word from the Lorentzian to the Lorentz–Finsler case, and since the weight is not used, to the weighted Lorentz–Finsler case. These topological proofs can then be read from reviews of Lorentzian causality theory, e.g., [Min7, Theorem 6.23] includes the above statement.

It is important to understand that what we shall be doing in the following sections is to generalize Step I. Step II has been already adapted to the Lorentz–Finsler setting in [Min4], and hence to the weighted Lorentz–Finsler setting since it does not use the weight. Step III was also already generalized in [Min6] to frameworks broader than that of this work. In this sense we are not considering the most general situation, and we do not intend to make a full list of applications. We wish to show that singularity theorems can be generalized to the weighted Lorentz–Finsler case, by presenting several singularity theorems for the sake of illustrating the general strategy. Once Steps I and II are established, by selecting a different causality core statement in Step III, one can obtain other singularity theorems not explicitly considered in this article (we refer to [Min4, Section 8], [Min7, Section 6.6], and [Min6, Section 2.15] for further singularity theorems as well as more general statements).

1.2   Notations and organization of the paper

Let us fix some terminologies and notations. Riemannian and Finsler manifolds have positive-definite metrics. The analogous structures in the Lorentzian signature will be called Lorentzian manifolds and Lorentz–Finsler manifolds. Lorentz–Finsler manifolds are also known as Lorentz–Finsler spaces in other references, for example, [Min6]. The Lorentzian signature we use is $(-, +, \ldots, +)$. We stress that the dimension of the space-time manifold is always $n + 1$, and the indices will be taken as $\alpha = 0, 1, \ldots, n$.

This article is organized as follows. In Sections 2 and 3, we introduce necessary notions of Finsler spacetimes, including some causality conditions and the flag and Ricci curvatures. We then introduce the weighted Ricci curvature in Section 4. In Sections 5 and 6, we study the timelike and null Raychaudhuri equations, respectively, which are applied in Section 7 to investigate the existence of conjugate points along geodesics. Finally, Section 8 is devoted to the proofs of some notable singularity theorems, along the strategy outlined in Subsection 1.1 above.

2   Finsler spacetimes

2.1   Lorentz–Finsler manifolds

Let M be a connected C^∞-manifold of dimension $n + 1$ without boundary. Given local coordinates $(x^\alpha)_{\alpha=0}^n$ on an open set $U \subset M$, we will use the fiber-wise linear coordinates $(x^\alpha, v^\beta)_{\alpha,\beta=0}^n$ of TU such that

$$v = \sum_{\beta=0}^n v^\beta \frac{\partial}{\partial x^\beta}\Big|_x, \quad x \in U.$$

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We employ Beem's definition of Lorentz–Finsler manifolds [Be] (see Remark 2.6 below for the relation with the other definitions).

Definition 2.1 (Lorentz–Finsler structure) A *Lorentz–Finsler structure* of M will be a function $L\colon TM \longrightarrow \mathbb{R}$ satisfying the following conditions:

(1) $L \in C^\infty(TM \setminus \{0\})$;

(2) $L(cv) = c^2 L(v)$ for all $v \in TM$ and $c > 0$;

(3) For any $v \in TM \setminus \{0\}$, the symmetric matrix

$$g_{\alpha\beta}(v) := \frac{\partial^2 L}{\partial v^\alpha \partial v^\beta}(v), \quad \alpha, \beta = 0, 1, \ldots, n, \tag{2.1}$$

is non-degenerate with signature $(-, +, \ldots, +)$.

We will call (M, L) a *Lorentz–Finsler manifold* or a *Lorentz–Finsler space*.

We stress that the homogeneity condition (2) is imposed only in the positive direction $(c > 0)$, thus $L(-v) \neq L(v)$ is allowed. We say that L is *reversible* if $L(-v) = L(v)$ for all $v \in TM$. The matrix $(g_{\alpha\beta}(v))_{\alpha,\beta=0}^n$ in (2.1) defines the Lorentzian metric g_v of T_xM by

$$g_v\left(\sum_{\alpha=0}^n a^\alpha \frac{\partial}{\partial x^\alpha}\Big|_x, \sum_{\beta=0}^n b^\beta \frac{\partial}{\partial x^\beta}\Big|_x \right) := \sum_{\alpha,\beta=0}^n a^\alpha b^\beta g_{\alpha\beta}(v). \tag{2.2}$$

By construction g_v is the second order approximation of $2L$ at v. Similarly to the positive-definite case, the metric g_v and Euler's homogeneous function theorem (see [BCS, Theorem 1.2.1]) will play a fundamental role in our argument. We have for example

$$g_v(v, v) = \sum_{\alpha,\beta=0}^n v^\alpha v^\beta g_{\alpha\beta}(v) = 2L(v).$$

Definition 2.2 (Timelike vectors) We call $v \in TM$ a *timelike vector* if $L(v) < 0$ and a *null vector* if $L(v) = 0$. A vector v is said to be *lightlike* if it is null and nonzero. The *spacelike vectors* are those for which $L(v) > 0$ or $v = 0$. The *causal* (or *non-spacelike*) vectors are those which are lightlike or timelike ($L(v) \leq 0$ and $v \neq 0$). The set of timelike vectors will be denoted by

$$\Omega_x' := \{v \in T_xM \mid L(v) < 0\}, \qquad \Omega' := \bigcup_{x \in M} \Omega_x'.$$

Sometimes we shall make use of the function $F : \overline{\Omega'} \longrightarrow [0, +\infty)$ defined by

$$F(v) := \sqrt{-g_v(v, v)} = \sqrt{-2L(v)}, \tag{2.3}$$

which measures the 'length' of causal vectors. The structure of the set of timelike vectors was studied in [Be]. We summarize fundamental properties in the next lemma, see also [Be, Per, Min3] for more detailed investigations.

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Lemma 2.3 (Properties of Ω'_x) *Let (M, L) be a Lorentz–Finsler manifold and $x \in M$.*

(i) *We have $\Omega'_x \neq \emptyset$.*

(ii) *For each $c < 0$, $T_x M \cap L^{-1}(c)$ is nonempty and positively curved with respect to the linear structure of $T_x M$.*

(iii) *Every connected component of Ω'_x is a convex cone.*

Proof. (i) If $L(v) > 0$ for all $v \in T_x M \setminus \{0\}$, then $T_x M \cap L^{-1}(1)$ is compact and L is nonnegative-definite at an extremal point of $T_x M \cap L^{-1}(1)$. This contradicts Definition 2.1(3). If $L \geq 0$ on $T_x M$ and there is $v \in T_x M \setminus \{0\}$ with $L(v) = 0$, then L is again nonnegative-definite at v and we have a contradiction. Therefore we conclude $\Omega'_x \neq \emptyset$.

(ii) The first assertion $T_x M \cap L^{-1}(c) \neq \emptyset$ is straightforward from (i) and the homogeneity of L. The second assertion is shown by comparing L and its second order approximation g_v at $v \in T_x M \cap L^{-1}(c)$ (see [Be, Lemma 1]).

(iii) This is a consequence of (ii). $\qquad\square$

In the 2-dimensional case $(n + 1 = 2)$, the number of connected components of Ω'_x is not necessarily 2, even when L is reversible (i.e., $L(-v) = L(v)$).

Example 2.4 (Beem's example, [Be]) Let us consider the Euclidean plane $\mathbb{R}^2$. Given $k \in \mathbb{N}$, we define $L : \mathbb{R}^2 \longrightarrow \mathbb{R}$ in the polar coordinates by $L(r, \theta) := r^2 \cos k\theta$. Then $\mathrm{Hess}\, L(r, \theta)$ has the negative determinant for $r > 0$, and the number of connected components of $\{x \in \mathbb{R}^2 \,|\, L(x) < 0\}$ is k. Notice that $L(r, \theta + \pi) = L(r, \theta)$ (reversible) if k is even, and $L(r, \theta + \pi) = -L(r, \theta)$ (non-reversible) if k is odd.

This phenomenon could be regarded as a drawback of the formulation of Definition 2.1 from the view of theoretical physics, since it is difficult to interpret the causal structure of such multi-cones (see [Min3] for further discussions). However, in the reversible case, it turned out that such an ill-posedness occurs only when $n + 1 = 2$ ([Min3, Theorem 7]).

Theorem 2.5 (Well-posedness for $n + 1 \geq 3$) *Let (M, L) be a reversible Lorentz–Finsler manifold of dimension $n + 1 \geq 3$. Then, for any $x \in M$, the set Ω'_x has exactly two connected components.*

The key difference between $n + 1 = 2$ and $n + 1 \geq 3$ used in the proof is that the sphere $\mathbb{S}^n$ is simply-connected if and only if $n \geq 2$ (see [Min3, Theorem 6]). One may think of taking the product of $(\mathbb{R}^2, L)$ in Example 2.4 and $\mathbb{R}$, that is, $\overline{L}(r, \theta, z) := r^2 \cos k\theta + z^2$. This Lagrangian $\overline{L}$ is, however, twice differentiable at $(0, 0, 1)$ if and only if $k = 2$.

Remark 2.6 (Definitions of Lorentz–Finsler structures) The analogue to Theorem 2.5 in the non-reversible case is an open problem. Nevertheless, it is in many cases acceptable to consider L as defined just inside the future cone, as in the approach by Asanov [As]. That is to say, we consider a smooth family of convex cones, $\{\Omega_x\}_{x \in M}$ with $\Omega_x \subset T_x M \setminus \{0\}$, and L is defined only on $\bigcup_{x \in M} \overline{\Omega}_x$ such that $L < 0$ on Ω_x, $L = 0$ on $\partial\Omega_x$, and having the Lorentzian signature (studies of increasing functions for cone distributions can be found in [FS, BS] and their general causality theory is developed in [Min6]). In

this case, under the natural assumption that $dL \neq 0$ on $\partial\Omega_x$, we can extend L to $\widetilde{L}$ on the whole tangent bundle TM such that the set of timelike vectors of $\widetilde{L}$ has exactly two connected components in each tangent space (see [Min5, Theorem 1], $\widetilde{L}$ may not be reversible). Therefore assuming that L is globally defined as in Definition 2.1 costs no generality. Furthermore, in most arguments, given a (future-directed) timelike vector v, we make use of g_v from (2.2) instead of L itself.

2.2 Causality theory

We recall some fundamental concepts in causality theory on a Lorentz–Finsler manifold (M, L). A continuous vector field X on M is said to be *timelike* if $L(X(x)) < 0$ for all $x \in M$. If (M, L) admits a timelike smooth vector field X, then (M, L) is said to be *time oriented* by X, or simply *time oriented*. We will call a time oriented Lorentz–Finsler manifold a *Finsler spacetime*.

A causal vector $v \in T_xM$ is said to be *future-directed* if it lies in the same connected component of $\overline{\Omega'_x} \setminus \{0\}$ as $X(x)$. We will denote by $\Omega_x \subset \Omega'_x$ the set of future-directed timelike vectors, and set

$$\Omega := \bigcup_{x \in M} \Omega_x, \qquad \overline{\Omega} := \bigcup_{x \in M} \overline{\Omega}_x, \qquad \overline{\Omega} \setminus \{0\} := \bigcup_{x \in M} (\overline{\Omega}_x \setminus \{0\}).$$

A C^1-curve in (M, L) is said to be timelike (resp. causal, lightlike, spacelike) if its tangent vector is always timelike (resp. causal, lightlike, spacelike). All causal curves will be future-directed in this article. Given distinct points $x, y \in M$, we write $x \ll y$ if there is a future-directed timelike curve from x to y. Similarly, $x < y$ means that there is a future-directed causal curve from x to y, and $x \leq y$ means that $x = y$ or $x < y$.

The *chronological past* and *future* of x are defined by

$$I^-(x) := \{y \in M \mid y \ll x\}, \qquad I^+(x) := \{y \in M \mid x \ll y\},$$

and the *causal past* and *future* are defined by

$$J^-(x) := \{y \in M \mid y \leq x\}, \qquad J^+(x) := \{y \in M \mid x \leq y\}.$$

For a general set $S \subset M$, we define $I^-(S), I^+(S), J^-(S)$ and $J^+(S)$ analogously.

Definition 2.7 (Causality conditions) Let (M, L) be a Finsler spacetime.

(1) (M, L) is said to be *chronological* if $x \notin I^+(x)$ for all $x \in M$.

(2) We say that (M, L) is *causal* if there is no closed causal curve.

(3) (M, L) is said to be *strongly causal* if, for all $x \in M$, every neighborhood U of x contains another neighborhood V of x such that no causal curve intersects V more than once.

(4) We say that (M, L) is *globally hyperbolic* if it is strongly causal and, for any $x, y \in M$, $J^+(x) \cap J^-(y)$ is compact.

Clearly strong causality implies causality, and a causal spacetime is chronological. The chronological condition implies that the spacetime is non-compact. The following concept plays an essential role in the study of the geodesic incompleteness in general relativity.

Definition 2.8 (Inextendibility) A future-directed causal curve $\eta : (a, b) \longrightarrow M$ is said to be *future* (resp. *past*) *inextendible* if $\eta(t)$ does not converge as $t \to b$ (resp. $t \to a$). We say that η is *inextendible* if it is both future and past inextendible.

Global hyperbolicity can be characterized in many ways. Here we mention one of them in terms of Cauchy hypersurfaces (see [Min4, Proposition 6.12], [FS, Theorem 1.3]).

Definition 2.9 (Cauchy hypersurfaces) A hypersurface $S \subset M$ is called a *Cauchy hypersurface* if every future-directed inextendible causal curve intersects S exactly once.

Proposition 2.10 *A Finsler spacetime (M, L) is globally hyperbolic if and only if it admits a smooth Cauchy hypersurface.*

2.3 Geodesics

Next we introduce some geometric concepts. Define the *Lorentz–Finsler length* of a piecewise C^1-causal curve $\eta : [a, b] \longrightarrow M$ by (recall (2.3) for the definition of F)

$$\ell(\eta) := \int_a^b F\big(\dot{\eta}(t)\big)\, dt.$$

Then, for $x, y \in M$, we define the *Lorentz–Finsler distance* $d(x, y)$ from x to y by

$$d(x, y) := \sup_\eta \ell(\eta),$$

where η runs over all piecewise C^1-causal curves from x to y. We set $d(x, y) := 0$ if there is no causal curve from x to y. We remark that, under the assumption of global hyperbolicity, d is finite and continuous ([Min4, Proposition 6.8]). A causal curve $\eta : I \longrightarrow M$ is said to be *maximizing* if, for every $t_1, t_2 \in I$ with $t_1 < t_2$, we have $d(\eta(t_1), \eta(t_2)) = \ell(\eta|_{[t_1,t_2]})$.

The *Euler–Lagrange equation* for the *action* $\mathcal{S}(\eta) := \int_a^b L(\dot{\eta}(t))\, dt$ provides the *geodesic equation*

$$\ddot{\eta}^\alpha + \sum_{\beta,\gamma=0}^n \widetilde{\Gamma}_{\beta\gamma}^\alpha(\dot{\eta})\dot{\eta}^\beta\dot{\eta}^\gamma = 0, \tag{2.4}$$

where we define

$$\widetilde{\Gamma}_{\beta\gamma}^\alpha(v) := \frac{1}{2}\sum_{\delta=0}^n g^{\alpha\delta}(v)\left(\frac{\partial g_{\delta\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\delta}}{\partial x^\gamma} - \frac{\partial g_{\beta\gamma}}{\partial x^\delta}\right)(v) \tag{2.5}$$

for $v \in TM \setminus \{0\}$ and $(g^{\alpha\beta}(v))$ denotes the inverse matrix of $(g_{\alpha\beta}(v))$.

We say that a C^∞-causal curve $\eta : [a, b] \longrightarrow \mathbb{R}$ is *geodesic* if (2.4) holds for all $t \in (a, b)$. Since $L(\dot{\eta})$ is constant by (2.4), a causal geodesic is indeed either a timelike geodesic or

a lightlike geodesic. Given $v \in \overline{\Omega}_x$, if there is a geodesic $\eta : [0,1] \longrightarrow M$ with $\dot{\eta}(0) = v$, then the *exponential map* $\exp_x$ is defined by $\exp_x(v) := \eta(1)$.

Locally maximizing causal curves coincide with causal geodesics up to reparametrizations ([Min2, Theorem 6]). Under very weak differentiability assumptions on the metric, this local maximization property can be used to define the notion of causal geodesics (see [Min6]). We remark that, under Definition 2.1, due to a classical result by Whitehead, the manifold admits convex neighborhoods. Ultimately, this single fact makes it possible to work out much of causality theory for Lorentz–Finsler manifolds in analogy with that for Lorentzian manifolds (we refer to [Min2, Min4]).

3 Covariant derivatives and curvatures

In this section, along the argument in [Sh, Chapter 6] (see also [Oh5]) in the positive-definite case, we introduce covariant derivatives (associated with the Chern connection) and Jacobi fields by analyzing the behavior of geodesics. Then we define the flag and Ricci curvatures in the spacetime context. We refer to [Min4, Section 2] for a further account.

Similarly to the previous section, (M, L) will denote a Finsler spacetime and all causal curves and vectors are future-directed. In this section, however, this is merely for simplicity and the time-orientability plays no role. Everything is local and can be readily generalized to general causal vectors and geodesics on Lorentz–Finsler manifolds.

3.1 Covariant derivatives

We first introduce the coefficients of the *geodesic spray* and the *nonlinear connection* as

$$G^\alpha(v) := \frac{1}{2} \sum_{\beta,\gamma=0}^n \widetilde{\Gamma}^\alpha_{\beta\gamma}(v) v^\beta v^\gamma, \qquad N^\alpha_\beta(v) := \frac{\partial G^\alpha}{\partial v^\beta}(v)$$

for $v \in TM \setminus \{0\}$, and $G^\alpha(0) = N^\alpha_\beta(0) := 0$. Note that G^α is positively 2-homogeneous and N^α_β is positively 1-homogeneous, and $2G^\alpha(v) = \sum_{\beta=0}^n N^\alpha_\beta(v) v^\beta$ holds by the homogeneous function theorem. The geodesic equation (2.4) is now written as $\ddot{\eta}^\alpha + 2G^\alpha(\dot{\eta}) = 0$. In order to define covariant derivatives, we need to modify $\widetilde{\Gamma}^\alpha_{\beta\gamma}$ in (2.5) as

$$\Gamma^\alpha_{\beta\gamma}(v) := \widetilde{\Gamma}^\alpha_{\beta\gamma}(v) - \frac{1}{2} \sum_{\delta,\mu=0}^n g^{\alpha\delta}(v) \left(\frac{\partial g_{\delta\gamma}}{\partial v^\mu} N^\mu_\beta + \frac{\partial g_{\beta\delta}}{\partial v^\mu} N^\mu_\gamma - \frac{\partial g_{\beta\gamma}}{\partial v^\mu} N^\mu_\delta \right)(v)$$

for $v \in TM \setminus \{0\}$. Notice that these formulas are the same as those in [Sh] (while $G^i(v)$ in [Oh5] corresponds to $2G^\alpha(v)$ in this article).

Definition 3.1 (Covariant derivatives) For a C^1-vector field X on M, $x \in M$ and $v, w \in T_x M$ with $w \neq 0$, we define the *covariant derivative* of X by v with reference (support) vector w by

$$D^w_v X := \sum_{\alpha,\beta=0}^n \left\{ v^\beta \frac{\partial X^\alpha}{\partial x^\beta}(x) + \sum_{\gamma=0}^n \Gamma^\alpha_{\beta\gamma}(w) v^\beta X^\gamma(x) \right\} \frac{\partial}{\partial x^\alpha}\Big|_x.$$

The reference vector will be usually chosen as $w = v$ or $w = X(x)$. The following result is shown in the same way as [Sh, Section 6.2] (see also [Oh3, Lemma 2.3]).

Proposition 3.2 (Riemannian characterization) *If V is a nowhere vanishing C^∞-vector field such that all integral curves of V are geodesic, then we have*

$$D_V^V X = D_V^{g_V} X, \qquad D_X^V V = D_X^{g_V} V$$

for any differentiable vector field X, where D^{g_V} denotes the covariant derivative with respect to the Lorentzian structure g_V induced from V via (2.2).

Along a C^∞-curve η with $\dot{\eta} \neq 0$, one can consider the covariant derivative along η,

$$D_{\dot\eta}^\eta X(t) := \sum_{\alpha=0}^n \left(\dot{X}^\alpha + \sum_{\beta,\gamma=0}^n \Gamma_{\beta\gamma}^\alpha(\dot\eta)\dot\eta^\beta X^\gamma \right)(t) \left. \frac{\partial}{\partial x^\alpha} \right|_{\eta(t)},$$

for vector fields X along η, where $X(t) = \sum_{\alpha=0}^n X^\alpha(t)(\partial/\partial x^\alpha)|_{\eta(t)}$. Then the geodesic equation (2.4) coincides with $D_{\dot\eta}^\eta \dot\eta = 0$.

For a nonconstant causal geodesic η and C^∞-vector fields X, Y along η, we have

$$\frac{d}{dt}\big[g_{\dot\eta}(X, Y) \big] = g_{\dot\eta}(D_{\dot\eta}^\eta X, Y) + g_{\dot\eta}(X, D_{\dot\eta}^\eta Y) \tag{3.1}$$

(see, e.g., [BCS, Exercise 5.2.3]). One also has, for nowhere vanishing X,

$$\frac{d}{dt}\big[g_X(X, Y) \big] = g_X(D_{\dot\eta}^X X, Y) + g_X(X, D_{\dot\eta}^X Y) \tag{3.2}$$

(see [BCS, Exercise 10.1.2]).

3.2 Jacobi fields

Next we introduce Jacobi fields. Let $\zeta : [a, b] \times (-\varepsilon, \varepsilon) \longrightarrow M$ be a C^∞-map such that $\zeta(\cdot, s)$ is a causal geodesic for each $s \in (-\varepsilon, \varepsilon)$. Put $\eta(t) := \zeta(t, 0)$ and consider the variational vector field $Y(t) := \partial\zeta/\partial s(t, 0)$. Then we have

$$
\begin{aligned}
&D_{\dot\eta}^\eta D_{\dot\eta}^\eta Y \\
&= \sum_{\alpha,\beta=0}^n \left\{ -2\frac{\partial G^\alpha}{\partial x^\beta}(\dot\eta) + \sum_{\gamma=0}^n \left(\frac{\partial N_\beta^\alpha}{\partial x^\gamma}(\dot\eta)\dot\eta^\gamma - 2\frac{\partial N_\beta^\alpha}{\partial v^\gamma}(\dot\eta)G^\gamma(\dot\eta) + N_\gamma^\alpha(\dot\eta)N_\beta^\gamma(\dot\eta) \right) \right\} Y^\beta \left. \frac{\partial}{\partial x^\alpha} \right|_\eta.
\end{aligned}
$$

Now, we define

$$R_\beta^\alpha(v) := 2\frac{\partial G^\alpha}{\partial x^\beta}(v) - \sum_{\gamma=0}^n \left(\frac{\partial N_\beta^\alpha}{\partial x^\gamma}(v)v^\gamma - 2\frac{\partial N_\beta^\alpha}{\partial v^\gamma}(v)G^\gamma(v) \right) - \sum_{\gamma=0}^n N_\gamma^\alpha(v)N_\beta^\gamma(v)$$

for $v \in \overline{\Omega}$ (note that $R_\beta^\alpha(0) = 0$), and

$$R_v(w) := \sum_{\alpha,\beta=0}^n R_\beta^\alpha(v)w^\beta \left. \frac{\partial}{\partial x^\alpha} \right|_x \tag{3.3}$$

for $v \in \overline{\Omega}_x$ and $w \in T_x M$. Then we arrive at the *Jacobi equation*

$$D_{\dot\eta}^\eta D_{\dot\eta}^\eta Y + R_{\dot\eta}(Y) = 0. \tag{3.4}$$

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Definition 3.3 (Jacobi fields) A solution Y to (3.4) is called a *Jacobi field* along a causal geodesic η.

We recall two important properties of R_v, see [Min4, Proposition 2.4] for a detailed account.

Proposition 3.4 (Properties of R_v)   (i) *We have $R_v(v) = 0$ for every $v \in \overline{\Omega}_x$.*

   (ii) *R_v is symmetric in the sense that*

$$g_v\big(w_1, R_v(w_2)\big) = g_v\big(R_v(w_1), w_2\big) \tag{3.5}$$

    holds for all $v \in \overline{\Omega}_x \setminus \{0\}$ and $w_1, w_2 \in T_xM$.

Along a nonconstant causal geodesic $\eta : [a, b] \longrightarrow M$, if there is a nontrivial Jacobi field Y such that $Y(a) = Y(t) = 0$ for some $t \in (a, b]$, then we call $\eta(t)$ a *conjugate point* of $\eta(a)$ along η. The existence of conjugate points is a key issue throughout this article.

3.3 Curvatures

The flag and Ricci curvatures are defined by using R_v in (3.3) as follows. The flag curvature corresponds to the sectional curvature in the Riemannian or Lorentzian context.

Definition 3.5 (Flag curvature) For $v \in \Omega_x$ and $w \in T_xM$ linearly independent of v, define the *flag curvature* of the 2-plane $v \wedge w$ (a *flag*) spanned by v, w with *flagpole* v as

$$\mathbf{K}(v, w) := -\frac{g_v(R_v(w), w)}{g_v(v, v)g_v(w, w) - g_v(v, w)^2}. \tag{3.6}$$

We remark that this is the opposite sign to [BEE], while the Ricci curvature will be the same. The flag curvature $\mathbf{K}(v, w)$ depends only on the 2-plane $v \wedge w$ and the choice of the flagpole $\mathbb{R}_+ v$ in it.

Note that, for v timelike, the denominator in the right-hand side of (3.6) is negative. The flag curvature is not defined for v lightlike, for in this case the denominator could vanish. Thus we define the Ricci curvature directly as the trace of R_v in (3.3).

Definition 3.6 (Ricci curvature) For $v \in \overline{\Omega}_x \setminus \{0\}$, the *Ricci curvature* or *Ricci scalar* is defined as the trace of R_v, i.e., $\mathrm{Ric}(v) := \mathrm{trace}(R_v)$.

Since $\mathrm{Ric}(v)$ is positively 2-homogeneous, we can set $\mathrm{Ric}(0) := 0$ by continuity. We say that $\mathrm{Ric} \geq K$ holds *in timelike directions* for some $K \in \mathbb{R}$ if we have $\mathrm{Ric}(v) \geq K F(v)^2 = -2KL(v)$ for all $v \in \Omega$. For v lightlike, since $L(v) = 0$, only the nonnegative curvature condition $\mathrm{Ric}(v) \geq 0$ makes sense.

For a normalized timelike vector $v \in \Omega_x$ with $F(v) = 1$, $\mathrm{Ric}(v)$ can be given as $\mathrm{Ric}(v) = \sum_{i=1}^{n} \mathbf{K}(v, e_i)$, where $\{v\} \cup \{e_i\}_{i=1}^{n}$ is an orthonormal basis with respect to g_v, i.e., $g_v(e_i, e_j) = \delta_{ij}$ and $g_v(v, e_i) = 0$ for all $i, j = 1, \ldots, n$.

We deduce from Proposition 3.2 the following important feature of the Finsler curvature. This is one of the main driving forces behind the recent developments of comparison geometry and geometric analysis on Finsler manifolds (see [Sh, Oh1, Oh5]).

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Theorem 3.7 (Riemannian characterizations) *Given a timelike vector $v \in \Omega_x$, take a C^1-vector field V on a neighborhood of x such that $V(x) = v$ and every integral curve of V is geodesic. Then, for any $w \in T_x M$ linearly independent of v, the flag curvature $\mathbf{K}(v, w)$ coincides with the sectional curvature of $v \wedge w$ for the Lorentzian metric g_V. Similarly, the Ricci curvature $\mathrm{Ric}(v)$ coincides with the Ricci curvature of v for g_V.*

Proof. Let $\eta : (-\delta, \delta) \longrightarrow M$ be the geodesic with $\dot\eta(0) = v$ and observe that $V(\eta(t)) = \dot\eta(t)$ by the condition imposed on V. Take a C^∞-variation $\zeta : (-\delta, \delta) \times (-\varepsilon, \varepsilon) \longrightarrow M$ of η such that $\partial_s \zeta(0, 0) = w$ and that each $\zeta(\cdot, s)$ is an integral curve of V. Then by the hypothesis, $\zeta(\cdot, s)$ is geodesic for all s and hence $Y(t) := \partial_s \zeta(t, 0)$ is a Jacobi field along η. Hence we deduce from the Jacobi equation (3.4) that

$$\mathbf{K}(v, w) = -\frac{g_v(R_v(w), w)}{g_v(v, v) g_v(w, w) - g_v(v, w)^2} = \frac{g_v(D_{\dot\eta}^{\dot\eta} D_{\dot\eta}^{\dot\eta} Y(0), w)}{g_v(v, v) g_v(w, w) - g_v(v, w)^2}.$$

Now we compare this observation with the Lorentzian counterpart for g_V. Since ζ is also a geodesic variation for g_V (by Proposition 3.2), Y is a Jacobi field also for g_V. Moreover, it follows from Proposition 3.2 that $D_{\dot\eta}^{\dot\eta} D_{\dot\eta}^{\dot\eta} Y(0) = D_{\dot\eta}^{g_V} D_{\dot\eta}^{g_V} Y(0)$. This shows the first assertion, and the second assertion is obtained by taking the trace. $\qquad\square$

This observation is particularly helpful when we consider comparison theorems, see Subsection 5.5 for some instances.

4   Weighted Ricci curvature

In this section we introduce the main ingredient of our results, the *weighted Ricci curvature*, for a triple (M, L, ψ) where (M, L) is a Finsler spacetime and $\psi : \overline{\Omega} \setminus \{0\} \longrightarrow \mathbb{R}$ is a *weight function* which is C^∞ and positively 0-homogeneous, i.e., $\psi(cv) = \psi(v)$ for all $c > 0$.

Let $\pi : \overline{\Omega} \setminus \{0\} \longrightarrow M$ be the bundle of causal vectors. The function ψ can be used to define a section of the pullback bundle $\pi^*[\bigwedge^{n+1}(T^*M)] \longrightarrow \overline{\Omega} \setminus \{0\}$ as

$$\Phi(x, v)\, dx^0 \wedge dx^1 \wedge \cdots \wedge dx^n, \qquad \Phi(x, v) := \mathrm{e}^{-\psi(v)} \sqrt{-\det\big[\big(g_{\alpha\beta}(v)\big)_{\alpha,\beta=0}^n\big]},$$

provided that M is orientable. In other words, we can consider a similar formula (even when M is not orientable) as follows: For every causal vector field V on M,

$$\mathfrak{m}_V(dx) := \Phi\big(x, V(x)\big)\, dx^0 dx^1 \cdots dx^n = \mathrm{e}^{-\psi(V(x))} \mathrm{vol}_{g_V}(dx)$$

defines a measure $\mathfrak{m}_V$ on M, where vol_{g_V} is the volume measure induced from g_V.

This structure (M, L, ψ) generalizes that of a *Lorentz–Finsler measure space*, which means a triple $(M, L, \mathfrak{m})$ where $\mathfrak{m}$ is a positive C^∞-measure on M in the sense that, in each local coordinates $(x^\alpha)_{\alpha=0}^n$, $\mathfrak{m}$ is written as $\mathfrak{m}(dx) = \Phi(x)\, dx^0 dx^1 \cdots dx^n$ (see [Oh1] for the positive-definite case). In this setting the function ψ is defined so as to satisfy, for $v \in \overline{\Omega}_x \setminus \{0\}$,

$$\Phi(x) = \mathrm{e}^{-\psi(v)} \sqrt{-\det\big[\big(g_{\alpha\beta}(v)\big)_{\alpha,\beta=0}^n\big]}.$$

Notice that $g_{\alpha\beta}(v)$ depends on the direction v in the Lorentz–Finsler (or Finsler) case. This is the reason why we consider a function on $\overline{\Omega} \setminus \{0\}$, instead of a function on M as in the Lorentzian case. Our approach here, considering a general function ψ not necessarily induced from a measure, represents a further generalization which allows us to identify the *unweighted case*: We shall say that we are in the unweighted case if ψ is constant. (There may not exist any measure such that ψ is constant, see [Oh2] for a related study in the positive-definite case.) Since all the following calculations involve only the derivatives of ψ, we can regard the choice $\psi = 0$ as the only unweighted case.

We need to modify $\text{Ric}(v)$ defined in Definition 3.6 according to the choice of ψ, so as to generalize the definition of [Oh1] for the Finsler measure space case. As a matter of notation, given a causal geodesic $\eta(t)$ we shall write

$$\psi_\eta(t) := \psi\big(\dot{\eta}(t)\big). \tag{4.1}$$

Definition 4.1 (Weighted Ricci curvature) On (M, L, ψ) with $\dim M = n+1$, given a nonzero causal vector $v \in \overline{\Omega}_x \setminus \{0\}$, let $\eta : (-\varepsilon, \varepsilon) \longrightarrow M$ be the geodesic with $\dot{\eta}(0) = v$. Then, for $N \in \mathbb{R} \setminus \{n\}$, we define the *weighted Ricci curvature* by

$$\text{Ric}_N(v) := \text{Ric}(v) + \psi_\eta''(0) - \frac{\psi_\eta'(0)^2}{N - n}. \tag{4.2}$$

As the limits of $N \to +\infty$ and $N \downarrow n$, we also define

$$\text{Ric}_\infty(v) := \text{Ric}(v) + \psi_\eta''(0), \qquad \text{Ric}_n(v) := \begin{cases} \text{Ric}(v) + \psi_\eta''(0) & \text{if } \psi_\eta'(0) = 0, \\ -\infty & \text{if } \psi_\eta'(0) \neq 0. \end{cases}$$

Remark 4.2 Because of our notation $\dim M = n + 1$, Ric_N in this article corresponds to Ric_{N+1} in [Oh1, Oh5] or Ric_f^{N+1} in [WW1, WW2].

Similarly to Definition 3.6, we say that $\text{Ric}_N \geq K$ holds in timelike directions for some $K \in \mathbb{R}$ if we have $\text{Ric}_N(v) \geq KF(v)^2$ for all $v \in \Omega$, and $\text{Ric}_N \geq 0$ in null directions means that $\text{Ric}_N(v) \geq 0$ for all lightlike vectors v.

The weighted Ricci curvature Ric_N is also called the *Bakry–Émery–Ricci curvature*, due to the pioneering work by Bakry–Émery [BE] in the Riemannian situation (we refer to the book [BGL] for further information). The Finsler version was introduced in [Oh1] as we mentioned, and we refer to [Ca] for the case of Lorentzian manifolds.

Remark 4.3 (Remarks on Ric_N) (a) In the unweighted case, we have $\text{Ric}_N(v) = \text{Ric}(v)$ for every $N \in (-\infty, +\infty]$. In general, it is clear by definition that Ric_N is monotone non-decreasing in the ranges $[n, +\infty]$ and $(-\infty, n)$, and we have

$$\text{Ric}_n(v) \leq \text{Ric}_N(v) \leq \text{Ric}_\infty(v) \leq \text{Ric}_{N'}(v) \tag{4.3}$$

for any $N \in (n, +\infty)$ and $N' \in (-\infty, n)$.

(b) The study of the case where $N \in (-\infty, n)$ is rather recent. The above monotonicity in N implies that $\text{Ric}_N \geq K$ with $N < n$ is a weaker condition than $\text{Ric}_\infty \geq K$. Nevertheless, one can generalize a number of results to this setting, see [KM, Mil, Oh4, Wy] for the positive-definite case and [WW1, WW2] for the Lorentzian case.

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(c) The Riemannian characterization as in Theorem 3.7 is valid also for the weighted Ricci curvature. Take a C^1-vector field V such that $V(x) = v$ and all integral curves of V are geodesic. Then V induces the metric g_V and the weight function $\psi_V := \psi(V)$ on a neighborhood of x, thus we can calculate the weighted Ricci curvature $\mathrm{Ric}_N^{(g_V, \psi_V)}(v)$ for (M, g_V, ψ_V). Since η is geodesic also for g_V and $\dot\eta(t) = V(\eta(t))$ by construction, we deduce that $\mathrm{Ric}_N(v)$ in (4.2) coincides with the Lorentzian counterpart:

$$\mathrm{Ric}_N^{(g_V, \psi_V)}(v) = \mathrm{Ric}^{g_V}(v) + (\psi_V \circ \eta)''(0) - \frac{(\psi_V \circ \eta)'(0)^2}{N - n}.$$

5 Weighted Raychaudhuri equation

Next we consider the *Raychaudhuri equation* on weighted Finsler spacetimes. In the unweighted case, the Finsler Raychaudhuri equation was established in [Min4] along with corresponding singularity theorems. Our approach is inspired by [Ca] on the weighted Lorentzian setting. (A counterpart to the Raychaudhuri equation in the positive-definite case is the Bochner–Weitzenböck formula; for that we refer to [OS] in the Finsler context.)

5.1 Weighted Jacobi and Riccati equations

We begin with the notion of Jacobi and Lagrange tensor fields. We say that a timelike geodesic η has *unit speed* if $F(\dot\eta) \equiv 1$ ($L(\dot\eta) \equiv -1/2$). For simplicity, the covariant derivative of a vector field X along η will be denoted by X'. Observe that this time-differentiation by acting linearly passes to the tensor bundle over η and in particular to endomorphisms E as $E'(P) := E(P)' - E(P')$. We denote by $N_\eta(t) \subset T_{\eta(t)}M$ the n-dimensional subspace $g_{\dot\eta(t)}$-orthogonal to $\dot\eta(t)$.

Definition 5.1 (Jacobi, Lagrange tensor fields) Let $\eta : I \longrightarrow M$ be a timelike geodesic of unit speed.

(1) A smooth tensor field J, giving an endomorphism $\mathsf{J}(t) : N_\eta(t) \longrightarrow N_\eta(t)$ for each $t \in I$, is called a *Jacobi tensor field along η* if we have

$$\mathsf{J}'' + \mathsf{R}\mathsf{J} = 0 \tag{5.1}$$

and $\ker(\mathsf{J}(t)) \cap \ker(\mathsf{J}'(t)) = \{0\}$ holds for all $t \in I$, where $\mathsf{R}(t) := R_{\dot\eta(t)} : N_\eta(t) \longrightarrow N_\eta(t)$ is the curvature endomorphism.

(2) A Jacobi tensor field J is called a *Lagrange tensor field* if

$$(\mathsf{J}')^\mathsf{T}\mathsf{J} - \mathsf{J}^\mathsf{T}\mathsf{J}' = 0 \tag{5.2}$$

holds on I, where the transpose T is taken with respect to $g_{\dot\eta}$.

For $t \in I$ where $\mathsf{J}(t)$ is invertible, note that (5.2) is equivalent to the $g_{\dot\eta}$-symmetry of $\mathsf{J}'\mathsf{J}^{-1}$. At those points we define $\mathsf{B} := \mathsf{J}'\mathsf{J}^{-1}$. Then, multiplying (5.1) by J^{-1} from right, we arrive at the *Riccati equation*

$$\mathsf{B}' + \mathsf{B}^2 + \mathsf{R} = 0. \tag{5.3}$$

For thoroughness, let us explain the precise meaning of (5.1) and (5.2).

Remark 5.2 (a) The equation (5.1) means that, for any $g_{\dot{\eta}}$-parallel vector field $P(t) \in N_\eta(t)$ along η (namely $P' \equiv 0$), $Y(t) := \mathsf{J}(t)(P(t))$ is a Jacobi field along η such that $g_{\dot{\eta}}(Y, \dot{\eta}) = 0$. Then we find from (3.1) that $g_{\dot{\eta}}(Y', \dot{\eta}) = 0$. Thus we have $\mathsf{J}' : N_\eta(t) \longrightarrow N_\eta(t)$ and $\mathsf{B} : N_\eta(t) \longrightarrow N_\eta(t)$.

(b) Proposition 3.4 ensures $R_{\dot{\eta}(t)}(v) \in N_\eta(t)$ for all $v \in T_{\eta(t)}M$. The $g_{\dot{\eta}}$-symmetry in (5.2) means that, given two $g_{\dot{\eta}}$-parallel vector fields $P_1(t), P_2(t) \in N_\eta(t)$ along η, the Jacobi fields $Y_i := \mathsf{J}(P_i)$ satisfy

$$g_{\dot{\eta}}(Y_1', Y_2) - g_{\dot{\eta}}(Y_1, Y_2') = 0 \tag{5.4}$$

on I. Since (5.1) and (3.5) imply $[g_{\dot{\eta}}(Y_1', Y_2) - g_{\dot{\eta}}(Y_1, Y_2')]' \equiv 0$, we have (5.4) for all t if it holds at some t.

We introduce fundamental quantities in the analysis of Jacobi tensor fields along the Lorentz–Finsler treatment of [Min4].

Definition 5.3 (Expansion, shear tensor) Let J be a Jacobi tensor field along a timelike geodesic $\eta : I \longrightarrow M$ of unit speed. For $t \in I$ where $\mathsf{J}(t)$ is invertible, we define the *expansion scalar* by

$$\theta(t) := \mathrm{trace}\big(\mathsf{B}(t)\big)$$

and the *shear tensor* (the traceless part of B) by

$$\sigma(t) := \mathsf{B}(t) - \frac{\theta(t)}{n} I_n,$$

where I_n represents the identity of $N_\eta(t)$.

We proceed to the weighted situation. Recall that ψ is a function on $\overline{\Omega} \setminus \{0\}$ and, along a causal geodesic η, we set $\psi_\eta := \psi(\dot{\eta})$ (see (4.1)). For a Jacobi tensor field J along a timelike geodesic $\eta : I \longrightarrow M$ of unit speed, define the *weighted Jacobi endomorphism* by

$$\mathsf{J}_\psi(t) := \mathrm{e}^{-\psi_\eta(t)/n} \mathsf{J}(t). \tag{5.5}$$

Now we introduce an auxiliary time, the ϵ-*proper time*, defined by

$$\tau_\epsilon := \int \mathrm{e}^{\frac{2(\epsilon-1)}{n}\psi_\eta(t)} \, dt, \tag{5.6}$$

where t is the usual proper time parametrization. Notice that τ_ϵ coincides with the usual proper time for $\epsilon = 1$, and the case of $\epsilon = 0$ was introduced in [WW1]. For brevity the (covariant) derivative in τ_ϵ will be denoted by $*$. For instance,

$$\eta^*(t) := \frac{d[\eta \circ \tau_\epsilon^{-1}]}{d\tau_\epsilon}\big(\tau_\epsilon(t)\big) = \mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta(t)} \dot{\eta}(t).$$

Let us also introduce a weighted counterpart to the *curvature endomorphism*:

$$\mathsf{R}_{(N,\epsilon)}(t) := \mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta(t)} \left\{ \mathsf{R}(t) + \frac{1}{n}\left(\psi_\eta''(t) - \frac{\psi_\eta'(t)^2}{N-n}\right)I_n \right\} \tag{5.7}$$

for $N \neq n$ (compare this with $R_f(t)$ in [Ca, Definition 2.7]). This expression is chosen in such a way that

$$\mathrm{trace}(\mathsf{R}_{(N,\epsilon)}) = \mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta} \mathrm{Ric}_N(\dot{\eta}) = \mathrm{Ric}_N(\eta^*).$$

A straightforward calculation shows the following *weighted Jacobi equation*, which generalizes (5.1).

Lemma 5.4 (Weighted Jacobi equation) *With the notations as above, we have*

$$\mathsf{J}_\psi^{**} + \frac{2\epsilon}{n}\psi_\eta^*\mathsf{J}_\psi^* + \mathsf{R}_{(0,\epsilon)}\mathsf{J}_\psi = 0. \tag{5.8}$$

Proof. Recalling the definition of J_ψ in (5.5), we observe

$$\mathsf{J}_\psi^* = \mathrm{e}^{-\psi_\eta/n}\left(\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{J}' - \frac{\psi_\eta^*}{n}\mathsf{J}\right), \tag{5.9}$$

$$\mathsf{J}_\psi^{**} = \mathrm{e}^{-\psi_\eta/n}\left\{\mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta}\mathsf{J}'' + \frac{1-2\epsilon}{n}\psi_\eta^*\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{J}' - \frac{\psi_\eta^{**}}{n}\mathsf{J} - \frac{\psi_\eta^*}{n}\left(\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{J}' - \frac{\psi_\eta^*}{n}\mathsf{J}\right)\right\}$$

$$= \mathrm{e}^{-\psi_\eta/n}\left(\mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta}\mathsf{J}'' - \frac{2\epsilon}{n}\psi_\eta^*\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{J}' - \frac{\psi_\eta^{**}}{n}\mathsf{J} + \frac{(\psi_\eta^*)^2}{n^2}\mathsf{J}\right).$$

Moreover,

$$\mathsf{R}_{(0,\epsilon)} = \mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta}\mathsf{R} + \frac{1}{n}\left(\psi_\eta^{**} - \frac{2(1-\epsilon)}{n}(\psi_\eta^*)^2 + \frac{(\psi_\eta^*)^2}{n}\right)I_n. \tag{5.10}$$

Therefore we have, with the help of $\mathsf{J}'' + \mathsf{R}\mathsf{J} = 0$ in (5.1),

$$\mathsf{J}_\psi^{**} + \mathsf{R}_{(0,\epsilon)}\mathsf{J}_\psi = -\mathrm{e}^{-\psi_\eta/n}\left(\frac{2\epsilon}{n}\psi_\eta^*\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{J}' - \frac{2\epsilon}{n^2}(\psi_\eta^*)^2\mathsf{J}\right) = -\frac{2\epsilon}{n}\psi_\eta^*\mathsf{J}_\psi^*.$$

$\square$

For $t \in I$ where $\mathsf{J}(t)$ is invertible, we define

$$\mathsf{B}_\epsilon(t) := \mathsf{J}_\psi^*(t)\mathsf{J}_\psi(t)^{-1} = \mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta(t)}\mathsf{B}(t) - \frac{\psi_\eta^*(t)}{n}I_n, \tag{5.11}$$

where we used (5.9) and suppressed the dependence on ψ. Similarly to Lemma 5.4 above, one can show the *weighted Riccati equation* generalizing (5.3) as follows.

Lemma 5.5 (Weighted Riccati equation) *With the notations as above, we have*

$$\mathsf{B}_\epsilon^* + \frac{2\epsilon}{n}\psi_\eta^*\mathsf{B}_\epsilon + \mathsf{B}_\epsilon^2 + \mathsf{R}_{(0,\epsilon)} = 0. \tag{5.12}$$

Proof. We deduce from

$$\mathsf{B}_\epsilon^* = \mathrm{e}^{\frac{4(1-\epsilon)}{n}\psi_\eta}\mathsf{B}' + \frac{2(1-\epsilon)}{n}\psi_\eta^*\mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{B} - \frac{\psi_\eta^{**}}{n}I_n,$$

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(5.10) and $\mathsf{B}' + \mathsf{B}^2 + \mathsf{R} = 0$ in (5.3) that

$$\mathsf{B}_\epsilon^* + \mathsf{B}_\epsilon^2 + \mathsf{R}_{(0,\epsilon)} = \frac{2(1-\epsilon)}{n}\psi_\eta^* e^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{B} - 2e^{\frac{2(1-\epsilon)}{n}\psi_\eta}\frac{\psi_\eta^*}{n}\mathsf{B} + \frac{(\psi_\eta^*)^2}{n^2}I_n - \frac{1-2\epsilon}{n^2}(\psi_\eta^*)^2 I_n$$

$$= -\frac{2\epsilon}{n}\psi_\eta^* e^{\frac{2(1-\epsilon)}{n}\psi_\eta}\mathsf{B} + \frac{2\epsilon}{n^2}(\psi_\eta^*)^2 I_n = -\frac{2\epsilon}{n}\psi_\eta^*\mathsf{B}_\epsilon.$$

$\square$

Observe that for $\epsilon = 0$ both the weighted Jacobi and Riccati equations are simplified to have the same forms as the unweighted situation (compare this with [Ca, Proposition 2.8], adding the factor $e^{\frac{2(1-\epsilon)}{n}\psi_\eta}$ enabled us to remove the extra term appearing there). We define the ϵ-*expansion scalar* by

$$\theta_\epsilon(t) := \mathrm{trace}\big(\mathsf{B}_\epsilon(t)\big) = e^{\frac{2(1-\epsilon)}{n}\psi_\eta(t)}\theta(t) - \psi_\eta^*(t) = e^{\frac{2(1-\epsilon)}{n}\psi_\eta(t)}\big(\theta(t) - \psi_\eta'(t)\big). \qquad (5.13)$$

For $\epsilon = 0$, we may also write $\theta_\psi := \theta_0 = e^{\frac{2}{n}\psi_\eta}(\theta - \psi_\eta')$. Define the ϵ-*shear tensor* by

$$\sigma_\epsilon(t) := \mathsf{B}_\epsilon(t) - \frac{\theta_\epsilon(t)}{n}I_n = e^{\frac{2(1-\epsilon)}{n}\psi_\eta(t)}\sigma(t). \qquad (5.14)$$

Since B is $g_{\dot\eta}$-symmetric, so are B_ϵ and σ_ϵ.

5.2 Raychaudhuri equation

Taking the trace of the weighted Riccati equation (5.12), we obtain the *weighted Raychaudhuri equation* displaying Ric_0 and after a straightforward manipulation the versions displaying Ric_N.

Theorem 5.6 (Timelike weighted Raychaudhuri equation) *Let J be a nonsingular Lagrange tensor field along a future-directed timelike geodesic $\eta : I \longrightarrow M$ of unit speed. Then, for $N = 0$, the ϵ-expansion θ_ϵ satisfies*

$$\theta_\epsilon^* + \frac{2\epsilon}{n}\psi_\eta^*\theta_\epsilon + \frac{\theta_\epsilon^2}{n} + \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_0(\eta^*) = 0 \qquad (5.15)$$

on I. For $N \in (-\infty, +\infty)\backslash\{0, n\}$, θ_ϵ satisfies

$$\theta_\epsilon^* + \left(1 - \epsilon^2\frac{N-n}{N}\right)\frac{\theta_\epsilon^2}{n} + \frac{N(N-n)}{n}\left(\frac{\epsilon\theta_\epsilon}{N} + \frac{\psi_\eta^*}{N-n}\right)^2 + \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_N(\eta^*) = 0, \quad (5.16)$$

and for $N = +\infty$, θ_ϵ satisfies

$$\theta_\epsilon^* + (1 - \epsilon^2)\frac{\theta_\epsilon^2}{n} + \frac{1}{n}(\epsilon\theta_\epsilon + \psi_\eta^*)^2 + \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_\infty(\eta^*) = 0. \qquad (5.17)$$

Proof. The first equation (5.15) is obtained as the trace of (5.12) by noticing

$$\mathrm{trace}(\mathsf{B}_\epsilon^2) = \mathrm{trace}\left(\sigma_\epsilon^2 + \frac{2\theta_\epsilon}{n}\sigma_\epsilon + \frac{\theta_\epsilon^2}{n^2}I_n\right) = \mathrm{trace}(\sigma_\epsilon^2) + \frac{\theta_\epsilon^2}{n}.$$

Then (5.16) follows from (5.15) by comparing Ric_0 and Ric_N. The expression (5.17) for $N = +\infty$ can be derived again from (5.15), or as the limiting case of (5.16). $\square$

The usefulness of (5.16) and (5.17) stands in the possibility of controlling the positivity of the coefficient in front of θ_ϵ^2, as we shall see. Though we did not have a Raychaudhuri equation with this property for $N = n$, we do have a meaningful *Raychaudhuri inequality*.

Proposition 5.7 (Timelike weighted Raychaudhuri inequality) *Let J be a nonsingular Lagrange tensor field along a timelike geodesic $\eta : I \longrightarrow M$ of unit speed. For every $\epsilon \in \mathbb{R}$ and $N \in (-\infty, 0) \cup [n, +\infty]$, we have on I*

$$\theta_\epsilon^* \leq -\mathrm{Ric}_N(\eta^*) - \mathrm{trace}(\sigma_\epsilon^2) - c\theta_\epsilon^2, \tag{5.18}$$

where

$$c = c(N, \epsilon) := \frac{1}{n}\left(1 - \epsilon^2 \frac{N - n}{N}\right). \tag{5.19}$$

Moreover, for $\epsilon = 0$ one can take $N \to 0$ and (5.18) holds with $c = c(0, 0) := 1/n$.

Proof. For $N \in (-\infty, 0) \cup (n, +\infty]$, the inequality (5.18) readily follows from (5.16) or (5.17). The case of $N = n$ is obtained by taking the limit $N \downarrow n$. The case of $N = \epsilon = 0$ is immediate from (5.15). $\qquad\square$

Looking at the condition for $c > 0$, we arrive at a key step for singularity theorems.

Proposition 5.8 (Timelike ϵ-range for convergence) *Given $N \in (-\infty, 0] \cup [n, +\infty]$, take $\epsilon \in \mathbb{R}$ such that*

$$\epsilon = 0 \text{ for } N = 0, \quad |\epsilon| < \sqrt{\frac{N}{N - n}} \text{ for } N \neq 0. \tag{5.20}$$

Let $\eta : (a, b) \longrightarrow M$ be a timelike geodesic of unit speed. Assume that $\mathrm{Ric}_N(\eta^*) \geq 0$ holds on (a, b), and let J be a Lagrange tensor field along η such that for some $t_0 \in (a, b)$ we have $\theta_\epsilon(t_0) < 0$. Then we have $\det \mathsf{J}(t) = 0$ for some $t \in [t_0, t_0 + s_0]$ provided that $t_0 + s_0 < b$, where we set, with $c = c(N, \epsilon) > 0$ in (5.19),

$$s_0 := \tau_\epsilon^{-1}\left(\tau_\epsilon(t_0) - \frac{1}{c\theta_\epsilon(t_0)}\right) - t_0. \tag{5.21}$$

Similarly, if $\theta_\epsilon(t_0) > 0$, then we have $\det \mathsf{J}(t) = 0$ for some $t \in [t_0 + s_0, t_0]$ provided that $t_0 + s_0 > a$ for s_0 above.

Note that the assumption $\mathrm{Ric}_N(\eta^*) \geq 0$ is equivalent to $\mathrm{Ric}_N(\dot{\eta}) \geq 0$, and that $\theta_\epsilon(t_0) < 0$ is equivalent to $\theta_\psi(t_0) < 0$ (corresponding to $\epsilon = 0$). When $N = n$, the condition (5.20) is void and we can take any $\epsilon \in \mathbb{R}$.

Proof. Let us consider the former case of $\theta_\epsilon(t_0) < 0$, then $s_0 > 0$. Observe that $\theta_\epsilon(t_0)^{-1} = c(\tau_\epsilon(t_0) - \tau_\epsilon(t_0 + s_0))$. Assume to the contrary that $[t_0, t_0 + s_0] \subset (a, b)$ and $\det \mathsf{J}(t) \neq 0$ for all $t \in [t_0, t_0 + s_0]$. Since σ_ϵ is $g_{\dot{\eta}}$-symmetric, we deduce from (5.18) that $\theta_\epsilon^* \leq -c\theta_\epsilon^2 \leq 0$. Hence we have $\theta_\epsilon < 0$ on $[t_0, b)$ and, moreover, $[\theta_\epsilon^{-1}]^* \geq c$. Integrating this inequality from t_0 to $t \in (t_0, t_0 + s_0)$ yields

$$\theta_\epsilon(t) \leq \frac{1}{\theta_\epsilon(t_0)^{-1} + c(\tau_\epsilon(t) - \tau_\epsilon(t_0))} = \frac{1}{c(\tau_\epsilon(t) - \tau_\epsilon(t_0 + s_0))} < 0.$$

This implies $\lim_{t \uparrow t_0 + s_0} \theta_\epsilon(t) = -\infty$. Then, since

$$\theta_\epsilon = \mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta} \operatorname{trace}(\mathsf{B}) - \psi_\eta^* = \mathrm{e}^{\frac{2(1-\epsilon)}{n}\psi_\eta} \frac{(\det \mathsf{J})'}{\det \mathsf{J}} - \psi_\eta^*,$$

it necessarily holds that $\det \mathsf{J}(t_0 + s_0) = 0$, a contradiction. The case of $\theta_\epsilon(t_0) > 0$ (where $s_0 < 0$) is proved analogously. $\qquad\square$

Remark 5.9 (Admissible range of ϵ) The condition (5.20) for ϵ gives an important insight on the relation between N and the admissible range of ϵ. Observe that $\epsilon = 0$ as in [WW1, WW2] is allowed for any $N \in (-\infty, 0] \cup [n, +\infty]$, while $\epsilon = 1$ corresponding to the usual proper time is allowed only for $N \in [n, +\infty)$.

5.3 Completenesses

Inspired by Proposition 5.8, we introduce a completeness condition associated with the ϵ-proper time in (5.6).

Definition 5.10 (Timelike ϵ-completeness) Let $\eta : (a, b) \longrightarrow M$ be an inextendible timelike geodesic. We say that η is *future ϵ-complete* if $\lim_{t \to b} \tau_\epsilon(t) = +\infty$. Similarly, we say that it is *past ϵ-complete* if $\lim_{t \to a} \tau_\epsilon(t) = -\infty$. The spacetime (M, L, ψ) is said to be *future timelike ϵ-complete* if every inextendible timelike geodesic is future ϵ-complete, and similar in the past case.

If $\epsilon = 1$ one simply speaks of the (geodesic) completeness with respect to the usual proper time (namely $b = +\infty$), while if $\epsilon = 0$ one speaks of the *ψ-completeness* introduced by Wylie [Wy] in the Riemannian case and by Woolgar–Wylie [WW1, WW2] in the Lorentzian case. Note also that the ϵ-completeness was tacitly assumed in [Ca, GW] through the upper boundedness of ψ (see Lemma 5.12 below). The following corollary is immediate from Proposition 5.8.

Corollary 5.11 *Let $N \in (-\infty, 0] \cup [n, +\infty]$ and J be a Lagrange tensor field along a future inextendible timelike geodesic $\eta : (a, b) \longrightarrow M$ satisfying $\operatorname{Ric}_N(\dot\eta) \geq 0$. Assume that η is future ϵ-complete for some $\epsilon \in \mathbb{R}$ that belongs to the timelike ϵ-range in (5.20), and that $\theta_\epsilon(t_0) < 0$ for some $t_0 \in (a, b)$. Then η develops a point $t \in (t_0, b)$ where $\det \mathsf{J}(t) = 0$.*

Proof. It suffices to show that one can always find $s_0 \in (0, b - t_0)$ satisfying $\theta_\epsilon(t_0)^{-1} = c(\tau_\epsilon(t_0) - \tau_\epsilon(t_0 + s_0))$. This clearly holds true under the future ϵ-completeness. $\qquad\square$

We remark that the future ϵ-completeness clearly requires the future inextendability, but not necessarily $b = +\infty$. The next lemma is an immediate consequence of Definition 5.10, see [WW1, Lemma 1.3].

Lemma 5.12 *Let $\epsilon < 1$. If ψ is bounded above, then the future (resp. past) completeness implies the future (resp. past) ϵ-completeness. If ψ_η is non-increasing along every timelike geodesic η, then the future completeness implies the future ϵ-completeness. Similarly, if ψ_η is non-decreasing along every timelike geodesic η, then the past completeness implies the past ϵ-completeness.*

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5.4   Timelike geodesic congruence from a point

In this subsection, we study timelike geodesic congruences issued from a point. A similar analysis can be applied to timelike geodesic congruences that are orthogonal to a spacelike hypersurface. Our objective is to show that they determine Lagrange tensor fields. This subsection does not use the weight.

Proposition 5.13 *Let (M, L) be a Finsler spacetime, and let $\eta : [0, l] \longrightarrow M$ be a timelike geodesic of unit speed. Suppose that there is no point conjugate to $\eta(0)$ along η. Then there exists a Lagrange tensor field $\mathsf{J}(t) : N_\eta(t) \longrightarrow N_\eta(t)$ such that $\mathsf{J}(0) = 0$, $\mathsf{J}'(0) = I_n$ and $\det \mathsf{J}(t) > 0$ for all $t \in (0, l]$.*

Proof. Let $x = \eta(0)$ and $v = \dot\eta(0)$. For each $w \in T_x M$, we consider the vector field $Y_w := d(\exp_x)_{tv}(tw) \in T_{\eta(t)}M$. By construction it is a Jacobi field along η satisfying $Y_w(0) = 0$ and $Y_w'(0) = w$, where we denote by Y_w' the covariant derivative $D_{\dot\eta}^{\dot\eta} Y_w$ along η.

We shall define an endomorphism $\mathsf{J}(t) : N_\eta(t) \longrightarrow N_\eta(t)$ (in a way similar to Remark 5.2). Given $w \in N_\eta(t)$, we extend it to the $g_{\dot\eta}$-parallel vector field P along η (namely $P' \equiv 0$), and then define $\mathsf{J}(t)(w) := Y_{P(0)}(t)$. Note that the image of $\mathsf{J}(t)$ is indeed included in $N_\eta(t)$, since it follows from (3.1), (3.4) and Proposition 3.4 that

$$\frac{d^2}{dt^2}\big[g_{\dot\eta}(\dot\eta, Y_{P(0)})\big] = -g_{\dot\eta}\big(\dot\eta, R_{\dot\eta}(Y_{P(0)})\big) \equiv 0.$$

Since P is $g_{\dot\eta}$-parallel, we have

$$\mathsf{J}'(P) = \mathsf{J}(P)' - \mathsf{J}(P') = Y_{P(0)}', \qquad \mathsf{J}''(P) = (\mathsf{J}'(P))' - \mathsf{J}'(P') = Y_{P(0)}'' = -R_{\dot\eta}(Y_{P(0)}).$$

Therefore J satisfies the equation $\mathsf{J}'' + \mathsf{R}\mathsf{J} = 0$. Since $\eta(0)$ has no conjugate point by hypothesis and $Y_{P(0)}(0) = 0$, the map $\mathsf{J}(t)$ has maximum rank and hence invertible for every $t \in (0, l]$. In particular, $\ker(\mathsf{J}(t)) \cap \ker(\mathsf{J}'(t)) = \{0\}$ for all $t \in [0, l]$, thus J is a Jacobi tensor field.

Next, we prove that $\mathsf{J}^\mathsf{T}\mathsf{J}'$ is $g_{\dot\eta}$-symmetric. To this end, observe that

$$\frac{d}{dt}\big[g_{\dot\eta}(Y_{w_1}', Y_{w_2}) - g_{\dot\eta}(Y_{w_1}, Y_{w_2}')\big] = -g_{\dot\eta}\big(R_{\dot\eta}(Y_{w_1}), Y_{w_2}\big) + g_{\dot\eta}\big(Y_{w_1}, R_{\dot\eta}(Y_{w_2})\big) = 0$$

for $w_1, w_2 \in N_\eta(0)$, where we used (3.5). Combining this with $Y_{w_1}(0) = Y_{w_2}(0) = 0$ yields $g_{\dot\eta}(Y_{w_1}', Y_{w_2}) = g_{\dot\eta}(Y_{w_1}, Y_{w_2}')$. This shows that $\mathsf{J}^\mathsf{T}\mathsf{J}'$ is indeed symmetric (and hence J is a Lagrange tensor field) because, for the $g_{\dot\eta}$-parallel vector field P_i with $P_i(0) = w_i$ ($i = 1, 2$),

$$g_{\dot\eta}\big(P_1, \mathsf{J}^\mathsf{T}\mathsf{J}'(P_2)\big) = g_{\dot\eta}(Y_{w_1}, Y_{w_2}') = g_{\dot\eta}(Y_{w_1}', Y_{w_2}) = g_{\dot\eta}\big(\mathsf{J}^\mathsf{T}\mathsf{J}'(P_1), P_2\big).$$

Finally, we find by construction that $\mathsf{J}(0) = 0$ and $\mathsf{J}'(0) = I_n$, where I_n is the identity of $N_\eta(0)$. Thus we obtain, for t sufficiently close to 0, $\det \mathsf{J}(t) = \det(tI_n + o(t)) > 0$. By the continuity and non-degeneracy of J, $\det \mathsf{J}(t)$ is indeed positive for every t. $\qquad\square$

5.5 Comparison theorems

This subsection is devoted to the (weighted) Lorentz–Finsler analogues of two fundamental comparison theorems in Riemannian geometry, the *Bonnet–Myers* and *Cartan–Hadamard theorems*. We refer to [Ch] for the Riemannian case, [BCS] for the Finsler case, and to [BEE, Chapter 11] for the Lorentzian case.

Though we will give precise proofs, it is also possible to reduce those theorems to the (weighted) Lorentzian setting by using Theorem 3.7. We refer to [Oh5] for details.

Proposition 5.14 (Weighted Bishop inequality) *Let* J *be a nonsingular Lagrange tensor field along a timelike geodesic* $\eta : I \longrightarrow M$ *of unit speed. Let* $N \in (-\infty, 0] \cup [n, +\infty]$ *and* $\epsilon \in \mathbb{R}$ *be in the timelike* ϵ-range *as in* (5.20). *Defining* $\xi := |\det \mathsf{J}_\psi|^c$ *with* $c > 0$ *in* (5.19)*, we have on* I

$$\xi^{**} \leq -c\xi \operatorname{Ric}_N(\eta^*).$$

Proof. Note that J being nonsingular ensures that $\det \mathsf{J}_\psi$ is always positive or always negative. If $\det \mathsf{J}_\psi > 0$, then we deduce from $\log \xi = c \log(\det \mathsf{J}_\psi)$ that

$$\frac{\xi^*}{\xi} = c\frac{(\det \mathsf{J}_\psi)^*}{\det \mathsf{J}_\psi} = c \operatorname{trace}(\mathsf{J}_\psi^* \mathsf{J}_\psi^{-1}) = c \operatorname{trace}(\mathsf{B}_\epsilon) = c\theta_\epsilon.$$

Thus $\xi^{**}\xi - (\xi^*)^2 = c\theta_\epsilon^* \xi^2$, and then the weighted Raychaudhuri inequality (5.18) yields

$$\xi^{**} \leq -c\xi\{\operatorname{Ric}_N(\eta^*) + \operatorname{trace}(\sigma_\epsilon^2)\} \leq -c\xi \operatorname{Ric}_N(\eta^*).$$

In the case of $\det \mathsf{J}_\psi < 0$, we have $\log \xi = c \log(-\det \mathsf{J}_\psi)$ and can argue similarly. $\square$

An interesting case is $N \in [n, +\infty)$, $\epsilon = 1$ and $c = 1/N$, for it corresponds to the usual proper time parametrization and leads us to the weighted Bonnet–Myers theorem.

We are going to need some auxiliary geometric properties of Finsler spacetimes. The existence of convex neighborhoods implies that several standard proofs from causality theory, originally developed for Lorentzian spacetimes, pass unaltered to the Lorentz–Finsler framework (we refer to [Min2]). An important result is a generalization of the *Avez–Seifert connectedness theorem* as follows (see [Min4, Proposition 6.9], it actually holds under much weaker regularity assumptions on the metric as in [Min6, Theorem 2.55]).

Theorem 5.15 (Avez–Seifert theorem) *In a globally hyperbolic Finsler spacetime, any two causally related points are connected by a maximizing causal geodesic.*

It should be recalled here that in a Finsler spacetime two points connected by a causal curve which is not a lightlike geodesic are necessarily connected by a timelike curve, see [Min2, Lemma 2] or [Min6, Theorem 2.16]. Thus a lightlike curve which is maximizing is necessarily a lightlike geodesic.

We also need the following (see [Min4, Proposition 5.1] and also [Min7, Theorem 6.16]).

Proposition 5.16 (Beyond conjugate points) *In a Finsler spacetime, a causal geodesic* $\eta : [a, b] \longrightarrow M$ *cannot be maximizing if it contains an internal point conjugate to* $\eta(a)$. *Similarly, a causal geodesic* $\eta : (a, b) \longrightarrow M$ *cannot be maximizing if it contains a pair of mutually conjugate points.*

Define the *timelike diameter* of a Finsler spacetime (M, L) by

$$\text{diam}(M) := \sup\{d(x, y) \mid x, y \in M\}.$$

By the definition of the distance function, given $x, y \in M$ and any causal curve η from x to y, we have $\ell(\eta) \leq d(x, y)$. Hence, if $\text{diam}(M) < \infty$, then every timelike geodesic has finite length and (M, L) is timelike geodesically incomplete (see [BEE, Remark 11.2]).

Now we state a weighted Lorentz–Finsler analogue of the Bonnet–Myers theorem.

Theorem 5.17 (Weighted Bonnet–Myers theorem) *Let (M, L, ψ) be globally hyperbolic of dimension $n + 1 \geq 2$. If $\text{Ric}_N \geq K$ holds in timelike directions for some $N \in [n, +\infty)$ and $K > 0$, then we have*

$$\text{diam}(M) \leq \pi\sqrt{\frac{N}{K}}.$$

Proof. Suppose that the claim is not true, then we can find two causally related points $x, y \in M$ such that $d(x, y) > \pi\sqrt{N/K}$. By Theorem 5.15, there is a timelike geodesic $\eta : [0, l] \longrightarrow M$ with $\eta(0) = x$, $\eta(l) = y$, $F(\dot\eta) = 1$ and $l = \ell(\eta) = d(x, y) > \pi\sqrt{N/K}$. We are going to prove that, due to $l > \pi\sqrt{N/K}$, there necessarily exists a conjugate point to $\eta(0)$. Then Proposition 5.16 gives the desired contradiction.

Now we assume that there is no conjugate point to $\eta(0)$. Then Proposition 5.13 applies and we have a Lagrange tensor field J with the properties given there. Define $\mathsf{J}_\psi = e^{-\psi_\eta/n}\mathsf{J}$ and $\xi = (\det \mathsf{J}_\psi)^{1/N}$ (i.e., $\epsilon = 1$), and notice that $\xi > 0$ for $t > 0$. Then by Proposition 5.14 with $\epsilon = 1$, we have

$$N\xi''(t) \leq -\xi(t)\text{Ric}_N\big(\dot\eta(t)\big) \leq -K\xi(t). \tag{5.22}$$

Putting $\mathsf{s}(t) := \sin(t\sqrt{K/N})$, we obtain $(\xi'\mathsf{s} - \xi\mathsf{s}')' \leq 0$.

Let us prove that $\lim_{t \to 0}(\xi'\mathsf{s} - \xi\mathsf{s}')(t) \leq 0$, from which it follows $\xi'\mathsf{s} - \xi\mathsf{s}' \leq 0$. Notice that $\xi \in C^2((0, l]) \cap C^0([0, l])$ and $\xi(0) = 0$ (by $\mathsf{J}(0) = 0$), so we need only to prove $\lim_{t \to 0} \xi'(t)t \leq 0$ where $\xi'(t)t$ needs not be C^1 at 0. We deduce from (5.22) that ξ is concave in t near $t = 0$. Let $f(t) := \xi(t) - t\xi'(t)$ be the ordinate of the intersection between the tangent to the graph of ξ at $(t, \xi(t))$ and the vertical axis. By the concavity of ξ, f is non-decreasing in t and $f(t) \geq \xi(0) = 0$. Therefore the limit $\lim_{t \to 0} f(t)$ exists and we obtain

$$\lim_{t \to 0} t\xi'(t) = -\lim_{t \to 0} f(t) \leq 0.$$

Since $\xi'\mathsf{s} - \xi\mathsf{s}' \leq 0$, the ratio $\xi(t)/\mathsf{s}(t)$ is non-increasing in $t \in (0, \pi\sqrt{N/K})$. Hence $\xi(t_0) = 0$ necessarily holds at some $t_0 \in (0, \pi\sqrt{N/K}]$. This contradicts the assumed absence of conjugate points, therefore we conclude that $\text{diam}(M) \leq \pi\sqrt{N/K}$. $\qquad\square$

We remark that the unweighted situation is included in the above theorem as $\psi = 0$ and then we have $\text{diam}(M) \leq \pi\sqrt{n/K} = \pi\sqrt{(\dim M - 1)/K}$.

We end this section by giving a Lorentz–Finsler version of the Cartan–Hadamard theorem, that we obtain in the unweighted case only.

Theorem 5.18 (Cartan–Hadamard theorem) *Let (M, L) be a globally hyperbolic Finsler spacetime whose flag curvature $\mathbf{K}(v, w)$ is nonpositive for every $v \in \Omega_x$ and linearly independent $w \in T_x M$. Then every causal geodesic does not have conjugate points.*

We remark that our flag curvature has the opposite sign to [BEE], thus we are considering the nonpositive curvature (similarly to the Riemannian or Finsler case).

Proof. Assume that there is a timelike geodesic $\eta : [0, l] \longrightarrow M$ and a nontrivial Jacobi field Y along η such that $Y(0)$ and $Y(l)$ vanish. We will denote by Y' the covariant derivative $D_{\dot\eta}^{\dot\eta} Y$ along η. We deduce from

$$\frac{d^2}{dt^2}\big[g_{\dot\eta}(\dot\eta, Y)\big] = -g_{\dot\eta}\big(\dot\eta, R_{\dot\eta}(Y)\big) = 0$$

that $g_{\dot\eta}(\dot\eta(t), Y(t))$ is affine in t, but it vanishes at $t = 0, l$. This implies that $g_{\dot\eta}(\dot\eta, Y) \equiv 0$ and $g_{\dot\eta}(\dot\eta, Y') \equiv 0$. Hence Y and Y' are $g_{\dot\eta}$-spacelike and, in particular, $g_{\dot\eta}(Y, Y) \geq 0$ as well as $g_{\dot\eta}(Y', Y') \geq 0$. The assumption implies $g_v\big(w, R_v(w)\big) \leq 0$ for $v \in \Omega_x$ and $w \in T_x M$ (which by continuity implies the same inequality for $v \in \overline{\Omega}_x$). Thus we have

$$\frac{d^2}{dt^2}\big[g_{\dot\eta}(Y, Y)\big] = 2g_{\dot\eta}(Y', Y') - 2g_{\dot\eta}\big(Y, R_{\dot\eta}(Y)\big) \geq 0.$$

Therefore $g_{\dot\eta}(Y, Y)$ is a nonnegative convex function vanishing at $t = 0, l$, and hence $g_{\dot\eta}(Y, Y) = 0$. This implies that Y vanishes on entire $[0, l]$, a contradiction.

For a lightlike geodesic η, we obtain $g_{\dot\eta}(Y, Y) = 0$ by the same argument, and then $Y \equiv 0$ if $Y(t)$ is $g_{\dot\eta}$-spacelike at some $t \in (0, l)$. In the case where Y is always $g_{\dot\eta}$-lightlike, since $g_{\dot\eta}(\dot\eta, Y) = 0$, we have $Y(t) = f(t)\dot\eta(t)$ for some function f with $f(0) = f(l) = 0$. Combining this with $f'' = 0$ following from the Jacobi equation of Y, we have $f \equiv 0$. □

In the Riemannian or Finsler setting, the absence of conjugate points yields that the exponential map $\exp_x : T_x M \longrightarrow M$ is a covering and, if M is simply-connected, $\exp_x$ is a diffeomorphism. The Lorentzian case is not as simple as such since Theorem 5.18 is concerned with only causal geodesics. See [BEE, Section 11.3] for further discussions.

6 Null case

The arguments in Subsections 5.1–5.3 can be extended to lightlike geodesics. We will keep the same notations τ_ϵ and c for quantities that are just analogous to those appearing in the timelike case (compare (6.2) and (6.5) in this section with (5.6) and (5.19), respectively), hoping that this choice will cause no confusion.

Let $\eta : I \longrightarrow M$ be a future-directed lightlike geodesic, i.e., $L(\dot\eta) = 0$ and $\dot\eta \neq 0$. Then $N_\eta(t) \subset T_{\eta(t)} M$ is similarly defined as the n-dimensional subspace $g_{\dot\eta(t)}$-orthogonal to $\dot\eta(t)$, but in this case $\dot\eta(t) \in N_\eta(t)$. Thus it is convenient to work with the quotient space

$$Q_\eta(t) := N_\eta(t)/\dot\eta(t).$$

The metric $g_{\dot\eta}$ induces the positive-definite metric h on this quotient bundle over η. It can be shown (see [Min4]) that the covariant derivative $D_{\dot\eta}^{\dot\eta}$ is well defined over this quotient,

and it can be extended linearly over the space of endomorphisms of $Q_\eta(t)$. It is important to observe that this vector space is $(n-1)$-dimensional, so its identity I_{n-1} has a trace which equals $n - 1$. This fact explains why in passing from the timelike to the null case we get the replacements $n \mapsto n - 1$ and $N \mapsto N - 1$ in several formulas. Jacobi and Lagrange tensor fields are endomorphisms of this space but are otherwise defined in the usual way (Definition 5.1). For instance, a Jacobi tensor field J satisfies

$$\mathsf{J}'' + \mathsf{R}\mathsf{J} = 0 \tag{6.1}$$

(where $'$ is the mentioned covariant derivative on the quotient space) and $\ker(\mathsf{J}(t)) \cap \ker(\mathsf{J}'(t)) = \{0\}$ (this 0 belongs to $Q_\eta(t)$, if we work with endomorphisms of $N_\eta(t)$ then we would have $\mathbb{R}\dot\eta(t)$ on the right-hand side). In (6.1), $\mathsf{R} : Q_\eta \longrightarrow Q_\eta$ is the h-symmetric curvature endomorphism. Then $\mathsf{B} := \mathsf{J}'\mathsf{J}^{-1}$ is also an h-symmetric endomorphism of Q_η, and σ and θ are its trace and traceless parts (similarly to Definition 5.3), see [Min4] for details.

Analogous to the ϵ-proper time (5.6) in the timelike case, along a lightlike geodesic η we define

$$\tau_\epsilon := \int \mathrm{e}^{\frac{2(\epsilon-1)}{n-1}\psi_\eta(t)} \, dt. \tag{6.2}$$

Similarly to the previous section, we denote by $*$ the (covariant) derivative in τ_ϵ, thus $\eta^*(t) = \mathrm{e}^{\frac{2(1-\epsilon)}{n-1}\psi_\eta(t)}\dot\eta(t)$. The *weighted Jacobi endomorphism*

$$\mathsf{J}_\psi(t) := \mathrm{e}^{-\psi_\eta(t)/(n-1)}\mathsf{J}(t)$$

and the *curvature endomorphism*

$$\mathsf{R}_{(N,\epsilon)}(t) := \mathrm{e}^{\frac{4(1-\epsilon)}{n-1}\psi_\eta(t)} \left\{ \mathsf{R}(t) + \frac{1}{n-1}\left(\psi_\eta''(t) - \frac{\psi_\eta'(t)^2}{N-n} \right)I_{n-1} \right\}$$

are defined in the same way as well. Notice that $\mathrm{trace}(\mathsf{R}_{(N,\epsilon)}) = \mathrm{Ric}_N(\eta^*)$. The same calculation as Lemma 5.4 yields the *weighted Jacobi equation*

$$\mathsf{J}_\psi^{**} + \frac{2\epsilon}{n-1}\psi_\eta^*\mathsf{J}_\psi^* + \mathsf{R}_{(1,\epsilon)}\mathsf{J}_\psi = 0,$$

where we remark that $\mathsf{R}_{(1,\epsilon)}$ is employed instead of $\mathsf{R}_{(0,\epsilon)}$ in (5.8).

For $t \in I$ where $\mathsf{J}(t)$ is invertible, we define

$$\mathsf{B}_\epsilon := \mathsf{J}_\psi^*\mathsf{J}_\psi^{-1} = \mathrm{e}^{\frac{2(1-\epsilon)}{n-1}\psi_\eta} \left(\mathsf{B} - \frac{\psi_\eta'}{n-1}I_{n-1} \right).$$

Then the *weighted Riccati equation*

$$\mathsf{B}_\epsilon^* + \frac{2\epsilon}{n-1}\psi_\eta^*\mathsf{B}_\epsilon + \mathsf{B}_\epsilon^2 + \mathsf{R}_{(1,\epsilon)} = 0 \tag{6.3}$$

is obtained similarly to Lemma 5.5. We also define the *ϵ-expansion scalar*

$$\theta_\epsilon(t) := \mathrm{trace}\big(\mathsf{B}_\epsilon(t)\big) = \mathrm{e}^{\frac{2(1-\epsilon)}{n-1}\psi_\eta(t)}\big(\theta(t) - \psi_\eta'(t)\big)$$

and the ϵ-*shear tensor*

$$\sigma_\epsilon(t) := \mathsf{B}_\epsilon(t) - \frac{\theta_\epsilon(t)}{n-1} I_{n-1} = \mathrm{e}^{\frac{2(1-\epsilon)}{n-1}\psi_\eta(t)}\sigma(t).$$

Taking the trace of the weighted Riccati equation (6.3), we get the *weighted Raychaudhuri equation*, in the same manner as Theorem 5.6.

Theorem 6.1 (Null weighted Raychaudhuri equation) *Let* J *be a nonsingular Lagrange tensor field along a future-directed lightlike geodesic* $\eta : I \longrightarrow M$. *Then, for* $N = 1$, *the* ϵ-*expansion* θ_ϵ *satisfies*

$$\theta_\epsilon^* + \frac{2\epsilon}{n-1}\psi_\eta^*\theta_\epsilon + \frac{\theta_\epsilon^2}{n-1} + \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_1(\eta^*) = 0$$

on I. *For* $N \in (-\infty, +\infty)\setminus\{1,n\}$, *it satisfies*

$$\theta_\epsilon^* + \left(1 - \epsilon^2\frac{N-n}{N-1}\right)\frac{\theta_\epsilon^2}{n-1} + \frac{(N-1)(N-n)}{n-1}\left(\frac{\epsilon\theta_\epsilon}{N-1} + \frac{\psi_\eta^*}{N-n}\right)^2$$
$$+ \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_N(\eta^*) = 0,$$

and for $N = +\infty$ *it satisfies*

$$\theta_\epsilon^* + (1-\epsilon^2)\frac{\theta_\epsilon^2}{n-1} + \frac{1}{n-1}(\epsilon\theta_\epsilon + \psi_\eta^*)^2 + \mathrm{trace}(\sigma_\epsilon^2) + \mathrm{Ric}_\infty(\eta^*) = 0.$$

Once again the usefulness of these equations stands in the possibility of controlling the positivity of the coefficient in front of θ_ϵ^2. The analogues to Propositions 5.7 and 5.8 hold as follows.

Proposition 6.2 (Null weighted Raychaudhuri inequality) *Let* J *be a nonsingular Lagrange tensor field along a lightlike geodesic* $\eta : I \longrightarrow M$. *For every* $\epsilon \in \mathbb{R}$ *and* $N \in (-\infty, 1) \cup [n, +\infty]$, *we have on* I

$$\theta_\epsilon^* \leq -\mathrm{Ric}_N(\eta^*) - \mathrm{trace}(\sigma_\epsilon^2) - c\theta_\epsilon^2, \tag{6.4}$$

where

$$c = c(N, \epsilon) = \frac{1}{n-1}\left(1 - \epsilon^2\frac{N-n}{N-1}\right). \tag{6.5}$$

Moreover, for $\epsilon = 0$ *one can take* $N \to 1$ *and* (6.4) *holds with* $c = c(1,0) = 1/(n-1)$.

Proposition 6.3 (Null ϵ-range for convergence) *Given* $N \in (-\infty, 1] \cup [n, +\infty]$, *take* $\epsilon \in \mathbb{R}$ *such that*

$$\epsilon = 0 \text{ for } N = 1, \quad |\epsilon| < \sqrt{\frac{N-1}{N-n}} \text{ for } N \neq 1. \tag{6.6}$$

Let $\eta : (a,b) \longrightarrow M$ *be a lightlike geodesic. Assume that* $\mathrm{Ric}_N(\eta^*) \geq 0$ *holds on* (a,b), *and let* J *be a Lagrange tensor field along* η *such that for some* $t_0 \in (a,b)$ *we have* $\theta_\epsilon(t_0) < 0$. *Then we have* $\det \mathsf{J}(t) = 0$ *for some* $t \in [t_0, t_0 + s_0]$ *provided that* $t_0 + s_0 < b$, *where* c *and* s_0 *are from* (6.5) *and* (5.21), *respectively.*

Similarly, if $\theta_\epsilon(t_0) > 0$, *then we have* $\det \mathsf{J}(t) = 0$ *for some* $t \in [t_0 + s_0, t_0]$ *provided that* $t_0 + s_0 > a$.

Similarly to Remark 5.9, note that in (6.6) $\epsilon = 0$ is allowed for any N, while $\epsilon = 1$ is allowed only for $N \in [n, +\infty)$. We proceed to the study of completeness conditions.

Definition 6.4 (Null ϵ-completeness) Let $\eta : (a, b) \longrightarrow M$ be an inextendible light-like geodesic. We say that η is *future ϵ-complete* (resp. *past ϵ-complete*) if $\lim_{t \to b} \tau_\epsilon(t) = +\infty$ (resp. $\lim_{t \to a} \tau_\epsilon(t) = -\infty$). The spacetime (M, L, ψ) is said to be *future null ϵ-complete* if every lightlike geodesic is future ϵ-complete, and similar in the past case.

The next corollary is obtained similarly to Corollary 5.11.

Corollary 6.5 *Let $N \in (-\infty, 1] \cup [n, +\infty]$ and J be a Lagrange tensor field along a future inextendible lightlike geodesic $\eta : (a, b) \longrightarrow M$ satisfying $\mathrm{Ric}_N(\dot{\eta}) \geq 0$. Assume that η is future ϵ-complete for some $\epsilon \in \mathbb{R}$ that satisfies (6.6), and that $\theta_\epsilon(t_0) < 0$ for some $t_0 \in (a, b)$. Then η develops a point $t \in (t_0, b)$ where $\det \mathsf{J}(t) = 0$.*

7 Incomplete or conjugate

In this section we show that, under some genericity and convergence conditions, every timelike or lightlike geodesic is either incomplete or including a pair of conjugate points. The following notion will play an essential role.

Definition 7.1 (Genericity conditions) Let $\eta : (a, b) \longrightarrow M$ be a timelike geodesic of unit speed. We say that the *genericity condition* holds along η if there exists $t_1 \in (a, b)$ such that $\mathsf{R}(t_1) \neq 0$, where $\mathsf{R}(t) = R_{\dot{\eta}(t)} : N_\eta(t) \longrightarrow N_\eta(t)$. We say that (M, L, ψ) satisfies the *timelike genericity condition* if the genericity condition holds along every inextendible timelike geodesic. Similarly, we define the *null genericity condition* where this time we use the curvature endomorphism on the quotient space Q_η. We say that (M, L, ψ) satisfies the *causal genericity condition* if it satisfies both the timelike and null genericity conditions.

Remark 7.2 This is the standard genericity condition for Lorentz–Finsler geometry (see [Min4]) which generalizes that of Lorentzian geometry (see for instance [BEE]).

In the timelike case, we need to introduce a weighted version only in the extremal case $N = 0$, where we replace R with $\mathsf{R}_{(0,0)}$ from (5.7) similarly to [Ca, WW2], see Remarks 7.5, 7.13 for further discussions. Also for $N \neq 0$, we could use the weighted version in the next results, Lemma 7.4 and Proposition 7.6, with no alteration in the conclusions. This is because in the relevant step of the proof one observes that $\psi'_\eta = 0$ and hence all the curvature endomorphisms coincide up to a multiplicative factor.

In the null case, we need a weighted version only in the extremal case $N = 1$, where we replace R with $\mathsf{R}_{(1,0)}$. Again for $N \neq 1$, we could use the weighted version in the next results with no alteration in the conclusions.

Definition 7.3 Let $\eta : (a, b) \longrightarrow M$ be an inextendible timelike geodesic of unit speed. For $t \in (a, b)$, define $L_+(t)$ (resp. $L_-(t)$) as the collection of all Lagrange tensor fields J along η such that $\mathsf{J}(t) = I_n$ and $\theta_1(t) \geq 0$ (resp. $\theta_1(t) \leq 0$).

Recall from (5.13) that $\theta_1 = \theta - \psi'_\eta$ and that $\theta_1(t) \geq 0$ is equivalent to $\theta_\epsilon(t) \geq 0$ regardless of the choice of ϵ.

Lemma 7.4 *Let $N \in (-\infty, 0) \cup [n, +\infty]$ and $\eta : (a, b) \longrightarrow M$ be an inextendible timelike geodesic of unit speed such that $\mathrm{Ric}_N(\dot{\eta}) \geq 0$ on (a, b) and $\mathsf{R}(t_1) \neq 0$ for some $t_1 \in \mathbb{R}$.*

(i) *Suppose that η is future ϵ-complete where $\epsilon \in \mathbb{R}$ belongs to the timelike ϵ-range in (5.20). Then, for any $\mathsf{J} \in L_-(t_1)$, there exists some $t \in (t_1, b)$ such that $\det \mathsf{J}(t) = 0$.*

(ii) *Similarly, if η is past ϵ-complete for ϵ in (5.20), then for any $\mathsf{J} \in L_+(t_1)$ there exists some $t \in (a, t_1)$ such that $\det \mathsf{J}(t) = 0$.*

Proof. Since the proofs are similar, we prove only (i). The condition $\mathsf{J} \in L_-(t_1)$ implies $\theta_\epsilon(t_1) \leq 0$. If there is some $t_0 \geq t_1$ such that $\theta_\epsilon(t_0) < 0$, then Corollary 5.11 shows the existence of $t > t_1$ with $\det \mathsf{J}(t) = 0$. Thus we assume $\theta_\epsilon(t) \geq 0$ for all $t \geq t_1$.

It follows from the Raychaudhuri inequality (5.18) that $\theta'_\epsilon(t) \leq 0$, hence $\theta_\epsilon(t) = 0$ for all $t \geq t_1$. Then the Raychaudhuri equation (5.16) or (5.17) implies that $\mathrm{Ric}_N(\dot{\eta}(t)) = 0$, $\mathrm{trace}(\sigma_\epsilon(t)^2) = 0$ and $\psi'_\eta(t) = 0$ for all $t \geq t_1$. (For the case $N = n$, we take $N' \in (n, \infty)$ such that ϵ belongs to the timelike ϵ-range of N' and apply (5.16) for N' with the help of $\mathrm{Ric}_{N'} \geq \mathrm{Ric}_n$ from (4.3)). Since σ_ϵ is $g_{\dot{\eta}}$-symmetric, we have $\sigma_\epsilon(t) = 0$ for all $t \geq t_1$. Moreover, we deduce from (5.13), (5.14) and (5.11) that $\theta(t) = 0$, $\sigma(t) = 0$ and $\mathsf{B}(t) = 0$ for all $t \geq t_1$. Then we obtain from the unweighted Riccati equation $\mathsf{B}' + \mathsf{B}^2 + \mathsf{R} = 0$ in (5.3) that $\mathsf{R}(t) = 0$ for all $t \geq t_1$, a contradiction that completes the proof. $\square$

Remark 7.5 ($N = 0$ case) In the extremal case of $N = 0$ (and hence $\epsilon = 0$), the same argument as Lemma 7.4 shows $\theta_\epsilon(t) = 0$ and it implies $\mathrm{Ric}_0(\dot{\eta}(t)) = 0$, $\sigma_\epsilon(t) = 0$ and $\mathsf{B}_\epsilon(t) = 0$, but not $\psi'_\eta(t) = 0$ (see (5.15)). Nonetheless, the weighted Riccati equation (5.12) yields $\mathsf{R}_{(0,0)}(t) = 0$, therefore we obtain the same conclusion as Lemma 7.4 by replacing the hypothesis $\mathsf{R}(t_1) \neq 0$ with the weighted genericity condition $\mathsf{R}_{(0,0)}(t_1) \neq 0$ similar to [Ca, WW2]. This phenomenon could be compared with Wylie's observation in the splitting theorems: One obtains the isometric splitting for $N \in (-\infty, 0) \cup [n, +\infty]$, while for $N = 0$ only the weaker *warped product splitting* holds true. We refer to [Wy] for the Riemannian case and [WW2] for the Lorentzian case (where $N = 1$ is the extremal case due to the difference from our notation, recall Remark 4.2).

The following proposition is the next key step towards singularity theorems.

Proposition 7.6 (Generating conjugate points) *Let $N \in (-\infty, 0) \cup [n, +\infty]$ and $\epsilon \in \mathbb{R}$ belong to the timelike ϵ-range in (5.20). Let $\eta : (a, b) \longrightarrow M$ be an ϵ-complete timelike geodesic satisfying the genericity condition and $\mathrm{Ric}_N(\dot{\eta}) \geq 0$ on (a, b). Then η necessarily has a pair of conjugate points.*

To prove the proposition, we need two lemmas on Lagrange tensor fields shown in the same way as the Lorentzian setting. Indeed, everything can be calculated in terms of $g_{\dot{\eta}}$, thereby one can follow the same lines as [BEE, Lemmas 12.12, 12.13].

Lemma 7.7 *Let $\eta : (a, b) \longrightarrow M$ be a timelike geodesic of unit speed having no conjugate points. Take $t_1 \in (a, b)$ and let J be the unique Lagrange tensor field along η such that*

$\mathsf{J}(t_1) = 0$ *and* $\mathsf{J}'(t_1) = I_n$. *Then, for each* $s \in (t_1, b)$, *the Lagrange tensor field* D_s *with* $\mathsf{D}_s(t_1) = I_n$ *and* $\mathsf{D}_s(s) = 0$ *satisfies the equation*

$$\mathsf{D}_s(t) = \mathsf{J}(t) \int_t^s (\mathsf{J}^\mathsf{T}\mathsf{J})(r)^{-1} \, dr \tag{7.1}$$

for all $t \in (t_1, b)$. *Moreover,* $\mathsf{D}_s(t)$ *is nonsingular for all* $t \in (t_1, s)$.

Proof. Recall that J' means $D^{\dot\eta}_{\dot\eta}\mathsf{J}$. Note first that by the standard ODE theory the Jacobi tensor field J is uniquely determined by the boundary condition $\mathsf{J}(t_1) = 0$ and $\mathsf{J}'(t_1) = I_n$. Moreover, $\mathsf{J}(t_1) = 0$ ensures that J is a Lagrange tensor field (recall Remark 5.2).

The endomorphism in the right-hand side of (7.1),

$$\mathsf{X}(t) := \mathsf{J}(t) \int_t^s (\mathsf{J}^\mathsf{T}\mathsf{J})(r)^{-1} \, dr, \qquad t \in (t_1, b),$$

is well defined since η has no conjugate points and $\mathsf{J}(t_1) = 0$. We shall see that X is a Lagrange tensor field satisfying the same boundary condition as D_s at s, which implies $\mathsf{D}_s = \mathsf{X}$. The condition $\mathsf{X}'' + \mathsf{R}\mathsf{X} = 0$ for X being a Jacobi tensor field is proved by using the symmetry (5.2) for J. Since $\mathsf{X}(s) = 0$ clearly holds, X is indeed a Lagrange tensor field. Moreover, we deduce from $[(\mathsf{J}^\mathsf{T})'\mathsf{D}_s - \mathsf{J}^\mathsf{T}\mathsf{D}'_s]' \equiv 0$ (by the symmetry (3.5)), $\mathsf{J}(t_1) = 0$ and $\mathsf{J}'(t_1) = \mathsf{D}_s(t_1) = I_n$ that $(\mathsf{J}^\mathsf{T})'\mathsf{D}_s - \mathsf{J}^\mathsf{T}\mathsf{D}'_s \equiv I_n$. Hence

$$\mathsf{X}'(s) = -\mathsf{J}(s) \cdot (\mathsf{J}^\mathsf{T}\mathsf{J})(s)^{-1} = -\mathsf{J}^\mathsf{T}(s)^{-1} = \mathsf{D}'_s(s).$$

Therefore we obtain $\mathsf{D}_s = \mathsf{X}$. The nonsingularity for $t \in (t_1, s)$ is seen by noticing that $(\mathsf{J}^\mathsf{T}\mathsf{J})(r)^{-1}$ is positive-definite. □

Lemma 7.8 *Let* $\eta : (a, b) \longrightarrow M$ *be a timelike geodesic of unit speed without conjugate points. For* $t_1 \in (a, b)$ *and* $s \in (t_1, b)$, *let* J *and* D_s *be the Lagrange tensor fields as in Lemma 7.7. Then* $\mathsf{D}(t) := \lim_{s \to b} \mathsf{D}_s(t)$ *exists and is a Lagrange tensor field along* η *such that* $\mathsf{D}(t_1) = I_n$ *and* $\mathsf{D}'(t_1) = \lim_{s \to b} \mathsf{D}'_s(t_1)$. *Moreover,* $\mathsf{D}(t)$ *is nonsingular for all* $t \in (t_1, b)$.

Proof. We can argue along the lines of [BEE, Lemma 12.13] (by replacing a in that proof with any $a' \in (a, t_1)$ in our notation) and find that $\lim_{s \to b} \mathsf{D}'_s(t_1)$ exists and D is the Lagrange tensor field such that $\mathsf{D}(t_1) = I_n$ and $\mathsf{D}'(t_1) = \lim_{s \to b} \mathsf{D}'_s(t_1)$, represented as

$$\mathsf{D}(t) = \mathsf{J}(t) \int_t^b (\mathsf{J}^\mathsf{T}\mathsf{J})(r)^{-1} \, dr, \qquad t \in (t_1, b).$$

The nonsingularity is shown in the same way as Lemma 7.7. □

We are ready to prove Proposition 7.6. Notice that we will use both (i) and (ii) of Lemma 7.4, so that both the future and past ϵ-completenesses are required.

Proof of Proposition 7.6. Suppose that η has no conjugate points and fix $t_1 \in (a, b)$ such that $\mathsf{R}(t_1) \neq 0$. Let $\mathsf{D} := \lim_{s \to b} \mathsf{D}_s$ be the Lagrange tensor field given in Lemma 7.8, and

$\theta_1(t)$ be the 1-expansion associated to D. Thanks to Lemma 7.4(i) and the nonsingularity of D on (t_1, b), we have $\mathsf{D} \notin L_-(t_1)$ and hence $\theta_1(t_1) > 0$. Since $\mathsf{D}(t_1) = \lim_{s \to b} \mathsf{D}_s(t_1)$ and $\mathsf{D}'(t_1) = \lim_{s \to b} \mathsf{D}'_s(t_1)$, $\theta_1(t_1) > 0$ still holds for D_s with sufficiently large $s > t_1$. Then it follows from Lemma 7.4(ii) that there exists $t_2 < t_1$ such that $\det \mathsf{D}_s(t_2) = 0$.

Now, take $v \in N_\eta(t_2) \setminus \{0\}$ with $\mathsf{D}_s(t_2)(v) = 0$ and let P be the $g_{\dot\eta}$-parallel vector field along η with $P(t_2) = v$. Then, $Y := \mathsf{D}_s(P)$ is a Jacobi field (recall Remark 5.2) and we have

$$Y(t_2) = \mathsf{D}_s(t_2)(v) = 0, \qquad Y(s) = 0, \qquad Y(t_1) = P(t_1) \neq 0.$$

Therefore $\eta(s)$ is conjugate to $\eta(t_2)$, a contradiction. This completes the proof. $\qquad\square$

An analogous proof gives the following result for null geodesics.

Proposition 7.9 *Let $N \in (-\infty, 1) \cup [n, +\infty]$ and $\epsilon \in \mathbb{R}$ belong to the null ϵ-range in (6.6). Let $\eta : (a, b) \longrightarrow M$ be an ϵ-complete lightlike geodesic satisfying the genericity condition and $\mathrm{Ric}_N(\dot\eta) \geq 0$ on (a, b). Then η necessarily has a pair of conjugate points.*

We summarize the outcomes of Propositions 7.6 and 7.9 by using the following notion.

Definition 7.10 (Convergence conditions) We say that (M, L, ψ) satisfies the *timelike N-convergence condition* (resp. the *null N-convergence condition*) for $N \in (-\infty, +\infty]$ if we have $\mathrm{Ric}_N(v) \geq 0$ for all timelike vectors $v \in \Omega$ (resp. for all $v \in \partial\Omega$).

By continuity, the timelike N-convergence condition is equivalent to $\mathrm{Ric}_N(v) \geq 0$ for all causal vectors $v \in \overline{\Omega}$, so it can also be called the *causal N-convergence condition*.

Theorem 7.11 *Let (M, L, ψ) be a Finsler spacetime of dimension $n + 1 \geq 2$, satisfying the timelike genericity and timelike N-convergence conditions for some $N \in (-\infty, 0) \cup [n, +\infty]$. Then every future-directed timelike geodesic is either including a pair of conjugate points or ϵ-incomplete for any $\epsilon \in \mathbb{R}$ belonging to the timelike ϵ-range (5.20).*

By the *ϵ-incompleteness*, we mean that (at least) one of the future and past ϵ-completenesses fails. In the null case we have similarly the next result.

Theorem 7.12 *Let (M, L, ψ) be a Finsler spacetime of dimension $n + 1 \geq 3$, satisfying the null genericity and null N-convergence conditions for some $N \in (-\infty, 1) \cup [n, +\infty]$. Then every future-directed lightlike geodesic is either including a pair of conjugate points or ϵ-incomplete for any $\epsilon \in \mathbb{R}$ belonging to the null ϵ-range (6.6).*

Remark 7.13 (Extremal cases) Due to Remark 7.5, when $N = 0$ in the timelike case or $N = 1$ in the null case, we have the analogues to Theorems 7.11, 7.12 under the modified genericity conditions $\mathsf{R}_{(0,0)}(t_1) \neq 0$ or $\mathsf{R}_{(1,0)}(t_1) \neq 0$ at some $t_1 \in (a, b)$.

8   Singularity theorems

We finally discuss several singularity theorems derived from the results in the previous sections (recall Subsection 1.1 for the general strategy). Our presentation follows [Min4] based on causality theory (see also [AJ]). We also refer to [Min6, Min7] for singularity theorems in causality theory. Recall Subsection 2.2 for some notations in causality theory.

8.1 Trapped surfaces

We first introduce the notion of trapped surfaces. Let $S \subset M$ be a co-dimension 2, orientable, compact C^2-spacelike submanifold without boundary. By this we mean that for each $x \in S$, $T_x S \cap \overline{\Omega}_x = \{0\}$. By the convexity of the cone $\overline{\Omega}_x$ there are exactly two hyperplanes $H_x^\pm \subset T_x M$ containing $T_x S$ and tangent to $\overline{\Omega}_x$. These hyperplanes determine two future-directed lightlike vectors $v^\pm$ in the sense that $H_x^\pm$ intersects $\overline{\Omega}_x$ in the ray $\mathbb{R}_+ v^\pm$. This fact can be seen as a consequence of the bijectivity of the Legendre map, and we have $H_x^\pm = \ker g_{v^\pm}(v^\pm, \cdot)$ (see [Min3, Proposition 3] and also [AJ, Proposition 5.2]). A C^1-choice of the vector field $v^\pm$ over S will be denoted by $V^\pm$. It exists by the orientability provided that the spacetime is orientable in a neighborhood of S, and is uniquely determined up to a point-wise rescaling, $V^\pm \mapsto f V^\pm$, with $f > 0$.

Now we consider the *geodesic congruence* generated by V^+, namely the family of lightlike geodesics emanating from S with the initial condition V^+. Let $\eta : [0, b) \longrightarrow M$, with $x := \eta(0) \in S$ and $\dot{\eta}(0) = V^+(x)$, be one such geodesic. Then we consider the Jacobi tensor field J along η associated with the geodesic congruence, namely $\mathsf{J}(0) = I_{n-1}$ and $\mathsf{J}'(0)(w) = D_w^{V^+} V^+$ for each $w \in Q_\eta(0)$. (We remark that this is an endomorphism left unchanged by the above rescaling (thereby well defined), i.e., invariant under the replacements $w \mapsto w + f V^+(x)$. Hence it is enough to consider $w \in T_x S$. More intuitively, given $w \in T_x S$ and the $g_{\dot{\eta}}$-parallel vector field P with $P(0) = w$, the Jacobi field $Y_w := \mathsf{J}(P)$ satisfies $Y_w(0) = w$ and $Y_w'(0) = D_w^{V^+} V^+$ so that Y_w is the variational vector field of a geodesic variation $\zeta : [0, b) \times (-\varepsilon, \varepsilon) \longrightarrow M$ such that $\zeta(0, \cdot)$ is a curve in S with $\partial_s \zeta(0, 0) = w$ and $\zeta(\cdot, s)$ is the geodesic with initial vector $V^+(\zeta(0, s))$ for each s.

One can show that J is in fact a Lagrange tensor field (see [Min4, Section 4], and this could be compared with the symmetry of the Hessian in the positive-definite case as in [OS, Lemma 2.3]). That is, let $w_1, w_2 \in T_x S$ and extend them to two vector fields W_1, W_2 tangent to S and commuting at x (i.e., $[W_1, W_2](x) = 0$). Next extend them to a neighborhood U of S with no focal points. Let us also extend V^+ to a vector field on U, and let us keep the same notations for the extended fields. Since W_i is tangent to S, we have $\partial_{w_i} g_{V^+}(V^+, W_j) = 0$ for $i, j = 1, 2$. Then it follows from (3.2) and $[W_1, W_2](x) = 0$ that

$$g_{V^+}(w_1, D_{w_2}^{V^+} V^+) = -g_{V^+}(D_{w_2}^{V^+} W_1, V^+) = -g_{V^+}(D_{w_1}^{V^+} W_2, V^+) = g_{V^+}(w_2, D_{w_1}^{V^+} V^+).$$

This together with $\mathsf{J}(0) = I_{n-1}$ implies the symmetry of $\mathsf{B} = \mathsf{J}' \mathsf{J}^{-1}$, and hence the Lagrange property for J (recall Remark 5.2).

Focal points of S are those at which $\det \mathsf{J} = 0$. In Lorentz–Finsler geometry it has been proved in [Min4, Proposition 5.1] (see [Min7, Theorem 6.16] for the analogous Lorentzian proof) that every geodesic of the congruence including a focal point necessarily enters the set $I^+(S)$ defined in Subsection 2.2 (this result does not use the weight and so passes to our case). A future *lightlike S-ray* is a future inextendible, lightlike geodesic $\eta : [0, b) \longrightarrow M$ such that $\eta(0) \in S$ and $d(S, \eta(t)) = \ell(\eta|_{[0,t]})$ for all $t \in (0, b)$. Then η issues necessarily orthogonally from S, and does not intersect $I^+(S)$. Note also that, if every geodesic of the congruence develops a focal point, then there are no future lightlike S-rays.

The *expansion* $\theta^+ : S \longrightarrow \mathbb{R}$ of S will be the expansion of the geodesic congruence defined by

$$\theta^+(x) := \operatorname{trace}(\mathsf{J}' \mathsf{J}^{-1})(0) = \operatorname{trace}(w \mapsto D_w^{V^+} V^+),$$

where J is the Lagrange tensor field along the geodesic η with $\dot{\eta}(0) = V^+(x)$ as above. The right-hand side can be interpreted as the trace of the shape operator of S. Similarly, define the ϵ-*expansion* $\theta_\epsilon^+ : S \longrightarrow \mathbb{R}$ of S by

$$\theta_\epsilon^+(x) := \mathrm{trace}(\mathsf{J}_\psi^* \mathsf{J}_\psi^{-1})(0) = e^{\frac{2(1-\epsilon)}{n-1}\psi_\eta(0)}\big(\theta^+(x) - \psi_\eta'(0)\big).$$

The factor on the right-hand side is in most cases of no importance, since what really matters is the sign of θ_ϵ^+. For instance the constant ϵ does not appear in the next definition. We define θ^- and θ_ϵ^- associated with V^- in the same manner.

Definition 8.1 (Trapped surfaces) We say that S is a ψ-*trapped surface* if $\theta_1^+ < 0$ and $\theta_1^- < 0$ on S.

By the null Raychaudhuri equation of Theorem 6.1, more precisely by Corollary 6.5, we obtain the following.

Proposition 8.2 *Let (M, L, ψ) be a Finsler spacetime of dimension $n+1 \geq 3$, satisfying the null N-convergence condition for some $N \in (-\infty, 1] \cup [n, +\infty]$. Let S be a ψ-trapped surface. Then every lightlike S-ray is necessarily future ϵ-incomplete for any $\epsilon \in \mathbb{R}$ that belongs to the null ϵ-range (6.6).*

Proof. Assume to the contrary that a lightlike S-ray is future ϵ-complete for some ϵ satisfying (6.6). By Corollary 6.5 it develops a focal point, hence by [Min4, Proposition 5.1] it enters $I^+(S)$, which contradicts the definition of a future lightlike S-ray. $\square$

8.2 Singularity theorems

In the previous sections we generalized Step I according to the general strategy outlined in Subsection 1.1. Thus, we are ready to obtain some notable singularity theorems in the weighted Lorentz–Finsler framework.

We say that $S \subset M$ is *achronal* if $I^+(S) \cap S = \emptyset$ (namely, no two points in S are connected by a timelike curve). A nonempty set $S \subset M$ is called a *future trapped set* if the *future horismos* $E^+(S) := J^+(S) \setminus I^+(S)$ of S is nonempty and compact. Recall Definition 2.9 for the definition of Cauchy hypersurfaces.

As mentioned in Subsection 1.1 we have the next causality core statement which is valid for our Finsler spacetimes (and also for less regular spaces, [Min6, Theorem 2.67]).

Theorem 8.3 *Let (M, L) be a Finsler spacetime admitting a non-compact Cauchy hypersurface. Then every nonempty compact set S admits a future lightlike S-ray.*

Joining Proposition 8.2 (as Steps I and II) with Theorem 8.3 (as Step III), we obtain our first singularity theorem, which is a generalization of Penrose's theorem (analogous to [Min7, Theorem 6.25]). Recall that a ψ-trapped surface is compact.

Theorem 8.4 (Weighted Finsler Penrose's theorem) *Let (M, L, ψ) be a Finsler spacetime of dimension $n+1 \geq 3$, admitting a non-compact Cauchy hypersurface and satisfying the null N-convergence condition for some $N \in (-\infty, 1] \cup [n, +\infty]$. Suppose that there is a ψ-trapped surface S. Then there exists a lightlike geodesic issued from S which is future ϵ-incomplete for every $\epsilon \in \mathbb{R}$ that belongs to the null ϵ-range in (6.6).*

As another example of causality core statement, we consider the following theorem corresponding to [Min6, Theorem 2.64] or [Min7, Theorem 4.106]. Recall that a *time function* is a continuous function that increases over every causal curve. For closed cone structures and hence for Finsler spacetimes, the existence of a time function is equivalent to the *stable causality*, i.e., the possibility of widening the causal cones without introducing closed causal curves (see [Min6, Theorem 2.30]). A *lightlike line* is an inextendible lightlike geodesic for which no two points can be connected by a timelike curve (i.e., achronality).

Theorem 8.5 *Let (M, L) be a chronological Finsler spacetime. If there are no lightlike lines, then there exists a time function and hence (M, L) is stably causal.*

Joining this with Theorem 7.12 and Proposition 5.16 (as Steps I and II, respectively), we have a generalization of a singularity theorem obtained by the second author in [Min1].

Theorem 8.6 (Absence of time implies singularities) *Let (M, L, ψ) be a chronological Finsler spacetime of dimension $n + 1 \geq 3$, satisfying the null genericity and the null N-convergence conditions for some $N \in (-\infty, 1) \cup [n, +\infty]$. If there are no time functions, then there exists a lightlike line which is ϵ-incomplete for every $\epsilon \in \mathbb{R}$ belonging to the null ϵ-range (6.6).*

In the case of $N = 1$, we have the same conclusion by replacing the genericity condition with the weighted one $\mathsf{R}_{(1,0)} \neq 0$ (recall Remarks 7.5, 7.13).

The next lemma from [Min7, Corollary 2.117] passes word-for-word to the Lorentz–Finsler case. We say that $S \subset M$ is *future null* (resp. *causally*) *araying* if there are no future-directed lightlike (resp. causal) S-rays.

Lemma 8.7 *Let (M, L) be a stably causal Finsler spacetime. A nonempty compact set S is a future trapped set if and only if it is future null araying.*

Let us come to the causality core statement, found in [Min6, Theorem 2.71] or [Min7, Theorem 6.43], behind Hawking and Penrose's theorem.

Theorem 8.8 *Chronological Finsler spacetimes (M, L) without causal lines do not admit nonempty, compact, future null araying sets.*

Notice that a chronological spacetime without lightlike lines is stably causal by Theorem 8.5 and hence, by Lemma 8.7, future null araying sets in this statement can be equivalently replaced by future trapped sets. Then the following is an analogue to [Min7, Theorem 6.44]. Given an achronal set $S \subset M$, we define its *edge* edge(S) as the set of points $x \in \overline{S}$ such that, for every neighborhood U of x, there are $y \in I^-(x; U) \setminus S$, $z \in I^+(x; U) \setminus S$ and a timelike curve in $U \setminus S$ from y to z. We denoted by $I^-(x; U)$ (resp. $I^+(x; U)$) the set of points $y \in U$ such that there is a smooth timelike curve in U from y to x (resp. from x to y). An achronal set is a closed topological hypersurface if and only if its edge is empty (see [ON, Corollary 14.26]).

Theorem 8.9 (Weighted Finsler Hawking–Penrose's theorem) *Let (M, L, ψ) be a chronological Finsler spacetime satisfying the causal genericity and the causal N-convergence conditions for some $N \in (-\infty, 0) \cup [n, +\infty]$. Suppose that there exists one of the following:*

(i) *a compact achronal set without edge* (*e.g., a compact achronal spacelike hypersurface*),

(ii) *a ψ-trapped surface,*

(iii) *a point x such that, on every lightlike geodesic emanating from x, its expansion θ_1 becomes negative at some point* (*i.e., the lightlike geodesic is* reconverging).

Then (M, L, ψ) admits a timelike geodesic which is ϵ-incomplete for every $\epsilon \in \mathbb{R}$ belonging to the timelike ϵ-range (5.20), *or a lightlike geodesic which is ϵ-incomplete for every $\epsilon \in \mathbb{R}$ satisfying* (6.6). *In particular, it is ψ-incomplete* (*and incomplete in the usual sense if $N \in [n, \infty)$*).

Proof. Suppose that the claim is not true. Then, by Theorems 7.11 and 7.12, every causal geodesic has conjugate points and hence is not maximizing, thereby it is not a causal line. A chronological spacetime without causal lines is stably causal (by Theorem 8.5), thus compact future trapped sets and future null araying sets are the same (Lemma 8.7).

(i) A result of causality theory whose proof passes word-for-word to the Lorentz–Finsler case states that every compact achronal set without edge is a future trapped set (see [Min7, Corollary 2.145]), hence a compact future null araying set. This goes in contradiction with Theorem 8.8.

(ii) Since a ψ-trapped surface is necessarily future null araying due to Proposition 8.2 and the hypothesis, this also goes in contradiction with Theorem 8.8.

(iii) By Corollary 6.5 and [Min4, Proposition 5.1], every lightlike geodesic issued from x enters $I^+(x)$, namely the singleton $\{x\}$ is a compact future null araying set. Therefore we have a contradiction again with Theorem 8.8. $\qquad\square$

Remark 8.10 ($N = 0$ case) A version for $N = 0$ holds true, there we assume the standard null genericity condition and the weighted timelike genericity condition demanding $\mathsf{R}_{(0,0)} \neq 0$ in place of $\mathsf{R} \neq 0$ at a point on each timelike geodesic (recall Remark 7.13).

We say that $S \subset M$ is *acausal* if it does not admit $x, y \in S$ with $x < y$, namely no causal curve meets S more than once. An acausal set is clearly achronal. A *partial Cauchy hypersuface* is by definition an acausal set without edge (see [Min7, Definition 3.35]). The causal core statement which corresponds to Hawking's singularity theorem is the following (see [Min7, Theorem 6.48]).

Theorem 8.11 *On a Finsler spacetime (M, L) there is no compact partial Cauchy hypersurface S which is future causally araying.*

The concepts involved in this statement being dependent on the notion of Lorentz–Finsler length are not purely causal. Nevertheless, the proof uses only the existence of convex neighborhoods and does indeed pass word-for-word to the Finsler setting. We need a definition which is the analog of Definition 8.1 in the timelike case.

Definition 8.12 (Contraction and expansion) Let S be a C^2-spacelike hypersurface, and V be its future-directed normal vector field, namely $V(x) \in \Omega_x$ and $\ker g_V(V(x), \cdot) = T_x S$ for all $x \in S$. Consider the geodesic congruence generated by V, the expansions $\theta = \text{trace}(w \mapsto D_w^V V)$ and θ_ϵ on S in the same way as Subsection 8.1. Then we say that S

is *contracting* if $\theta < 0$ on S, and that S is ψ-*contracting* if $\theta_1 < 0$ on S. If the inequality is reversed, then one speaks of *expanding* and ψ-*expanding* hypersurfaces.

Theorem 8.13 (Weighted Finsler Hawking's theorem) *Let (M, L, ψ) be a Finsler spacetime satisfying the timelike N-convergence condition for some $N \in (-\infty, 0] \cup [n, +\infty]$. If M contains a compact C^2-spacelike hypersurface S which is ψ-contracting, then there exists a timelike geodesic issued normally from S which is future ϵ-incomplete for every $\epsilon \in \mathbb{R}$ that belongs to the timelike ϵ-range* (5.20).

Proof. The proof goes as in [Min7, Theorem 6.49]. If S is not acausal, then one can pass to the Geroch covering spacetime M_G which contains an acausal homeomorphic copy of S (see [Min7, Section 2.15]). Since the other assumptions lift to the covering spacetime, and timelike geodesic ϵ-incompleteness projects to the base, we can assume that S is acausal. In particular, S is achronal and a partial Cauchy hypersuface.

Assume that each timelike geodesic orthogonal to S is future ϵ-complete for some $\epsilon \in \mathbb{R}$ that belongs to the timelike ϵ-range in (5.20). By Corollary 5.11 and the hypothesis $\theta_1 < 0$, every timelike geodesic issued normally from S develops a focal point in the future, thereby it cannot be a future causal S-ray (by [Min4, Proposition 5.1]). However, all future causal S-rays are necessarily orthogonal to S and hence timelike, therefore there are no future causal S-rays. This shows that S is future causally araying, a contradiction to Theorem 8.11. $\qquad\square$

Remark 8.14 (Past case via reverse structure) The past case of Theorem 8.13 can be seen by introducing the *reverse structure* $\overleftarrow{L}(v) := L(-v)$. Precisely, we consider the cone structure $\overleftarrow{\Omega}_x := -\Omega_x$ and the weight $\overleftarrow{\psi}(v) := \psi(-v)$. Then, for each timelike geodesic $\eta : (a, b) \longrightarrow M$ in (M, L), the reverse curve $\bar{\eta}(t) := \eta(-t)$ is a timelike geodesic in $(M, \overleftarrow{L})$, and $\overleftarrow{\mathrm{Ric}}_N(\dot{\bar{\eta}}(t)) = \overleftarrow{\mathrm{Ric}}_N(-\dot{\eta}(-t)) = \mathrm{Ric}_N(\dot{\eta}(-t))$. Now, assuming that S is ψ-expanding with respect to L, S is $\overleftarrow{\psi}$-contracting with respect to $\overleftarrow{L}$ and Theorem 8.13 yields a timelike geodesic which is future ϵ-incomplete for any ϵ in (5.20) with respect to $\overleftarrow{L}$. Then its reverse curve is a timelike geodesic past ϵ-incomplete with respect to L, this gives the past case of Theorem 8.13.

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