HOLOMORPHIC WIENER FUNCTION

H. SUGITA
Graduate School of Mathematics, Kyushu University,
Hakozaki, Fukuoka 812, Japan

This is an exposition of the recent development in the study of holomorphic functions on the complex Wiener space.

1 Introduction

Holomorphic functions of infinitely many complex variables have been studied since long ago, for example, see [10][15]. Bargmann[1][2] considered a Hilbert space of the square integrable holomorphic functions, in particular, in [2], he considered it on an infinite dimensional space which is essentially equivalent to the complex Wiener space. Shigekawa's work[19] is closely related to [2], but he discussed things in the L^p-setting utilizing methods developed in the Malliavin calculus.

This exposition deals with the development in this subject made by stochastic analysis, including [4][18][19][23][24][25] and [26]. But we do not refer to works done by authors in the field of complex analysis. (See, for example, [16].)

The organization of this exposition is as follows: In § 2, we introduce our framework and basic notions. In § 3, we present the Taylor expansion (or Itô–Wiener expansion) for holomorphic Wiener functions. In § 4, we introduce a way to get a good version for each holomorphic Wiener function. In § 5, utilizing the results of § 4, we establish the definitions of the skeleton and the contraction operation for holomorphic Wiener functions. In § 6, we summarize properties of distribution laws of holomorphic Wiener functions. In § 7, we show intimate relations between the infinite dimensional Brownian motion and holomorphic Wiener functions. Here, the results of § 4 play an essential role. § 8 is a discussion.

Proofs are not fully given here, unless they are so short or not yet published.

2 Almost complex abstract Wiener space and holomorphic Wiener function

In this section, we introduce basic notions. See [19][23] and [26] for details.

2.1 *Almost complex abstract Wiener space*

First, we introduce our framework, almost complex abstract Wiener space.

Let (B, H, μ) be a *real* abstract Wiener space, i.e., B is a real separable Banach space (whose dimension is infinite), H is a real separable Hilbert space continuously and densely imbedded in B and μ is a Gaussian measure[1] satisfying

$$\int_B \exp\left(\sqrt{-1}\,\langle z, l\rangle\right)\mu(dz) = \exp\left(-\frac{1}{4}\parallel l\parallel^2_{H^*}\right), \quad l \in B^* \subset H^*.$$

Here B^* and H^* are the real dual spaces of B and H, respectively. We now introduce an *almost complex structure* $J : B \to B$ which is an isometry such that $J^2 = -1$ and that the restriction $J|_H : H \to H$ is also an isometry. The abstract Wiener space (B, H, μ) endowed with the almost complex structure J is called an *almost complex abstract Wiener space* and denoted by (B, H, μ, J). A function on B which is measurable with respect to μ is called a *Wiener function.*

Let $B^{*\mathbf{C}}$ be the complexification of B^*. Then define

$$\begin{aligned} B^{*(1,0)} &:= \{\varphi \in B^{*\mathbf{C}} \mid J^*\varphi = \sqrt{-1}\varphi\}, \\ B^{*(0,1)} &:= \{\varphi \in B^{*\mathbf{C}} \mid J^*\varphi = -\sqrt{-1}\varphi\}. \end{aligned}$$

In other words, $B^{*(1,0)}$ is the space of bounded *complex linear* functions on B and $B^{*(0,1)}$ is the space of bounded *complex anti-linear* functions on B. We see that $B^{*\mathbf{C}} = B^{*(1,0)} \oplus B^{*(0,1)}$. The Hilbert spaces $H^{*\mathbf{C}}$, $H^{*(1,0)}$ and $H^{*(0,1)}$ are similarly defined.

2.2 *Holomorphic polynomial*

Definition 1. (i) A function $G : B \to \mathbf{C}$ is called a *polynomial*, if it is expressed in the form

$$G(z) = g(\langle z, \varphi_1\rangle, \dots, \langle z, \varphi_n\rangle), \qquad z \in B, \tag{1}$$

where $n \in \mathbf{N}$, $g : \mathbf{C}^n \to \mathbf{C}$ is a polynomial with complex coefficients and $\varphi_1, \dots, \varphi_n \in B^{*\mathbf{C}}$. The class of all polynomials is denoted by $\mathcal{P}$.

[1]Note that the variance of μ here is $\frac{1}{4}\|\cdot\|^2_{H^*}$ but not $\frac{1}{2}\|\cdot\|^2_{H^*}$.

(ii) A function $G : B \to H^{*\mathbf{C}}$ is called an $H^{*\mathbf{C}}$*-valued polynomial*, if it is expressed in the form

$$G(z) = \sum_{\text{finite sum}} G_j(z)\psi_j, \qquad z \in B,$$

where $G_j \in \mathcal{P}$, $\psi_j \in H^{*\mathbf{C}}$. The class of all $H^{*\mathbf{C}}$-valued polynomials is denoted by $\mathcal{P}(H^{*\mathbf{C}})$.
(iii) A function $G : B \to \mathbf{C}$ is called a *holomorphic polynomial*, if it is expressed in the form (1) where $\varphi_1, \cdots, \varphi_n \in B^{*(1,0)}$. The class of all holomorphic polynomials is denoted by $\mathcal{P}_h$.

Holomorphic Wiener functions are characterized by the Cauchy–Riemann equation. Let us formulate it in our setting. Let $\pi^*_{(1,0)}$ and $\pi^*_{(0,1)}$ be the orthogonal projections from $H^{*\mathbf{C}}$ to $H^{*(1,0)}$ and $H^{*(0,1)}$ respectively. Then operators ∂ and $\bar{\partial}$, which map $\mathcal{P}$ into $\mathcal{P}(H^{*\mathbf{C}})$, are defined by

$$\partial F(z) = \pi^*_{(1,0)} DF(z) = \frac{1}{2}\left(DF(z) - \sqrt{-1}\,J^* DF(z)\right), \tag{2}$$

$$\bar{\partial} F(z) = \pi^*_{(0,1)} DF(z) = \frac{1}{2}\left(DF(z) + \sqrt{-1}\,J^* DF(z)\right), \tag{3}$$

where D is the H-differential operator. It is easy to see that $F \in \mathcal{P}$ is holomorphic if and only if it satisfies the Cauchy–Riemann equation, i.e., $\bar{\partial} F = 0$.

Since each $G \in \mathcal{P}_h$ is everywhere defined and is essentially a holomorphic function on a finite dimensional complex space and since the measure μ is invariant under rotations around the origin, the following *mean value theorem* holds([23]).

$$G(z) = \int_B G(z + z')\mu(dz'), \qquad z \in B \tag{4}$$

Similarly we have the following relation.

$$G(\sqrt{t}\,z) = \int_B G\left(\sqrt{t}\,z + \sqrt{1-t}\,z'\right)\mu(dz'), \qquad z \in B \tag{5}$$

This is the simplest example of Mehler's formula, which is well-defined for general integrable Wiener functions(c.f.[5]).

2.3 L^p-holomorphic Wiener function

Now we proceed to defining the general class of holomorphic Wiener functions.

From the definitions (2) and (3), we see that for $G \in \mathcal{P}(H^{*\mathbf{C}})$,

$$\begin{aligned}\partial^* G(z) &= \frac{1}{2} D^* \left(G(z) - \sqrt{-1} J^* G(z) \right), \\ \bar{\partial}^* G(z) &= \frac{1}{2} D^* \left(G(z) + \sqrt{-1} J^* G(z) \right).\end{aligned}$$

Definition 2. A function $F \in L^p(B, \mu)$, $1 < p < \infty$, is called an *L^p-holomorphic Wiener function*[2], if F satisfies the Cauchy–Riemann equation in the weak sense, i.e.,

$$\int_B F(z) \overline{\bar{\partial}^* G(z)} \mu(dz) = 0, \quad G \in \mathcal{P}(H^{*\mathbf{C}}).$$

The class of all L^p-holomorphic Wiener functions is denoted by $\mathcal{H}^p$. The space $\mathcal{H}^p$ is a closed subspace of $L^p(B, \mu)$ and hence it is a Banach space with respect to the relative topology.

Though we have established the Cauchy–Riemann equation in such a complicated way, the space $\mathcal{H}^p$ itself can be obtained in a much simpler way: Just take the closure of the space $\mathcal{P}_h$ in $L^p(B, \mu)$, and you will obtain $\mathcal{H}^p$ ([19]). In general, an L^p-holomorphic Wiener function is neither continuous nor H-differentiable.

2.4 Splitting lemma

Holomorphic Wiener functions are holomorphic in each complex coordinate. We will here formulate this property.

Take any $l \in B^*$ so that $\| l \|_{H^*} = 1$. Let $\iota : H^* \to H$ be the Riesz isomorphism and let H_1 be the linear span of vectors $\iota(l)$ and $J\iota(l)$. Then define B_2 to be the closure in B of the orthogonal complement $H_1^\perp$ of H_1. It is easy to see that the orthogonal projection $\pi_2 : H \to H_1^\perp$ is continuously extended to $\pi_2 : B \to B_2$. Let $\mu^\perp$ be the image measure of π_2. Then $(B_2, H_1^\perp, \mu^\perp, J)$ is again an almost complex Wiener space. Under an isomorphism $\mathbf{C} \times B_2 \cong B$

$$\mathbf{C} \times B_2 \ni (\xi + \sqrt{-1}\eta, w) \mapsto \xi\iota(l) + \sqrt{-1}\eta J\iota(l) + w \in B,$$

we have the following *pseudo direct sum decomposition*.

$$(B, H, \mu, J) \cong (\mathbf{C}, \mathbf{C}, \mu_{\mathbf{C}}, \sqrt{-1}) \oplus (B_2, H_1^\perp, \mu^\perp, J), \tag{6}$$

[2] We always assume $1 < p < \infty$ in the sequel.

where $\mu_{\mathbf{C}}$ is a Gaussian measure on $\mathbf{C}$ given by

$$\mu_{\mathbf{C}}(d\zeta) = \frac{1}{\pi} e^{-\xi^2-\eta^2} d\xi d\eta, \qquad \zeta = \xi + \sqrt{-1}\eta.$$

If we consider a Wiener function $F(z)$ as a function on $\mathbf{C} \times B_2$, we will write it as $F(\zeta, w)$. The following lemma is due to Shigekawa [19].

Lemma 1. *If $F(z) = F(\zeta, w)$ be an L^p-holomorphic Wiener function, the function $F(\cdot, w)$ is holomorphic on $\mathbf{C}$ (by changing values on a $\mu_{\mathbf{C}}$-null set, if necessary), for $\mu^{\perp}$-a.e.$w \in B_2$.*

3 Taylor expansion

In this section, we will show that the Itô–Wiener expansion of L^p-holomorphic Wiener functions is nothing but the Taylor expansion in L^p-sense.

Let $\{\varphi_n\}_{n=1}^{\infty}$ be an arbitrary complete orthonormal system (CONS) of $H^{*(1,0)}$. We define the set Φ of multi-indices by

$$\Phi = \left\{ \mathbf{m} = (m_1, m_2, \dots) \in \mathbf{Z}_+^{\mathbf{N}} \,\middle|\, |\mathbf{m}| = \sum_j m_j < \infty \right\}.$$

For each $\mathbf{m} = (m_1, m_2, \dots) \in \Phi$, we put

$$G_{\mathbf{m}}(z) := \frac{1}{\sqrt{\mathbf{m}!}} \prod_{j=1}^{\infty} \langle z, \varphi_j \rangle^{m_j}, \qquad z \in B,$$

where $\mathbf{m}! = m_1! m_2! \dots$. Note that $G_{\mathbf{m}}$ is a holomorphic monomial with degree $|\mathbf{m}|$.

Theorem 1.([13][19]) *The collection $\{G_{\mathbf{m}}\}_{\mathbf{m} \in \Phi}$ forms a CONS of $\mathcal{H}^2$.*

In other words, for each $F \in \mathcal{H}^2$, we have the following expansion

$$F = \sum_{\mathbf{m} \in \Phi} c_{\mathbf{m}} G_{\mathbf{m}}, \qquad c_{\mathbf{m}} = \int_B F \bar{G}_{\mathbf{m}} d\mu. \tag{7}$$

Since $G_{\mathbf{m}}$'s are monomials, (7) may be called the *Taylor expansion* of F in L^2-sense. If we rewrite this formula to

$$F = \sum_{n=0}^{\infty} J_n F, \qquad J_n F = \sum_{|\mathbf{m}|=n} c_{\mathbf{m}} G_{\mathbf{m}}, \tag{8}$$

we will get the *Ito–Wiener expansion* of F, where J_nF is called the n-th homogeneous chaos component of F.

On the image subspace $J_n\mathcal{H}^2$ of J_n, the L^2-norm and the L^p-norm are equivalent([21]), and hence J_n is a bounded operator on $\mathcal{H}^p$.

To compute J_nF for $F \in \mathcal{H}^p$, it is not necessary to compute the coefficients $c_{\mathbf{m}}$ in (8). In fact, the following formula is known ([19]).

Theorem 2. *For $F \in \mathcal{H}^p$, we have*

$$J_nF = \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}U_\theta F d\theta, \qquad \mu\text{-a.e.} \tag{9}$$

where an isometry $U_\theta : L^p(B,\mu) \to L^p(B,\mu)$ is defined by

$$(U_\theta F)(z) := F\left((\cos\theta + J\sin\theta)\, z\right). \tag{10}$$

In particular, the operator norm of $J_n : \mathcal{H}^p \to \mathcal{H}^p$ is 1.

Proof. We will first give a proof for $F \in \mathcal{P}_h$. Then, since J_kF is a homogeneous holomorphic polynomial of order k, we have $U_\theta J_kF(z) = e^{\sqrt{-1}k\theta}J_kF(z)$. Consequently,

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}U_\theta F d\theta &= \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}U_\theta \sum_k J_kF d\theta \\
&= \sum_k \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}e^{\sqrt{-1}k\theta} J_kF d\theta \\
&= \sum_k \frac{1}{2\pi}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}e^{\sqrt{-1}k\theta} d\theta \times J_kF \\
&= J_nF.
\end{aligned}$$

Now (9) is valid for a generic $F \in \mathcal{H}^p$ by virtue of the continuity of J_n and $(2\pi)^{-1}\int_0^{2\pi} e^{-\sqrt{-1}n\theta}U_\theta d\theta$ in $L^p(B,\mu)$. ∎

By using this formula, Shigekawa[19] proved the following theorem.

Theorem 3. *For $F \in \mathcal{H}^p$, we have $\sum_{n=0}^N J_nF \to F$, in $\mathcal{H}^p$ as $N \to \infty$.*

4 Holomorphically exceptional set and regular version

Usually, we identify Wiener functions which coincide μ-a.e., but if we can construct a good representative among them by modifying values on a μ-null set, it must be useful. For example, the Sobolev imbedding theorem stands upon the same spirit. In the case of holomorphic Wiener functions, we can do it thanks to the strong structure of holomorphy.

We first define a suitable notion of exceptional set[3].

Definition 3. For a sequence $\{G_n\} \subset \mathcal{P}_h$ such that $\sum_n \| G_n \|_{L^p} < \infty$, we define a subset $N^p(\{G_n\})$ of B by

$$N^p(\{G_n\}) := \left\{ z \in B \,\middle|\, \sum_n |G_n(z)| = \infty \right\}. \tag{11}$$

A set $A \subset B$ is called an *L^p-holomorphically exceptional set*, if it is a subset of a set of the type $N^p(\{G_n\})$. We denote the class of all L^p-holomorphically exceptional sets by $\mathcal{N}_h^p$. If an assertion holds outside of an L^p-holomorphically exceptional set, we say that it holds "*a.e.*$(\mathcal{N}_h^p)$".

Any countable union of L^p-holomorphically exceptional sets is again an L^p-holomorphically exceptional set.

Let us show that holomorphically exceptional sets are μ-null sets(, in fact, we will show even stronger assertions). For each $t \geq 0$, we denote by μ_t the induced measure of μ by the mapping $B \ni z \mapsto \sqrt{t}z \in B$. Note that μ_t and μ are mutually singular, if $t \neq 1$.

Theorem 4. *Any L^p-holomorphically exceptional set is a μ_t-null set, if* $0 \leq t \leq 1$.

Proof. For any sequence $\{G_n\} \subset \mathcal{P}_h$ such that $\sum_n \| G_n \|_{L^p} < \infty$, we have

$$\int_B \sum_n |G_n(z)| \mu_t(dz) = \sum_n \int_B |G_n(\sqrt{t}z)| \mu(dz) \leq \sum_n \left(\int_B |G_n(\sqrt{t}z)|^p \mu(dz) \right)^{1/p}$$

Since Mehler's transform (5) is a contraction on $L^p(\mu)$ ([5]), we have

$$\text{R.H.S.} \leq \sum_n \| G_n \|_{L^p} < \infty.$$

[3] The author was deeply inspired by Itô[14] to think of Definition 3.

Thus we see $\mu_t(N^p(\{G_n\})) = 0$. ■

Theorem 5. (i) *For any $h \in H$, the one point set $\{h\}$ is not an L^p-holomorphically exceptional set.*
(ii) *For any $z \in B \setminus H$, the one point set $\{z\}$ is an L^p-holomorphically exceptional set.*

Proof. We give a proof of the assertion (i) only. Take any sequence $\{G_n\} \subset \mathcal{P}_h$ such that $\sum_n \| G_n \|_{L^p} < \infty$. By the mean value theorem (4), we see

$$G_n(h) = \int_B G_n(z+h)\mu(dz) = \int_B G_n(z)M(h,z)\mu(dz), \qquad h \in H,$$

where $M(h,z) = \exp(2\langle h, z\rangle - |h|_H^2)$ is the Cameron-Martin density, which has every moment. Consequently, taking $q > 1$ so that $1/p + 1/q = 1$, we have

$$\sum_n |G_n(h)| \leq \sum_n \int_B |G_n(z)M(h,z)|\mu(dz) \leq \sum_n \| G_n \|_{L^p} \| M(h,\cdot) \|_{L^q(\mu)} < \infty.$$

Thus we see $h \notin N^p(\{G_n\})$. ■

For each holomorphic Wiener function, we can construct a good version in the following way.

Theorem 6. (i) *Each $F \in \mathcal{H}^p$ has a version $\tilde{F}$ $(= F, \mu\text{-a.e.})$, called a* regular version, *which satisfies the following: There exists a sequence $\{G_n\}_n \subset \mathcal{P}_h$ such that*

$$\| G_n - F \|_{L^p} \to 0 \quad \textit{and} \quad G_n \to \tilde{F} \quad \textit{a.e.}(\mathcal{N}_h^p), \qquad \textit{as } n \to \infty.$$

(ii) *If both $\tilde{F}_1$ and $\tilde{F}_2$ are regular versions of F, then $\tilde{F}_1 = \tilde{F}_2$, a.e. $(\mathcal{N}_h^p)$.*
(iii) *If $F_n \to F$ in $\mathcal{H}^p$, then there exists a subsequence $\{F_{n_k}\}_k$ such that $\tilde{F}_{n_k} \to \tilde{F}$, a.e.$(\mathcal{N}_h^p)$. Here $\tilde{F}_{n_k}$ and $\tilde{F}$ are regular versions.*

Proof. (i) Take a sequence $\{G_n\}_n \subset \mathcal{P}_h$ so that $\| G_n - F \|_{L^p} \to 0$ and that $\sum_n \| G_{n+1} - G_n \|_{L^p} < \infty$. Define a version of F by

$$\tilde{F}(z) := \begin{cases} \lim_n G_n(z), & z \notin N^p(\{G_{n+1} - G_n\}) \\ 0, & z \in N^p(\{G_{n+1} - G_n\}). \end{cases} \tag{12}$$

Then $\tilde{F}$ is a regular version of F. The assertions (ii) and (iii) follow from standard arguments. ■

For $F \in \mathcal{H}^p$, its regular version has a Lusin-type property with respect to a certain set function which characterizes L^p-holomorphically exceptional sets ([26]).

Theorem 7. *Define a set function $C^p(A)$ for $A \subset B$ by*

$$C^p(A) = \inf\left\{ \sup_{z\in A} \exp\left(-\sum_n |G_n(z)|\right) \,\middle|\, \{G_n\}_{n=1}^{\infty} \subset \mathcal{P}_h, \sum_n \| G_n \|_{L^p} = 1 \right\}.$$

(i) *For any $a_k > 0$ with $\sum_k a_k = 1$ and any $\varepsilon_k > 0$,*

$$C^p\left(\bigcup_k A_k\right) \le \sum_k (C^p(A_k) + \varepsilon_k)^{a_k}.$$

(ii) *A set $A \subset B$ is an L^p-holomorphically exceptional set, if and only if $C^p(A) = 0$.*
(iii) *A regular version $\tilde{F}$ of $F \in \mathcal{H}^p$ has the Lusin-type property with respect to C^p. That is, for any $\varepsilon > 0$, there exists a closed set $E \subset B$ such that $C^p(B \setminus E) < \varepsilon$ and that $\tilde{F}|_E : E \to \mathbf{C}$ is continuous. In particular, if $F \in \mathcal{H}^p$ admits a continuous version, then it is a regular version.*

5 Skeleton and contraction operation

For our abstract Wiener space (B, H, μ), we have $\mu(H) = 0$ because $\dim B = \infty$. Hence it is of no measure-theoretical meaning to consider the values of Wiener functions on H. However some kind of Wiener functions can have intrinsic values on H by virtue of additional structures. Indeed, holomorphic Wiener functions are one of such examples.

Let $F \in \mathcal{H}^p$ and $\tilde{F}_1, \tilde{F}_2$ be any regular versions of F. Since $\{h\}$, $h \in H$, is not a holomorphically exceptional set, we see $\tilde{F}_1(h) = \tilde{F}_2(h)$. Thus, the function $\tilde{F}_1(h)$ on H is uniquely determined by F, and it is called the *skeleton* of F.

Similarly, since μ_t, $0 \le t < 1$, is singular relative to μ, it is of no measure-theoretical meaning to consider $F(\sqrt{t}z)$ for a Wiener function F. But for each $F \in \mathcal{H}^p$, it is of significant meaning. Again, let $\tilde{F}_1, \tilde{F}_2$ be any regular versions of F. Since $\tilde{F}_1 = \tilde{F}_2$, μ_t-a.e. (or equivalently, $\tilde{F}_1(\sqrt{t}z) = \tilde{F}_2(\sqrt{t}z)$, μ-a.e.) for $0 \le t < 1$, the function $\tilde{F}_1(\sqrt{t}z)$ is uniquely determined by F, and it is called the *contraction operation*.

The following theorem follows from (4), (5) and Theorem 6.

Theorem 8. *Let $F \in \mathcal{H}^p$ and $\tilde{F}$ be any regular version of F. Then we have*

$$\tilde{F}(h) = \int_B F(z+h)\mu(dz), \quad h \in H, \tag{13}$$

$$\tilde{F}(\sqrt{t}\, z) = \int_B F\left(\sqrt{t}\, z + \sqrt{1-t}\, z'\right)\mu(dz'), \quad \mu\text{-a.e.}, \quad 0 \le t \le 1. \tag{14}$$

Skeletons of L^p-holomorphic Wiener functions is infinitely Gâteaux - differentiable and their Gâteaux - derivatives are analytic in the sense of [11] (see [4]).

Each holomorphic Wiener function is reconstructed from its skeleton in the following way ([4]).

Theorem 9. *Let $\{l_n\} \in B^*$ be a CONS of H^*. Define a projection $\Pi_n : B \to H$ by*

$$\Pi_n z := \sum_{k=1}^{n} \{\langle z, l_k\rangle \iota(l_k) + \langle z, J^* l_k\rangle J\iota(l_k)\}.$$

Then for each $F \in \mathcal{H}^p$ and its regular version $\tilde{F}$, we have

$$\tilde{F}(\Pi_n z) \longrightarrow F, \qquad \text{in } L^p(B, \mu).$$

In particular, if two L^p-holomorphic Wiener functions have a same skeleton, they coincide μ-a.e.

Next we consider the Taylor expansion of the contraction operation. Since each $G_{\mathbf{m}}$ in (7) is a monomial of order $|\mathbf{m}|$, we have $G_{\mathbf{m}}(\sqrt{t}z) = \sqrt{t}^{\,|\mathbf{m}|} G_{\mathbf{m}}(z)$. So we can guess that $F(\sqrt{t}z)$ for $F \in \mathcal{H}^p$ should be

$$\sum_{\mathbf{m}\in\Phi} \sqrt{t}^{\,|\mathbf{m}|} c_{\mathbf{m}} G_{\mathbf{m}}(z) = \sum_{n=0}^{\infty} \sqrt{t}^{\,n} J_n F(z). \tag{15}$$

In fact, this is true if take a regular version $\tilde{F}$ of F, because (15) is nothing but the Taylor (Ito–Wiener) expansion of the right hand side of (14) in $\mathcal{H}^p$.

6 Distribution law

6.1 *Absolute continuity*

Although being called holomorphic, a generic element of $\mathcal{H}^p$ is neither continuous nor H-differentiable. Nevertheless, with the help of holomorphy, the following theorem holds.

Theorem 10. *If $F \in \mathcal{H}^p$ is not a constant function, its distribution law is absolutely continuous with respect to the Lebesgue measure on* $\mathbf{C}$.

Proof. We give a full proof which is a small modification of the proof of Theorem 4.3 in [19].

Let $A \subset \mathbf{C}$ be of Lebesgue measure zero. What to prove is that if $\mu \circ F^{-1}(A) > 0$ then F is a constant.

Take any $l \in B^*$ so that $\| l \|_{H^*} = 1$. By the pseudo direct sum decomposition (using the same notation as in § 2.4), we express $F(z)$ as

$$F(z) = F(\zeta, w), \qquad \zeta \in \mathbf{C}, \quad w \in B_2.$$

Then, by Lemma 1, we know that $F(\cdot, w)$ is holomorphic on $\mathbf{C}$ for $\mu^\perp$-a.e. $w \in B_2$. We readily see that

$$\mu_{\mathbf{C}}\left(\{\zeta \in \mathbf{C} \,|\, F(\zeta, w) \in A\}\right) = 0 \text{ or } 1, \qquad \mu^\perp\text{-a.e. } w \in B_2.$$

In both cases, we see that

$$\mu\left(F^{-1}(A) \triangle (F^{-1}(A) + \iota(l))\right) = 0,$$

where $\triangle$ stands for the symmetric difference. Since l is arbitrary, it follows from the ergodicity of μ with respect to the H-shift that $\mu \circ F^{-1}(A) = 1$, because we have assumed $\mu \circ F^{-1}(A) > 0$. Then by the Fubini theorem, we have

$$\mu_{\mathbf{C}}\left(\{\zeta \in \mathbf{C} \,|\, F(\zeta, w) \in A\}\right) = 1, \qquad \mu^\perp\text{-a.e. } w \in B_2.$$

Since $\mathbf{C}$ is of complex dimension one, we see that $F(\zeta, w)$ does not depend on ζ, namely, there exists a measurable function G on B_2 such that $F(\zeta, w) = G(w)$, for $\mu_{\mathbf{C}} \otimes \mu^\perp$-a.e. (ζ, w). Again since l is arbitrary, F is measurable with respect to the tail σ-field generated by all $l \in B^*$. This means that F is a constant. ∎

6.2 Large deviation principle

In the theory of SDE, the large deviation principle is stated in terms of the skeleton, i.e., the solution of ODE associated to SDE by replacing Brownian motions by smooth paths. In our case, Fang–Ren [4] showed the following theorem.

Theorem 11. *Let $F \in \mathcal{H}^p$ and $\tilde{F}$ be its regular version. Then we have :*
(i) *If $A \subset \mathbf{C}$ is closed then*

$$\limsup_{t \to 0} t \log \mu \left(F(\sqrt{t}z) \in A \right) \leq -\inf \left\{ \frac{1}{4} \parallel h \parallel_H^2 \middle| \tilde{F}(h) \in A \right\}.$$

(ii) *If $A \subset \mathbf{C}$ is open then*

$$\liminf_{t \to 0} t \log \mu \left(F(\sqrt{t}z) \in A \right) \geq -\inf \left\{ \frac{1}{4} \parallel h \parallel_H^2 \middle| \tilde{F}(h) \in A \right\}.$$

6.3 Approximate continuity

Approximate continuity, being first introduced to infinite dimensional stochastic analysis by [3], is the property shown in (16) below. It is much weaker than the continuity in $\| \cdot \|_B$, but strong enough to assure, for instance, support theorems ([20]). Since approximate continuity depends heavily on the topology (see [22]), it is necessary to specify which topology we will talk about.

Definition 4. (i) The norm $\| \cdot \|_B$ is said to be *rotation invariant*, if it satisfies that

$$\| (\cos\theta + J \sin\theta)\, z \|_B = \| z \|_B, \quad z \in B, \quad \theta \in [0, 2\pi).$$

(ii) The norm $\| \cdot \|_B$ is said to be *completely rotation invariant*, if it is rotation invariant and if there exists a CONS $\{l_k\} \subset B^*$ of H^* such that for any $l = l_k$,

$$\| \left(e^{\sqrt{-1}\theta} \zeta \right) \oplus w \|_B = \| \zeta \oplus w \|_B, \qquad \zeta \in \mathbf{C}, \quad w \in B_2, \quad \theta \in [0, 2\pi),$$

where we are considering the pseudo direct sum decomposition (6).

Let $\| \cdot \|_B$ be a given norm. Then a new norm $\| \cdot \|$ defined by

$$\| z \| := \sup_{0 \leq \theta < 2\pi} \| (\cos\theta + J \sin\theta) z \|_B, \qquad z \in B,$$

is a rotation invariant norm and it is equivalent to $\| \cdot \|_B$. Thus we may always assume the rotation invariance of the norm.

A typical example of completely rotation invariant measurable norm is the following: Let $\{\varphi_n\} \subset H^{*(1,0)}$ be a CONS and let $a_n > 0$ be such that $\sum_n a_n < \infty$. Define a norm on H by

$$\| h \| := \left(\sum_n a_n \, |\langle h, \varphi_n \rangle|^2 \right)^{1/2}$$

Then this norm is measurable ([6][8]) and completely rotation invariant.

Theorem 12. *Let $\| \cdot \|_B$ be completely rotation invariant and let B_r be the centered $\| \cdot \|_B$-ball with radius $r > 0$. Then for each $F \in \mathcal{H}^p$, $p > 2$,*

$$\lim_{r \to 0} \frac{1}{\mu(B_r)} \int_{B_r} |F(z+h) - \tilde{F}(h)|^2 \mu(dz) = 0, \qquad h \in H.$$

In particular,

$$\lim_{\delta \to 0} \mu(|F(z) - \tilde{F}(h)| > \varepsilon \, | \, \| z - h \|_B < \delta) = 0, \qquad \varepsilon > 0. \tag{16}$$

If the norm is not rotation invariant, we can construct a counter example to the theorem using the same method as [22]. But we do not know whether the *complete* rotation invariance is necessary or not.

7 Fine continuity with respect to Brownian motion

In finite dimensions, holomorphic functions and the Brownian motion are closely related via the notion of conformal martingales. Namely, if f is holomorphic on $\mathbf{C}^n$ and $(z_t)_{t \geq 0}$ is a Brownian motion on $\mathbf{C}^n$, then the composite $(f(z_t))_{t \geq 0}$ is a conformal martingale. In this section, we will show that this relation can be extended to the infinite dimensional case.

Let $(Z_t)_{t \geq 0}$ be a B-valued independent increment process defined on a probability space $(\Omega, \mathcal{F}, P)$ such that $Z_0 = 0$ and the distribution of $Z_t - Z_s, t > s$, is μ_{t-s}. Then the process $(Z_t)_{t \geq 0}$ becomes a diffusion process on B and it is called a *B-valued Brownian motion* (see, for example, [8]).

Let F and F' be Wiener functions such that $F = F'$, μ-a.e. Then, in general, for $0 \leq t < 1$, we cannot expect $F(Z_t) = F'(Z_t)$, P-a.e., because the distribution law of Z_t is μ_t, which is singular relative to μ. But if F is holomorphic, its regular version $\tilde{F}$ is uniquely determined μ_t-a.e., and hence $\tilde{F}(Z_t)$ is well-defined P-a.e. Thus we get a stochastic process $(\tilde{F}(Z_t))_{0\le t\le 1}$ on the probability space $(\Omega, \mathcal{F}, P)$.

Theorem 13. (i) *The process* $(Z_t)_{0\le t\le 1}$ *does not hit any* $N \in \mathcal{N}_h^p$ *with probability* 1. *Namely,*

$$P(Z_t \notin N \text{ for } \forall t \in [0,1]) = 1.$$

(ii) *Let* $F \in \mathcal{H}^p$ *and* $\tilde{F}$ *be any regular version of* F. *Then the process* $(\tilde{F}(Z_t))_{0\le t\le 1}$ *is a continuous* L^p*-conformal martingale.*

Proof. (i) We will show that for any sequence $\{G_n\} \subset \mathcal{P}_h$ with $\sum_n \| G_n \|_{L^p} < \infty$,

$$\mathbf{E}\left[\sup_{0\le t\le 1} \sum_n |G_n(Z_t)|\right] < \infty, \tag{17}$$

where $\mathbf{E}$ stands for the expectation with respect to the probability P. Since $G_n \in \mathcal{P}_h$, the process $(G_n(Z_t))_{t\ge 0}$ is a continuous conformal martingale (see for example, [9], [12], Chapter IV-6). It therefore follows from Doob's inequality that $c_p > 0$ being some constant,

$$\begin{aligned}
\mathbf{E}\left[\sup_{0\le t\le 1} |G_n(Z_t) - G_n(0)|\right] &\le \mathbf{E}\left[\sup_{0\le t\le 1} |G_n(Z_t) - G_n(0)|^p\right]^{1/p} \\
&\le c_p \mathbf{E}\left[|G_n(Z_1) - G_n(0)|^p\right]^{1/p} \\
&= c_p \| G_n(\cdot) - G_n(0) \|_{L^p}.
\end{aligned}$$

Since $\sum_n |G_n(0)| \le \sum_n \| G_n \|_{L^1} < \infty$ by (4), we easily see (17).
(ii) Take a sequence $\{G_n\} \subset \mathcal{P}_h$ such that $\sum_n \| G_n - F \|_{L^p} < \infty$. On account of (i), it is sufficient to prove the assertion for the particular regular version $\tilde{F}$ defined by (12). Again by Doob's inequality, we have

$$\mathbf{E}\left[\sum_{n=1}^{\infty} \sup_{0\le t\le 1} |G_{n+1}(Z_t) - G_n(Z_t)|\right] \le c_p \sum_{n=1}^{\infty} \| G_{n+1} - G_n \|_{L^p} < \infty.$$

This implies that the sequence $\{(G_n(Z_t))_{0\le t\le 1}\}_n$ converges to a continuous L^p-conformal martingale, which coincide with $\tilde{F}(Z_t)$ for each $t \in [0,1]$, P-a.e. ∎

Thus, holomorphically exceptional sets and the property (ii) of Theorem 6 may be compared respectively to the polar sets and the fine continuity with respect to the Brownian motion (Z_t). However, since (Z_t) has no symmetrizing measure — if it had any, it would be the Feynmann measure (Lebesgue measure) on B —, no potential theoretical notion can be rigorously formulated. In fact, (Z_t) may hit holomorphically exceptional sets with positive probability after time 1: For example, define a holomorphically exceptional set $A = N^2(\{G_n\})$ by

$$G_n(z) := \frac{1}{n^2} \prod_{j=\frac{n(n-1)}{2}+1}^{\frac{n(n+1)}{2}} \langle z, \varphi_j \rangle, \quad n = 1, 2, \ldots,$$

where $\{\varphi_n\}_n \subset B^{*(1,0)}$ is an orthonormal system of $H^{*(1,0)}$. Then we can show that the first hitting time of (Z_t) to the set A is equal to e^γ a.s., $\gamma = 0.57721\ldots$ being Euler's constant ([25]).

8 Discussion

In all of the preceding sections, holomorphic Wiener functions are globally defined. Then how do we establish a theory of locally defined holomorphic Wiener functions under as little hypotheses as possible? There is an interesting answer given by Kusuoka–Taniguchi[18], to which we will give a brief introduction below.

In finite dimensions, locally defined holomorphic functions are closely related to the shapes of their defining domains. For example, it is known that the notions of pseudoconvex domain and domain of holomorphy are equivalent. In this connection, Kusuoka–Taniguchi[18] investigated stochastic differential equations (SDEs) with holomorphic coefficients. Note that if the coefficients are not linear but general holomorphic functions, the solution of the SDE may explode. Consider the set (say A) of Brownian paths for which the solution does not explode before time 1. Then Kusuoka–Taniguchi showed that the set A is considered to be pseudoconvex in the sense that $\bar{\partial}$-equations $\bar{\partial}u = F$ are solvable on A, and that the solution is holomorphic on A.

The next aim here should be to establish a theory of skeletons (and regular versions) defined on pseudoconvex domains. If it is achieved, more functional-analytic approaches would be possible.

In finite dimensions, holomorphic functions dwell also on complex manifolds. Then how do we construct them on complex Wiener–Riemannian manifolds? The answer is not yet definite to this point, but we expect that, for example, the works of Professors R. Léandre, B. Driver and L. Gross in this volume will have some connections to this question.

References

[1] V. Bargmann, On a Hilbert space of analytic functions and an associated integral transform, *Comm. Pure Appl. Math.*, **14**, 187–214 (1961).

[2] V. Bargmann, Remark on a Hilbert space of analytic functions, *Proc. Nat. Acad. Sci. USA*, **48**, 199–204 (1961).

[3] R. Carmona and D. Nualart, Traces of random variables on Wiener space and the Onsager-Machlup functional, preprint, 1991

[4] S. Fang and J. Ren, Sur le squelette et les dérivées de Malliavin des fonctions holomorphes sur espace de Wiener complexe, *Jour. of Math. Kyoto Univ.*, **33**, 749–764 (1993).

[5] D. Feyel and A. de La Pradelle, Opérateurs linéaires et espace de Sobolev sur l'espace de Wiener, *C. R. Acad. Sci. Paris*, **313**, Série I, 727–729 (1991).

[6] L. Gross, Measurable functions on Hilbert space, *Trans. Amer. Math. Soc.*, **105**, 372–390 (1962).

[7] L. Gross, Abstract Wiener spaces, in *Proc. of the Fifth Berkeley Symp. on Math. Statist. and Prob.* II, *Part* I, Univ. of Calif., 31–42 (1967).

[8] L. Gross, Potential theory on Hilbert space, *J. of Func. Anal.*, **1**, 123–181 (1967).

[9] R. K. Getoor and M. J. Sharpe, Conformal martingales, *Inventiones math.* **16**, 271–308 (1972).

[10] M. Hervé, *analytic and Plurisubharmonic Functions in Finite and Infinite Dimensional Spaces*, LMN **198**, Springer (1970).

[11] E. Hille and R.S. Phillips, *Functional Analysis and Semi-groups*, Colloquium Publications, **31**, AMS (1957).

[12] N. Ikeda and S. Watanabe, *Stochastic Differential Equations and Diffusion Processes*, second ed., North-Holland / Kodansha (1989).

[13] K. Itô, Complex multiple Wiener integral, *Japan J. Math.*, **22**, 63–86 (1953).

[14] K. Itô, Positive generalized functions on $(\mathbf{R}^\infty, B^\infty, N^\infty)$, in *White Noise Analysis, Mathematics and Applications*, eds. T. Hida, H. H. Kuo, J. Potthoff and L. Streit, World Scientific, Singapore, 166–179 (1990).

[15] P. Lelong (Ed.), *Séminaire Pierre Lelong*, LNM **474**, Springer (1975).

[16] J. Kajiwara et al.(Eds.), *Proceedings of First Korea–Japan Colloquium on Finite and Infinite Dimensioal Complex Analysis*, Pusan, Korea, 10-17 July (1993).

[17] S. Kusuoka, Analytic functionals of Wiener process and absolute continuity, in *Functional Analysis in Markov processes*, ed. by M.Fukushima, LNM **923**, Springer, 1–46 (1982).

[18] S. Kusuoka and S. Taniguchi, Pseudoconvex domain in almost complex abstract Wiener spaces, *J. of Func. Anal.*, **117**, 62–117 (1993).

[19] I. Shigekawa, Itô-Wiener expansions of holomorphic functions on the complex Wiener space, in *Stochastic Analysis*, ed. by E.Mayer et al, Academic press, San Diego, 459–473 (1991).

[20] D.W. Stroock and S.R.S. Varadhan, On the support of diffusion processes with applications to the strong maximum principle, in *Proc. of the Sixth Berkeley Symp. on Math. Statist. and Prob.* III, Univ. of Calif., 333–359 (1972).

[21] H. Sugita, Sobolev spaces of Wiener functionals and Malliavin's calculus, *J. Math. Kyoto Univ.*, **25**, 31–48 (1985).

[22] H. Sugita, Various topologies in the Wiener space and Lévy's stochastic area, *Probability Theory and Related fields*, **91**,283–296 (1992).

[23] H. Sugita, Properties of holomorphic Wiener functions — skeleton, contraction and local Taylor expansion, *Probab. Theo. and Rel. Fields*, **100**, 117–130 (1994).

[24] H. Sugita, Regular version of holomorphic Wiener function, *J. Math. Kyoto Univ.*, **34**, 849–857 (1994).

[25] H. Sugita and S. Takanobu, Accessibility of infinite dimensional Brownian motion to holomorphically exceptional set, to appear in *Proc. Japan Acad.* (1995)

[26] S. Taniguchi, On almost complex structures on abstract Wiener spaces, to appear in *Osaka Jour. Math.*, **33**, (1996)