Probabilistic proof of mean-value theorem for $[0,1]$-valued multiplicative function by means of adelic formulation

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Abstract

A probabilistic proof by means of the adelic formulation is given to the classical mean-value theorem for $[0,1]$-valued multiplicative arithmetical functions f. Then a more general mean-value theorem is derived for composite functions $\varphi(f)$ of f with (semi-)continuous functions φ.

1 Introduction

An arithmetical function $f : \mathbb{N} = \{1, 2, \ldots\} \to \mathbb{C}$ is called *multiplicative* if $f(1) = 1$ and $f(xy) = f(x)f(y)$ holds for every co-prime pair $x, y \in \mathbb{N}$. It has the following form:

$$f(n) = f\left(\prod_p p^{\alpha_p(n)}\right) = \prod_p f\left(p^{\alpha_p(n)}\right), \quad n \in \mathbb{N}. \tag{1}$$

Here and hereafter we let p denote a prime, and $\sum_p$ and $\prod_p$ denote a sum and a product over all primes, respectively, and

$$\alpha_p(n) := \sup\{k \in \mathbb{N} \cup \{0\}\,;\, p^k \mid n\}, \quad n \in \mathbb{N}.$$

For $g : \mathbb{N} \to \mathbb{C}$, we define

$$\mathbf{M}[g] := \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} g(n),$$

if the limit exists. For example, the indicator function $\mathbf{1}_{\{\alpha_p \geq m\}}$ of the set $\{n \in \mathbb{N}\,;\, \alpha_p(n) \geq m\}$ has the limit, and we have $\mathbf{M}\left[\mathbf{1}_{\{\alpha_p \geq m\}}\right] = p^{-m}$.

In this paper, we give a new proof to the following classical *mean-value theorem* due to Wirsing.

Theorem 1. *If $f : \mathbb{N} \to [0,1]$ is multiplicative, $\mathbf{M}[f]$ exists and we have*

$$\mathbf{M}[f] = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^m}\right). \tag{2}$$

In particular, $\mathbf{M}[f] = 0$ if and only if $\sum_p p^{-1}(1 - f(p)) = \infty$.

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If we could suppose $\mathbf{M}$ to be an expectation operator, under which $\{\alpha_p\}_p$ are independent, we would be able to show (2) in the following way: since f has the form (1), we would have

$$\begin{aligned}\mathbf{M}[f] &= \mathbf{M}\left[\prod_p f\left(p^{\alpha_p}\right)\right] = \mathbf{M}\left[\prod_p \sum_{m=0}^{\infty} f(p^m)\mathbf{1}_{\{\alpha_p=m\}}\right] \\ &= \prod_p \mathbf{M}\left[\sum_{m=0}^{\infty} f(p^m)\mathbf{1}_{\{\alpha_p=m\}}\right] = \prod_p \sum_{m=0}^{\infty} f(p^m)\mathbf{M}\left[\mathbf{1}_{\{\alpha_p=m\}}\right] \\ &= \prod_p \sum_{m=0}^{\infty} f(p^m)\left(\frac{1}{p^m} - \frac{1}{p^{m+1}}\right) = \prod_p \left(1 - \frac{1}{p}\right)\sum_{m=0}^{\infty} \frac{f(p^m)}{p^m}. \qquad (3)\end{aligned}$$

By $f(1) = 1$, we would see (2) holds. Unfortunately, since $\mathbf{M}$ is not countably additive, the above calculation (3) has no theoretical grounds. Thus all known rigorous proofs of Wirsing's theorem are quite different from (3).

In this paper, following the formulation of [8, 12], we extend f to a random variable, denoted by the same letter f, on the probability space $(\hat{\mathbb{Z}}, \mathcal{B}(\hat{\mathbb{Z}}), \lambda)$ (Definition 7), where $\hat{\mathbb{Z}}$ is *the ring of finite integral adeles*, a compact ring containing $\mathbb{N}$ densely, $\mathcal{B}(\hat{\mathbb{Z}})$ is its Borel σ-algebra, and λ is *the Haar probability measure*. Then we calculate the expectation $\mathbf{E}[f]$ of f along the lines of (3) (Proposition 9), and show $\mathbf{M}[f] = \mathbf{E}[f]$ to prove Theorem 1.

Remark 2. The assertion of Theorem 1 was first proved for multiplicative functions f with range $-1 \le f \le 1$ by Wirsing [14, 15], and it has been extended to $\mathbb{C}$-valued multiplicative functions f with $|f| \le 1$ under additional conditions (c.f. [3] Theorem 6.3). However, since our method developed in this paper needs the positivity of f, we assume $0 \le f \le 1$ in Theorem 1.

From the identity $\mathbf{M}[f] = \mathbf{E}[f]$, we can easily derive the following more general mean-value theorem.

Theorem 3. *Let $f : \mathbb{N} \to [0, 1]$ be multiplicative.*
(i) If $\varphi : [0, 1] \to \mathbb{C}$ is continuous, $\mathbf{M}[\varphi(f)]$ exists and it is equal to $\mathbf{E}[\varphi(f)]$.
(ii) If $\varphi : [0, 1] \to [-\infty, \infty)$ is upper semi-continuous, we have

$$\limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varphi(f(n)) \le \mathbf{E}[\varphi(f)]. \qquad (4)$$

If $\varphi : [0, 1] \to (-\infty, \infty]$ is lower semi-continuous, we have

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \varphi(f(n)) \ge \mathbf{E}[\varphi(f)]. \qquad (5)$$

(iii) For any $t \in [0, 1]$, we have

$$\begin{aligned}\lambda(f < t) &\le \liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[0,t)}(f(n)) \\ &\le \limsup_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathbf{1}_{[0,t]}(f(n)) \le \lambda(f \le t).\end{aligned}$$

In particular, if $\lambda(f = t) = 0$, then $\mathbf{M}[\mathbf{1}_{[0,t]}(f)]$ exists and it is equal to $\lambda(f \le t)$.

An example of f for which $\lambda(f = t) = 0$, $t \in [0, 1]$, is the extended random variable of $\phi(n)/n$, $n \in \mathbb{N}$, ϕ being *Euler's totient function* ([5] Chapter 4).

In § 2, we present a brief introduction to the adelic formulation, and basic facts about it. In § 3, we give a probabilistic proof by means of the adelic formulation to Theorem 1. Then a proof of Theorem 3 is given in § 4.

Other applications of the adelic formulation can be seen in [1, 2, 13]. The method of compactification of $\mathbb{N}$ for the investigation of the mean value or distribution problems of arithmetical functions was initiated by [10], and it has been studied by several papers and books, such as [4, 6, 7, 9, 11].

2 Ring of finite integral adeles and Haar probability measure

For each p, we define the p-adic metric d_p in $\mathbb{N}$ by

$$d_p(m, n) := \inf\{\, p^{-k}\,;\ p^k \mid (m-n),\ k \in \mathbb{N} \cup \{0\}\,\}, \quad m, n \in \mathbb{N}.$$

Let $\mathbb{Z}_p$ be the completion of $\mathbb{N}$ with respect to d_p. Then $\mathbb{Z}_p$ becomes a compact ring, and there exists a unique probability measure λ_p on the Borel σ-algebra $\mathcal{B}(\mathbb{Z}_p)$ of $\mathbb{Z}_p$ that is invariant with respect to addition. Thus we obtain a probability space $(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p), \lambda_p)$ for each p. We then consider the product of all the probability spaces $\{(\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p), \lambda_p)\}_p$:

$$(\hat{\mathbb{Z}}, \mathcal{B}(\hat{\mathbb{Z}}), \lambda) := \prod_p (\mathbb{Z}_p, \mathcal{B}(\mathbb{Z}_p), \lambda_p).$$

$\hat{\mathbb{Z}} := \prod_p \mathbb{Z}_p$ with the product topology is a compact ring with component-wise addition and multiplication, and it is called *the ring of finite integral adeles*. Note that the product of σ-algebras $\{\mathcal{B}(\mathbb{Z}_p)\}_p$ is nothing but the Borel σ-algebra $\mathcal{B}(\hat{\mathbb{Z}})$ of $\hat{\mathbb{Z}}$. The product probability measure $\lambda := \prod_p \lambda_p$ is obviously invariant with respect to addition and it is called *the Haar probability measure of* $\hat{\mathbb{Z}}$. We let $\mathbf{E}[g]$ denote the expectation of a random variable g defined on $(\hat{\mathbb{Z}}, \mathcal{B}(\hat{\mathbb{Z}}), \lambda)$.

Now, we list up some facts that will be used in the next section. For details, see [12] § 3.

Proposition 4. *(i) Identifying $\mathbb{N}$ with the diagonal set $\{(n, n, \ldots) \in \hat{\mathbb{Z}}\,;\ n \in \mathbb{N}\}$, it is dense in $\hat{\mathbb{Z}}$.*
(ii) For each $m \in \mathbb{N}$, we have $\hat{\mathbb{Z}} = \bigcup_{r=0}^{m-1}(m\hat{\mathbb{Z}} + r)$, which is a disjoint union.
(iii) For each $m \in \mathbb{N}$ and each $0 \le r < m$, the indicator function $\mathbf{1}_{m\hat{\mathbb{Z}}+r} : \hat{\mathbb{Z}} \to \{0, 1\}$ is continuous.

We can extend any periodic arithmetical function g to adeles. If it has a period m, it is of the form

$$g(n) = \sum_{r=0}^{m-1} g(r)\mathbf{1}_{m\mathbb{Z}+r}(n), \quad n \in \mathbb{N}.$$

Then we define its unique continuous extension to $\hat{\mathbb{Z}}$ as

$$g(x) := \sum_{r=0}^{m-1} g(r)\mathbf{1}_{m\hat{\mathbb{Z}}+r}(x), \quad x \in \hat{\mathbb{Z}}.$$

Proposition 5. *(i) If $g : \hat{\mathbb{Z}} \to \mathbb{R}$ is continuous, $\{g(n)\}_{n\in\mathbb{N}}$ is an* almost periodic sequence*, i.e., a uniform limit of periodic sequences. Conversely, if $\{g(n)\}_{n\in\mathbb{N}}$ is an almost periodic sequence, it is uniquely extended to a continuous function on $\hat{\mathbb{Z}}$.*
(ii) If $g : \hat{\mathbb{Z}} \to \mathbb{R}$ is continuous, we have $\mathbf{M}[g] = \mathbf{E}[g]$.

We write $m \mid x$ if $x \in m\hat{\mathbb{Z}}$. For each $x \in \hat{\mathbb{Z}}$, we define

$$\alpha_p(x) := \sup\{m \in \mathbb{N} \cup \{0\}\, ;\, p^m \mid x\} \leq \infty.$$

Proposition 6. *(i)* $\{\alpha_p\}_p$ *are independent random variables.*
(ii) $\mathbf{1}_{\{\alpha_p \geq m\}}$ *is periodic, and we have*

$$\mathbf{M}\left[\mathbf{1}_{\{\alpha_p \geq m\}}\right] = \mathbf{E}\left[\mathbf{1}_{\{\alpha_p \geq m\}}\right] = p^{-m}.$$

By Proposition 6 (ii), we have $\lambda(\alpha_p = \infty) = 0$, and hence we can extend any multiplicative function $f : \mathbb{N} \to [0, 1]$ to adeles by the following formula.

Definition 7.

$$f(x) := \prod_p f\left(p^{\alpha_p(x)}\right), \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}. \tag{6}$$

Then f is $\mathcal{B}(\hat{\mathbb{Z}})$-measurable, λ-integrable, and multiplicative. Indeed, if $x, y \in \hat{\mathbb{Z}}$ are co-prime, i.e., $\sum_p \alpha_p(x)\alpha_p(y) = 0$, then $f(xy) = f(x)f(y)$.

Note that (1) is a finite product for each $n \in \mathbb{N}$, whereas (6) is, in general, an infinite product. Indeed, since $\sum_p \lambda(\alpha_p \geq 1) = \sum_p p^{-1} = \infty$, Borel–Cantelli's second lemma implies $\lambda(\alpha_p \geq 1, \text{ i.o.}) = 1$. If f takes negative values, we may not be able to extend it to adeles by the formula (6). For example, the *Möbius function* μ, which is a $\{-1, 0, 1\}$-valued multiplicative function characterized by $\mu(p) = -1$, $\mu\left(p^k\right) = 0$, $k \geq 2$, for each p, cannot be extended to adeles by (6).

Remark 8. It is known that the shift $\hat{\mathbb{Z}} \ni x \mapsto x + 1 \in \hat{\mathbb{Z}}$ is ergodic. By the individual ergodic theorem, we have $\mathbf{M}[f(\bullet + x)] = \mathbf{E}[f]$, λ-a.e. $x \in \hat{\mathbb{Z}}$. In this context, Theorem 1 asserts that $x = 0$ is not an exceptional point in this ergodic theorem.

3 Proof of Theorem 1

Let $f : \hat{\mathbb{Z}} \to [0, 1]$ be the extended multiplicative function defined by (6).

Proposition 9.

$$\mathbf{E}[f] = \prod_p \left(1 - \frac{1}{p}\right)\left(1 + \sum_{m=1}^{\infty} \frac{f(p^m)}{p^m}\right). \tag{7}$$

Proof. Proposition 6 (i) and (ii) imply that $\mathbf{E}[f]$ can be computed in the same way as (3); we have only to replace all the $\mathbf{M}$'s in (3) with $\mathbf{E}$'s. □

Proposition 10. *The following three statements are equivalent to each other:*
(i) $\mathbf{E}[f] = 0$*; (ii)* $f = 0$*,* λ*-a.e.; (iii)* $\sum_p p^{-1}(1 - f(p)) = \infty$.

Proof. Since $f \geq 0$, it is obvious that (i) and (ii) are equivalent to each other. Again, since $f \geq 0$, each factor of the infinite product of (7) is not zero. The product is written as $\prod_p \left(1 - p^{-1} + f(p)p^{-1} + O\left(p^{-2}\right)\right)$, hence it diverges to 0 if and only if $\sum_p \left|p^{-1} - f(p)p^{-1} - O\left(p^{-2}\right)\right| = \infty$, but since $\sum_p \left|O(p^{-2})\right| < \infty$, the infinite product diverges to 0 if and only if $\sum_p p^{-1}(1 - f(p)) = \infty$. Thus (i) and (iii) are equivalent to each other. □

We now enumerate the set of all primes as $\{p_j\}_{j=1}^{\infty}$, and accordingly, we describe f as

$$f(x) = \prod_{j=1}^{\infty} f\left(p_j^{\alpha_j(x)}\right), \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}, \tag{8}$$

where α_j is an abbreviation of α_{p_j}.

Remark 11. Before going further, we make a remark on the continuity of the extended multiplicative function (6). If it is continuous, we have (2) immediately by Proposition 5 (ii). However, we cannot expect in general that it is continuous. For example, let us show that the extended function of $\psi(n) := \phi(n)/n$, which is written in the form (8) as

$$\psi(x) = \prod_{j=1}^{\infty} \left(1 - \frac{\mathbf{1}_{\{\alpha_j \geq 1\}}(x)}{p_j}\right), \quad x \in \hat{\mathbb{Z}}$$

is discontinuous at λ-a.e. $x \in \hat{\mathbb{Z}}$. We first show that ψ is not continuous at any $x \in \psi^{-1}(\{0\})$. Define a metric d on $\hat{\mathbb{Z}}$ as

$$d(x,y) := \sum_{j=1}^{\infty} 2^{-j} d_{p_j}(x_j, y_j), \quad x = (x_j),\ y = (y_j) \in \hat{\mathbb{Z}},$$

which is consistent with the topology of $\hat{\mathbb{Z}}$. For any $\varepsilon > 0$, there exist $\ell \in \mathbb{N}$ and $y \in \hat{\mathbb{Z}}$ such that $d(x,y) < \varepsilon$ and that $p_j | y$ for $j \geq \ell$. Then we have $0 \leq \psi(y) \leq \prod_{j=\ell}^{\infty}(1 - (1/p_j)) = 0$, i.e., $\psi(y) = 0$, which means ψ is not continuous at x unless $\psi(x) = 0$. By Kolmogorov's 0-1 law, $\lambda(\psi^{-1}(\{0\})) = 0$ or 1, and Proposition 10 implies $\lambda(\psi^{-1}(\{0\})) < 1$, we see $\psi(x) > 0$, λ-a.e. $x \in \hat{\mathbb{Z}}$. Thus ψ is discontinuous λ-a.e.

Definition 12. *For each $k \in \mathbb{N} \cup \{0\}$, let $d_j(k) \in \{0,1\}$ denote the j-th bit of k, i.e., $k = \sum_{j=1}^{\infty} 2^{j-1} d_j(k)$. Then for each $k \in \mathbb{N} \cup \{0\}$, we define*

$$f_k(x) := \prod_{j \in \mathbb{N};\, d_j(k)=0} f\left(p_j^{\alpha_j(x)}\right) \prod_{j \in \mathbb{N};\, d_j(k)=1} \left(1 - f\left(p_j^{\alpha_j(x)}\right)\right), \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}.$$

Here and hereafter any empty product is assumed to be 1 *by convention. Note that $f_0 = f$.*

Lemma 13. *(i) $\sum_{k=0}^{\infty} f_k(n) = 1$, $n \in \mathbb{N}$.*
(ii) If $\mathbf{E}[f] > 0$, then $\sum_{k=0}^{\infty} f_k(x) = 1$, λ-a.e. $x \in \hat{\mathbb{Z}}$, hence $\sum_{k=0}^{\infty} \mathbf{E}[f_k] = 1$.

Proof. (i) For each $n \in \mathbb{N}$, there exists $J \in \mathbb{N}$ such that $\alpha_j(n) = 0$, i.e., $f\left(p_j^{\alpha_j(n)}\right) = 1$, if $j > J$. Then for each $k \in \mathbb{N} \cup \{0\}$, we have

$$f_k(n) = \prod_{1 \leq j \leq J;\, d_j(k)=0} f\left(p_j^{\alpha_j(n)}\right) \prod_{j \in \mathbb{N};\, d_j(k)=1} \left(1 - f\left(p_j^{\alpha_j(n)}\right)\right).$$

If $k \geq 2^J$, there exists $j > J$ such that $d_j(k) = 1$ and $f\left(p_j^{\alpha_j(n)}\right) = 1$, which implies $f_k(n) = 0$. Therefore we have

$$\sum_{k=0}^{\infty} f_k(n) = \sum_{k=0}^{2^J-1} f_k(n) = \prod_{j=1}^{J} \left(f\left(p_j^{\alpha_j(n)}\right) + \left(1 - f\left(p_j^{\alpha_j(n)}\right)\right)\right) = 1.$$

(ii) The above argument leads to

$$\begin{aligned}\sum_{k=0}^{2^J-1} f_k(x) &= \prod_{j=1}^{J} \left(f\left(p_j^{\alpha_j(x)}\right) + \left(1 - f\left(p_j^{\alpha_j(x)}\right)\right)\right) \prod_{j=J+1}^{\infty} f\left(p_j^{\alpha_j(x)}\right) \\ &= \prod_{j=J+1}^{\infty} f\left(p_j^{\alpha_j(x)}\right), \quad J \in \mathbb{N}.\end{aligned}$$

This implies that the limit $\sum_{k=0}^{\infty} f_k(x)$, which is $\sigma(\{\alpha_j\}_{j=1}^{\infty})$-measurable, is independent of $\sigma(\alpha_1, \ldots, \alpha_J)$ for any $J \in \mathbb{N}$. Since $\{\alpha_j\}_{j=1}^{\infty}$ are independent, $\sum_{k=0}^{\infty} f_k(x)$ is equal to a constant $c \geq 0$, λ-a.e. $x \in \hat{\mathbb{Z}}$ by Kolmogorov's 0-1 law. Now $\mathbf{E}[f] > 0$ implies $\lambda(f_0 > 0) > 0$, and hence $c > 0$. Therefore for λ-a.e. $x \in \hat{\mathbb{Z}}$, there exists $J \in \mathbb{N}$ such that $\prod_{j=J+1}^{\infty} f\left(p_j^{\alpha_j(x)}\right) > 0$. From this, it follows that

$$c = \sum_{k=0}^{\infty} f_k(x) = \lim_{J \to \infty} \prod_{j=J+1}^{\infty} f\left(p_j^{\alpha_j(x)}\right) = 1, \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}.$$

□

Lemma 14.

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n) \leq \mathbf{E}\left[f_k\right], \quad k \in \mathbb{N} \cup \{0\}.$$

Proof. First we show the lemma for $k = 0$, i.e., for the function f. For each $L \in \mathbb{N}$, we define

$$\begin{aligned} f^L(x) &:= \prod_{j=1}^{L} f\left(p_j^{\alpha_j(x)\mathbf{1}_{\{\alpha_j \leq L\}}(x)}\right) \\ &= \prod_{j=1}^{L} \left(\sum_{m=0}^{L} f\left(p_j^m\right) \mathbf{1}_{\{\alpha_j = m\}}(x) + \mathbf{1}_{\{\alpha_j \geq L+1\}}(x) \right), \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}. \end{aligned}$$

Since $0 \leq f \leq 1$, we see $f(n) \leq f^L(n) \leq 1$, $n \in \mathbb{N}$. $f^L(n)$ is a periodic function and hence

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) \leq \mathbf{M}\left[f^L\right] = \mathbf{E}\left[f^L\right].$$

Since $f^L(x) \searrow f(x)$ as $L \to \infty$, for λ-a.e. $x \in \hat{\mathbb{Z}}$, we see $\mathbf{E}[f^L] \searrow \mathbf{E}\left[f\right]$ by the monotone convergence theorem, and that

$$\limsup_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} f(n) \leq \mathbf{E}\left[f\right].$$

For $k, L \in \mathbb{N}$ and λ-a.e. $x \in \hat{\mathbb{Z}}$, we define

$$\begin{aligned} f_k^L(x) &:= \prod_{1 \leq j \leq L;\, d_j(k)=0} f\left(p_j^{\alpha_j(x)\mathbf{1}_{\{\alpha_j \leq L\}}(x)}\right) \\ &\quad \times \prod_{1 \leq j \leq L;\, d_j(k)=1} \left(1 - \mathbf{1}_{\{\alpha_j \leq L\}}(x) f\left(p_j^{\alpha_j(x)}\right)\right) \\ &= \prod_{1 \leq j \leq L;\, d_j(k)=0} \left(\sum_{m=0}^{L} f\left(p_j^m\right) \mathbf{1}_{\{\alpha_j = m\}}(x) + \mathbf{1}_{\{\alpha_j \geq L+1\}}(x) \right) \\ &\quad \times \prod_{1 \leq j \leq L;\, d_j(k)=1} \left(\sum_{m=1}^{L} \left(1 - f\left(p_j^m\right)\right) \mathbf{1}_{\{\alpha_j = m\}}(x) + \mathbf{1}_{\{\alpha_j \geq L+1\}}(x) \right). \end{aligned}$$

Then the above proof for f can also be applied to f_k. □

Lemma 15. *If* $\mathbf{E}[f] > 0$*, for any* $K \in \mathbb{N} \cup \{0\}$*, we have*

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k=K}^{\infty} f_k(n) = \limsup_{N\to\infty} \sum_{k=K}^{\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n)$$
$$\leq \sum_{k=K}^{\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n). \tag{9}$$

Before proving Lemma 15, let us complete the proof of Theorem 1.

Proof of Theorem 1. If $\mathbf{E}[f] = 0$, Lemma 14 ($k = 0$) implies that

$$0 \leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n) \leq \mathbf{E}[f] = 0.$$

This and Proposition 9 show (2).

Now, let us assume $\mathbf{E}[f] > 0$. By Lemma 13 (i) and Lemma 15, we have

$$1 = \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k=0}^{\infty} f_k(n) \leq \sum_{k=0}^{\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n).$$

By Lemma 13 (ii) and Lemma 14,

$$\sum_{k=0}^{\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n) \leq \sum_{k=0}^{\infty} \mathbf{E}[f_k] = 1.$$

The above two imply that

$$\sum_{k=0}^{\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n) = \sum_{k=0}^{\infty} \mathbf{E}[f_k] = 1. \tag{10}$$

By Lemma 14 again, we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n) = \mathbf{E}[f_k], \quad k \in \mathbb{N} \cup \{0\}. \tag{11}$$

Then by Lemma 13 (i), $f = f_0$, Lemma 15, (10), and (11), we have

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(n) &= \liminf_{N\to\infty} \left(1 - \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{\infty} f_k(n) \right) \\
&= 1 - \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \sum_{k=1}^{\infty} f_k(n) \\
&\geq 1 - \sum_{k=1}^{\infty} \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f_k(n) \\
&= 1 - \sum_{k=1}^{\infty} \mathbf{E}[f_k] = \mathbf{E}[f].
\end{aligned} \tag{12}$$

From (11) ($k = 0$) and (12), it follows that $\mathbf{M}[f] = \mathbf{E}[f]$. Finally, by Proposition 9, we obtain (2). □

To prove Lemma 15, we use two technical lemmas.

Lemma 16. *Suppose that $h(n) = h_0(n) + \cdots + h_{m-1}(n)$, where each $\{h_l(n)\}_{n=1}^{\infty}$ is a non-negative periodic sequence of the form*

$$h_l(n) = a_l \mathbf{1}_{Q_l\mathbb{N}}(n), \quad n \in \mathbb{N}, \quad a_l \geq 0, \quad Q_l \in \mathbb{N}, \quad l = 0, \ldots, m-1. \tag{13}$$

Then h is periodic with period $\operatorname{lcm}(Q_0, \ldots, Q_{m-1})$, and we have

$$\frac{1}{N}\sum_{n=1}^{N} h(n) \leq \mathbf{M}[h], \quad N \in \mathbb{N}. \tag{14}$$

Proof. For any $N \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{N}\sum_{n=1}^{N} h(n) &= \frac{1}{N}\sum_{n=1}^{N}\sum_{l=0}^{m-1} h_l(n) = \sum_{l=0}^{m-1}\frac{1}{N}\sum_{n=1}^{N} h_l(n) = \sum_{l=0}^{m-1}\frac{1}{N}\left\lfloor\frac{N}{Q_l}\right\rfloor a_l \\ &\leq \sum_{l=0}^{m-1}\frac{1}{N}\cdot\frac{N}{Q_l}\cdot a_l = \sum_{l=0}^{m-1}\frac{a_l}{Q_l} = \sum_{l=0}^{m-1}\mathbf{M}[h_l] = \mathbf{M}[h].\end{aligned}$$

□

Lemma 17. *Let $f : \mathbb{N} \to [0,1]$ be a multiplicative function, let $q_1 < \ldots < q_r$ be a finite sequence of primes, and let*

$$h(n) := \prod_{i=1}^{r}\left(1 - \mathbf{1}_{\{\beta_i \leq 1\}}(n) f\left(q_i^{\beta_i(n)}\right)\right), \quad n \in \mathbb{N},$$

where $\beta_i(n) := \alpha_{q_i}(n)$. Then the sequence $\{h(n)\}_{n=1}^{\infty}$ is a finite sum of periodic sequences of the form (13).

Proof. For each $l = 0, 1, \ldots, 2^r - 1$, let $h_l(n) := a_l \mathbf{1}_{Q_l\mathbb{N}}(n)$, $n \in \mathbb{N}$, where

$$a_l := \prod_{1\leq i\leq r;\, d_i(l)=0} (1 - f(q_i)) \prod_{1\leq i\leq r;\, d_i(l)=1} f(q_i), \quad Q_l := \prod_{i=1}^{r} q_i^{d_i(l)+1}.$$

Then we have $h(n) = h_0(n) + \cdots + h_{2^r-1}(n)$, $n \in \mathbb{N}$, as shown below.

Let $n \in \mathbb{N}$. If $Q_0 \nmid n$, there exists $i \in \{1, \ldots, r\}$ such that $\beta_i(n) = 0$ and hence we have $h(n) = h_0(n) + \cdots + h_{2^r-1}(n) = 0$. If $Q_{2^r-1} \mid n$, i.e., $\beta_i(n) \geq 2$ for all $i \in \{1, \ldots, r\}$, we have $h(n) = 1$, on the other hand,

$$h_0(n) + \cdots + h_{2^r-1}(n) = \sum_{l=0}^{2^r-1} a_l = \prod_{i=1}^{r}((1 - f(q_i)) + f(q_i)) = 1.$$

If $Q_0 \mid n$ and $Q_{2^r-1} \nmid n$, there exists a unique $l_0 \in \{0, \ldots, 2^r - 2\}$ for which $Q_{l_0} \mid n$ and $Q_{l'} \nmid n$ for any $l' \neq l_0$ such that $Q_{l_0} \mid Q_{l'}$. Then we have

$$h(n) = \prod_{1\leq i\leq r;\, d_i(l_0)=0} (1 - f(q_i)).$$

On the other hand,

$$\begin{aligned}\sum_{l=0}^{2^r-1} h_l(n) &= \sum_{l=0}^{2^r-1} a_l \mathbf{1}_{Q_l\mathbb{N}}(n) = \sum_{\substack{0\leq l\leq 2^r-1 \\ Q_l \mid Q_{l_0}}} a_l \\ &= \sum_{\substack{0\leq l\leq 2^r-1 \\ Q_l \mid Q_{l_0}}} \prod_{1\leq i\leq r;\, d_i(l)=0} (1 - f(q_i)) \prod_{1\leq i\leq r;\, d_i(l)=1} f(q_i).\end{aligned}$$

If $Q_l \mid Q_{l_0}$ and $d_i(l_0) = 0$ then $d_i(l) = 0$. This means that if $Q_l \mid Q_{l_0}$, we have

$$\prod_{1\le i\le r;\, d_i(l)=0} (1 - f(q_i)) = \prod_{1\le i\le r;\, d_i(l_0)=0} (1 - f(q_i)) \times \prod_{1\le i\le r;\, d_i(l_0)=1,\, d_i(l)=0} (1 - f(q_i)).$$

Consequently,

$$\begin{aligned}
\sum_{l=0}^{2^r-1} h_l(n) &= \prod_{1\le i\le r;\, d_i(l_0)=0} (1 - f(q_i)) \\
&\quad \times \sum_{\substack{0\le l\le 2^r-1 \\ Q_l \mid Q_{l_0}}} \prod_{\substack{1\le i\le r \\ d_i(l_0)=1,\, d_i(l)=0}} (1 - f(q_i)) \prod_{\substack{1\le i\le r \\ d_i(l_0)=1,\, d_i(l)=1}} f(q_i) \\
&= \prod_{1\le i\le r;\, d_i(l_0)=0} (1 - f(q_i)) \prod_{1\le i\le r;\, d_i(l_0)=1} ((1 - f(q_i)) + f(q_i)) \\
&= \prod_{1\le i\le r;\, d_i(l_0)=0} (1 - f(q_i)).
\end{aligned}$$

Thus we see $h(n) = h_0(n) + \cdots + h_{2^r-1}(n)$ for all $n \in \mathbb{N}$. □

Proof of Lemma 15. Let $k \in \mathbb{N} \cup \{0\}$ and let

$$g_k(x) := \prod_{j\in\mathbb{N};\, d_j(k)=1} \left(1 - \mathbf{1}_{\{\alpha_j\le 1\}}(x) f\left(p_j^{\alpha_j(x)}\right)\right), \quad \lambda\text{-a.e. } x \in \hat{\mathbb{Z}}.$$

Then $f_k(n) \le g_k(n)$, and $\{g_k(n)\}_{n=1}^{\infty}$ is a finite sum of periodic sequences of the form (13) by Lemma 17. Therefore Lemma 16 implies that

$$\frac{1}{N}\sum_{n=1}^{N} f_k(n) \le \frac{1}{N}\sum_{n=1}^{N} g_k(n) \le \mathbf{M}[g_k] = \mathbf{E}[g_k] =: G_k, \quad N \in \mathbb{N}. \tag{15}$$

On the other hand, we have

$$\begin{aligned}
\sum_{k=0}^{2^J-1} G_k &= \sum_{k=0}^{2^J-1} \mathbf{E}[g_k] = \mathbf{E}\left[\sum_{k=0}^{2^J-1} g_k\right] \\
&= \mathbf{E}\left[\prod_{j=1}^{J} \left(1 + \left(1 - \mathbf{1}_{\{\alpha_j\le 1\}} f\left(p_j^{\alpha_j}\right)\right)\right)\right] \\
&= \prod_{j=1}^{J} \left(1 + \mathbf{E}\left[1 - \mathbf{1}_{\{\alpha_j\le 1\}} f\left(p_j^{\alpha_j}\right)\right]\right) \\
&= \prod_{j=1}^{J} \left(1 + \left((1 - f(p_j))\left(\frac{1}{p_j} - \frac{1}{p_j^2}\right) + \frac{1}{p_j^2}\right)\right) \\
&\le \prod_{j=1}^{J} \left(1 + \left(\frac{1 - f(p_j)}{p_j} + \frac{1}{p_j^2}\right)\right).
\end{aligned}$$

Now, if $\mathbf{E}[f] > 0$, i.e., $\sum_p p^{-1}(1 - f(p)) < \infty$ (Proposition 10), we see

$$\sum_{k=0}^{\infty} G_k \leq \prod_p \left(1 + \left(\frac{1 - f(p)}{p} + \frac{1}{p^2}\right)\right) < \infty. \tag{16}$$

By (15) and (16), we can apply the Lebesgue–Fatou lemma to obtain (9). $\square$

4 Proof of Theorem 3

Lemma 18. *Let $f : \mathbb{N} \to [0, 1]$ be multiplicative. If $\psi : [0, 1] \to \mathbb{C}$ is a polynomial function, $\mathbf{M}[\psi(f)]$ exists and it is equal to $\mathbf{E}[\psi(f)]$.*

Proof. For each $k \in \mathbb{N}$, since the k-th power f^k of f is also $[0, 1]$-valued and multiplicative, Theorem 1 and Proposition 9 imply that $\mathbf{M}[f^k]$ exists and $\mathbf{M}[f^k] = \mathbf{E}[f^k]$. Consequently, for any $c_0, \ldots, c_l \in \mathbb{C}$, we have

$$\mathbf{E}\left[\sum_{k=0}^{l} c_k f^k\right] = \sum_{k=0}^{l} c_k \mathbf{E}\left[f^k\right] = \sum_{k=0}^{l} c_k \mathbf{M}\left[f^k\right] = \mathbf{M}\left[\sum_{k=0}^{l} c_k f^k\right].$$

$\square$

Proof of Theorem 3. (i) Take any $\varepsilon > 0$. If $\varphi : [0, 1] \to \mathbb{C}$ is continuous, by the Weierstrass approximation theorem, there exists a $\mathbb{C}$-valued polynomial function φ_ε such that

$$\max\{\, |\varphi(t) - \varphi_\varepsilon(t)|\,;\, t \in [0, 1]\,\} < \frac{\varepsilon}{3}.$$

By Lemma 18, there exists $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$, we have

$$\left|\frac{1}{N} \sum_{n=1}^{N} \varphi_\varepsilon(f(n)) - \mathbf{E}[\varphi_\varepsilon(f)]\right| < \frac{\varepsilon}{3}.$$

Then for any $N \geq N_0$, we have

$$\begin{aligned}
&\left|\frac{1}{N} \sum_{n=1}^{N} \varphi(f(n)) - \mathbf{E}[\varphi(f)]\right| \\
&\quad \leq \left|\frac{1}{N} \sum_{n=1}^{N} \varphi(f(n)) - \frac{1}{N} \sum_{n=1}^{N} \varphi_\varepsilon(f(n))\right| + \left|\frac{1}{N} \sum_{n=1}^{N} \varphi_\varepsilon(f(n)) - \mathbf{E}[\varphi_\varepsilon(f)]\right| \\
&\qquad + |\mathbf{E}[\varphi_\varepsilon(f)] - \mathbf{E}[\varphi(f)]| \\
&\quad < \frac{1}{N} \sum_{n=0}^{N} |\varphi(f(n)) - \varphi_\varepsilon(f(n))| + \frac{\varepsilon}{3} + \mathbf{E}\left[|\varphi_\varepsilon(f) - \varphi(f)|\right] \\
&\quad < \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,
\end{aligned}$$

which completes the proof of (i).

(ii) If $\varphi : [0, 1] \to [-\infty, \infty)$ is upper semi-continuous, there exists a decreasing sequence of continuous functions $\{\varphi_k\}_{k=1}^{\infty}$ such that $\varphi_k(t) \searrow \varphi(t)$, $t \in [0, 1]$, as $k \to \infty$. Then for each $k \in \mathbb{N}$, by Lemma 18, we have

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi(f(n)) \leq \limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \varphi_k(f(n)) = \mathbf{M}[\varphi_k(f)] = \mathbf{E}[\varphi_k(f)]$$

and by the monotone convergence theorem, we have

$$\mathbf{E}[\varphi_k(f)] \searrow \mathbf{E}[\varphi(f)], \quad k \to \infty.$$

Thus we see (4). Similarly, we can prove (5).

(iii) Since $\mathbf{1}_{[0,t]}$ is upper semi-continuous and $\mathbf{1}_{[0,t)}$ is lower semi-continuous, (iii) follows from (ii). $\square$

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