Ryshkov domains of reductive algebraic groups

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Abstract

Let $G$ be a connected reductive algebraic group defined over a number field $k$. In this paper, we introduce the Ryshkov domain $R$ for the arithmetical minimum function $m_Q$ defined from a height function associated to a maximal $k$-parabolic subgroup $Q$ of $G$. The domain $R$ is a $Q(k)$-invariant subset of the adele group $G(A)$. We show that a fundamental domain $\Omega$ for $Q(k) \backslash R$ yields a fundamental domain for $G(k) \backslash G(A)$. We also see that any local maximum of $m_Q$ is attained in the boundary of $\Omega$.

Introduction Let $P_n$ be the cone of positive definite $n$ by $n$ real symmetric matrices, and let $m(A)$ be the arithmetical minimum $\min_{x \in \mathbb{Z}^n} A x x^t$ of $A \in P_n$. The function $f : A \mapsto m(A)/(\det A)^{1/n}$ on $P_n$ is called the Hermite invariant. Since the maximum of $f$ gives the Hermite constant $\gamma_n$ for dimension $n$, the determination of local maxima of $f$ is a fundamental problem of lattice sphere packings in Euclidean spaces and arithmetic theory of quadratic forms. Voronoi’s theorem [13, Théorème 17] states that $f$ attains a local maximum at a point $A$ if and only if $A$ is perfect and eutactic. Moreover, perfect forms play an essential role in Voronoi’s reduction theory of $P_n$ with respect to the action of $GL_n(\mathbb{Z})$ (see, e.g., [7], [9]). In [8], Ryshkov introduced a locally finite polyhedron $R(m)$ in $P_n$ defined by the condition $m(A) \geq 1$. It is not difficult to show that $A$ is perfect with $m(A) = 1$ if and only if $A$ is a vertex of the boundary of $R(m)$. In particular, any local maximum of the Hermite invariant $f$ is attained in the boundary of $R(m)$. In this sense, we can say that the Ryshkov polyhedron $R(m)$ is well matched with $f$.

Let $G$ be a connected isotropic reductive algebraic group defined over a number field $k$, and let $Q$ be a maximal $k$-parabolic subgroup of $G$. In previous papers [14] and [15], we investigated a constant $\gamma(G, Q, k)$ as a generalization of Hermite’s constant $\gamma_n$. Precisely, the constant $\gamma(G, Q, k)$ is defined to be the maximum of the function $m_Q(g) = \min_{x \in Q(k) \backslash G(k)} H_Q(xg) \det(xg)^{1/n}$ on $G(k) \backslash G(A)$, where $H_Q$ denotes the height function associated to $Q$. To prove the existence of the maximum of $m_Q$, we used Borel and Harish-Chandra’s reduction theory for the adele group $G(A)$ with respect to $G(k)$. However, a Siegel set in $G(A)$ is not well matched with $m_Q$ in a sense that one can not obtain any information on locations of extreme points of $m_Q$ in a Siegel set.

The purpose of this paper is to construct a fundamental domain of $G(A)$ with respect to $G(k)$ which is well matched with $m_Q$. We first consider an analog of the Ryshkov polyhedron. For a given $g \in G(A)$, we set $X_Q(g) = \{ x \in Q(k) \backslash G(k) : m_Q(g) = H_Q(xg) \}$. This is a finite subset of $Q(k) \backslash G(k)$ and is regarded as an analog of the set of minimal vectors of a positive definite real quadratic form. We define the domain $R(m_Q)$ as follows:

$$R(m_Q) = \{ g \in G(A) : \pi \in X_Q(g) \},$$
where \( \tau \) denotes the trivial class \( Q(k) \) in \( Q(k) \setminus G(k) \). The set \( R(m_Q) \) is a left \( Q(k) \)-invariant closed set with non-empty interior. The interior of \( R(m_Q) \) is just a subset \( R_1 \) consisting of \( g \in R(m_Q) \) such that \( \chi_Q(g) \) is the one-point set \( \{ \tau \} \). We denote by \( R_1^\tau \) the closure of \( R_1 \) in \( G(A) \). Both \( R_1 \) and \( R_1^\tau \) are also left \( Q(k) \)-invariant. By Baer and Levi’s theorem \( [1, \text{Satz 7}] \), there exists an open fundamental domain \( \Omega_Q \) of \( R_1^\tau \) with respect to \( Q(k) \), i.e., \( \Omega_Q \) is a relatively open subset of \( R_1^\tau \) satisfying

- \( Q(k)\Omega_Q = R_1^\tau \), where \( \Omega_Q \) denotes the closure of \( \Omega_Q \) in \( R_1^\tau \), and
- \( \gamma \Omega_Q \cap \Omega_Q = \emptyset \) for any \( \gamma \in Q(k) \setminus \{ e \} \).

Let \( \Omega_Q \) denote the interior of \( \Omega_Q \) in \( G(A) \). Then our main theorem is stated as follows:

**Theorem.** The set \( \Omega_Q^\circ \) is an open fundamental domain of \( G(A) \) with respect to \( G(k) \). Any local maximum of \( m_Q \) is attained in the intersection of the boundary of \( \Omega_Q^\circ \) and the boundary of \( R_1^\tau \).

If we denote by \( r_G \) the k-rank of the commutator subgroup of \( G \), then \( G \) has \( r_G \) standard maximal k-parabolic subgroups. Since \( \Omega_Q \) depends on \( Q \), we obtain \( r_G \) different kinds of fundamental domains of \( G(A) \) with respect to \( G(k) \). The method to construct \( \Omega_Q \) may be viewed as a generalization of the highest point method \( [5], [11, \text{§4,4}] \). For example, let \( k = Q \), \( G = \text{GL}_n \) and \( Q \) be a standard maximal \( Q \)-parabolic subgroup such that \( Q/G \) is a projective space. Then our construction gives a fundamental domain \( \Omega_Q \) whose Archimedean part is isomorphic with Grenier’s fundamental domain. If we choose another standard maximal \( Q \)-parabolic subgroup of \( \text{GL}_n \) as \( Q \), then the Archimedean part of \( \Omega_Q \) yields a new kind of fundamental domain of \( P_n \) with respect to \( \text{GL}_n(Z) \) (see Example 1 in §7).

**Notation.** For a given ring \( \mathfrak{A} \), the set of all \( n \) by \( k \) matrices with entries in \( \mathfrak{A} \) is denoted by \( M_{n,k}(\mathfrak{A}) \). We write \( M_n(\mathfrak{A}) \) for \( M_{n,n}(\mathfrak{A}) \). The transpose of a given matrix \( a \in M_{n,k}(\mathfrak{A}) \) is denoted by \( a^\top \). In this paper, \( k \) denotes an algebraic number field of finite degree over \( Q \) and \( \mathfrak{o} \) the ring of integers of \( k \). The sets of all infinite and finite places of \( k \) are denoted by \( p_{\infty} \) and \( p_f \), respectively. For \( \sigma \in p_{\infty} \cup p_f \), \( k_{\sigma} \) denotes the completion of \( k \) at \( \sigma \). For \( \sigma \in p_f \), \( \sigma_\infty \) denotes the closure of \( \sigma \) in \( k_{\sigma_\infty} \). The étale \( R \)-algebra \( k_{\sigma} = k \otimes_Q R \) is identified with \( \prod_{\sigma \in p_{\infty} \cup p_f} k_{\sigma} \). Let \( A \) and \( A^\times \) denote the adele ring and the idele group of \( k \), respectively. The idele norm of \( A^\times \) is denoted by \( | \cdot |_A \).

1. Height functions

Let \( G \) be a connected affine algebraic group defined over \( k \). For any k-algebra \( \mathfrak{A} \), \( G(\mathfrak{A}) \) stands for the set of \( \mathfrak{A} \)-rational points of \( G \). Let \( X^*(G)_k \) be the free \( \mathbb{Z} \)-modules consisting of all k-rational characters of \( G \). For each \( g \in G(A) \), we define the homomorphism \( \vartheta_G(g) : X^*(G)_k \rightarrow R_{\geq 0} \) by \( \vartheta_G(g)(\chi) = |\chi(g)|_A \) for \( \chi \in X^*(G)_k \). Then \( \vartheta_G \) is a homomorphism from \( G(A) \) into \( \text{Hom}_\mathbb{Z}(X^*(G)_k, R_{\geq 0}) \). We write \( G(A)^1 \) for the kernel of \( \vartheta_G \).

In the following, let \( G \) be a connected isotropic reductive group defined over \( k \). We fix a maximal k-split torus \( S \) of \( G \) and a minimal k-parabolic subgroup \( P_0 \) of \( G \) containing \( S \).

- Denote by \( \Phi_k \) and \( \Delta_k \) the relative root system of \( G \) with respect to \( S \) and the set of simple roots of \( \Phi_k \) corresponding to \( P_0 \), respectively. Let \( M_0 \) be the centralizer of \( S \) in \( G \). Then \( P_0 \) has a Levi decomposition \( P_0 = M_0U_0 \), where \( U_0 \) is the unipotent radical of \( P_0 \).
- A k-parabolic subgroup of \( G \) containing \( P_0 \) is called a standard k-parabolic subgroup of \( G \).
- Every standard k-parabolic subgroup \( R \) of \( G \) has a unique Levi subgroup \( M_R \) containing \( M_0 \). We
denote by $U_R$ the unipotent radical of $R$ and $Z_R$ the greatest central $k$-split torus in $M_R$. Throughout this paper, we fix a maximal compact subgroup $K = \prod_{\sigma \in \varphi,R} K_{\sigma} \times \prod_{\sigma \in \varphi,R} K_{\sigma}$ of $G(A)$ satisfying the following property: For every standard $k$-parabolic subgroup $R$ of $G$, $K \cap M_R(A)$ is a maximal compact subgroup of $M_R(A)$ and $M_R(A)$ possesses an Iwasawa decomposition $(M_R(A) \cap U_0(A))M_0(A)(K \cap M_R(A))$.

Let $Q$ be a standard proper maximal $k$-parabolic subgroup of $G$. There is an only one simple root $\alpha_0 \in \Delta_k$ such that the restriction of $\alpha_0$ to $Z_Q$ is non-trivial. Let $n_Q$ be the positive integer such that $n_Q^{-1} \alpha_0|Z_Q$ is a $Z$-basis of $X^*(Z_Q/Z_Q)_k$. We write $a_Q$ and $\hat{a}_Q$ for $n_Q^{-1} \alpha_0|Z_Q$ and $\hat{a}_Q n_Q^{-1} \alpha_0|Z_Q$, respectively, where $\hat{a}_Q = [X^*(Z_Q/Z_Q)_k : X^*(M_Q/Z_Q)_k]$. Then $\hat{a}_Q$ is a $Z$-basis of the submodule $X^*(M_Q/Z_Q)_k$ of $X^*(Z_Q/Z_Q)_k$. Define the map $z_Q : G(A) \rightarrow Z_G(A)M_Q(A)^1 \setminus M_Q(A)$ by $z_Q(g) = Z_G(A)M_Q(A)^1 m$ if $g = umh$, $u \in U_Q(A), m \in M_Q(A)$ and $h \in K$. This is well defined and left $Z_G(A)Q(A)^1$-invariant. Since $Z_G(A)^1 = Z_G(A) \cap G(A)^1 \subset M_Q(A)^1$, $z_Q$ gives rise to a map from $Y_Q = Q(A)^1 \setminus G(A)^1$ to $M_Q(A)^1 \setminus (M_Q(A) \cap G(A)^1)$. Namely, we have the following commutative diagram:

\[
\begin{array}{ccc}
Y_Q & \longrightarrow & M_Q(A)^1 \setminus (M_Q(A) \cap G(A)^1) \\
\downarrow & & \downarrow \\
Z_G(A)Q(A)^1 \setminus G(A) & \longrightarrow & Z_G(A)M_Q(A)^1 \setminus M_Q(A)
\end{array}
\]

Here both vertical arrows are natural maps. We define a height function $H_Q : G(A) \rightarrow \mathbb{R}_{\geq 0}$ by $H_Q(g) = |\hat{a}_Q(z_Q(g))|_{A}^2$ for $g \in G(A)$. We notice that the restriction of $H_Q$ to $M_Q(A)$ is a homomorphism from $M_Q(A)$ onto $\mathbb{R}_{\geq 0}$.

2. Twisted height functions restricted to one parameter subgroups

Let $N_G(S)$ be the normalizer of $S$ in $G$ and $W_G = N_G(S)(k)/M_0(k)$ the Weyl group of $G$ with respect to $S$. For a simple root $\alpha \in \Delta_k$, $s_\alpha \in W_G$ denotes the simple reflection corresponding to $\alpha$. Then $\{s_\alpha\}_{\alpha \in \Delta_k}$ generates $W_G$. We denote by $W_G^Q$ the subgroup of $W_G$ generated by $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{\alpha_0\}}$. For each $w \in W_G$, we use the same notation $w$ for a representative of $w$ in $N_G(S)(k)$. The following cell decomposition of $G(k)$ holds via Bruhat decomposition ([4, Proposition 4.10, Corollaire 5.20]):

$$G(k) = \bigsqcup_{[w] \in W_G^Q \setminus W_G/W_G^Q} Q(k)wQ(k),$$

where $[w]$ stands for the class $W_G^Q w W_G^Q$ in $W_G^Q \setminus W_G/W_G^Q$.

The Weyl group $W_G$ acts on $X^*(S)_k$ by $w : \chi \mapsto \chi(w^{-1}tw)$ for $w \in W_G$ and $\chi \in X^*(S)_k$. We consider the restriction $\hat{a}_Q|S$ of the rational character $\hat{a}_Q$ of $M_Q$ to $S$.

**Lemma 1.** The subgroup of $W_G$ fixing $\hat{a}_Q|S$ is equal to $W_G^Q$.

**Proof.** Put $W' = \{w \in W_G : w \cdot \hat{a}_Q|S = \hat{a}_Q|S\}$. Since a representative of $w \in W_G^Q$ is contained in $M_Q(k)$, we have $\hat{a}_Q(w^{-1}tw) = \hat{a}_Q(w)^{-1} \hat{a}_Q(t) \hat{a}_Q(w) = \hat{a}_Q(t)$ for all $t \in S$. Hence $W_G^Q$ is contained in $W'$. By [6, §1.12 Theorem (a) and (c)], $W'$ is generated by a subset $W' \cap \{s_\alpha\}_{\alpha \in \Delta_k}$ of simple reflections. From $W_G^Q \subset W'$, it follows $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{\alpha_0\}} \subset W' \cap \{s_\alpha\}_{\alpha \in \Delta_k} \subset \{s_\alpha\}_{\alpha \in \Delta_k}$. Since $\hat{a}_Q$ is non-trivial on $S/Z_Q$, $W' \cap \{s_\alpha\}_{\alpha \in \Delta_k}$ must be equal to $\{s_\alpha\}_{\alpha \in \Delta_k \setminus \{\alpha_0\}}$. Therefore $W'$ coincides with $W_G^Q$. \[\square\]
Let $X_*(S)_k$ be the free $\mathbb{Z}$-module consisting of all $k$-rational cocharacters of $S$. A natural pairing
\[
\langle \cdot, \cdot \rangle : X^*(S)_k \times X_*(S)_k \rightarrow \mathbb{Z}
\]
defined as in [3, §8.6] is a dual pairing over $\mathbb{Z}$.

**Lemma 2.** Let $w_1$ and $w_2$ be elements of $W_G$ such that $w_1^{-1} w_2^Q \neq w_2^{-1} W_G^Q$. Then there exist a cocharacter $\xi = \xi_{w_1,w_2} \in X_*(S)_k$ such that $H_Q(w_1 \xi(\lambda) w_1^{-1}) > H_Q(w_2 \xi(\lambda) w_2^{-1})$ holds for all $\lambda \in A^\times_{>0}$, where $A^\times_{>0}$ denotes the set of $\lambda \in A^\times$ satisfying $|\lambda|_A > 1$.

**Proof.** Since $w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S \neq 0$ by Lemma 1, there is an $\xi \in X_*(S)_k$ such that $\langle w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S, \xi \rangle < 0$. The value $\ell = \langle w_1^{-1} \cdot \widehat{\alpha}_Q|_S - w_2^{-1} \cdot \widehat{\alpha}_Q|_S, \xi \rangle$ is a negative integer. We have
\[
\widehat{\alpha}_Q(w_1 \xi(\lambda) w_1^{-1}) \cdot \widehat{\alpha}_Q(w_2 \xi(\lambda) w_2^{-1})^{-1} = \lambda^\ell
\]
for all $\lambda \in G_m$. Therefore,
\[
H_Q(w_1 \xi(\lambda) w_1^{-1}) H_Q(w_2 \xi(\lambda) w_2^{-1})^{-1} = |\lambda|_A^{\ell} > 1
\]
holds for all $\lambda \in A^\times_{>0}$. \(\square\)

3. The Hermite function and minimal points
We set $X_Q = Q(k) \backslash G(k)$, which is regarded as a subset of $Y_Q$. Let $\pi_X : G(k) \rightarrow X_Q$ be the natural quotient map. We write $\pi$ for $\pi_X(e) \in X_Q$. The Hermite function $m_Q : G(A)^1 \rightarrow R_{>0}$ is defined to be
\[
m_Q(g) = \min_{x \in X_Q} H_Q(xg).
\]
By definition, $m_Q$ is a positive valued continuous function on $G(k) \backslash G(A)^1 / K$.

For each $g \in G(A)^1$, we put
\[
X_Q(g) = \{ x \in X_Q : m_Q(g) = H_Q(xg) \},
\]
which is a finite subset of $X_Q$. Thus we can define the counting function $n_Q(g) = \sharp X_Q(g)$.

**Lemma 3.** For any $g \in G(A)^1$, $\gamma \in G(k)$ and $h \in K$, one has $X_Q(\gamma gh) = X_Q(g)\gamma^{-1}$. Especially, the counting function $n_Q$ is left $G(k)$-invariant and right $K$-invariant.

The following Lemma is proved by the same method as in [16, Proof of Proposition 4.1].

**Lemma 4.** For $g \in G(A)^1$, there is a neighbourhood $U$ of $g$ in $G(A)^1$ such that $X_Q(g') \subset X_Q(g)$ for all $g' \in U$.

4. The Ryshkov domain of $G$ associated to $Q$
We define the Ryshkov domain $R = R(m_Q)$ of $m_Q$ by
\[
R = R(m_Q) = \{ g \in G(A)^1 : m_Q(g) / H_Q(g) \geq 1 \}.
\]
Since $m_Q(g) \leq H_Q(g)$ holds for all $g \in G(A)^1$, we have
\[
R = \{ g \in G(A)^1 : m_Q(g) = H_Q(g) \}
\]
\[
= \{ g \in G(A)^1 : \varepsilon \in X_Q(g) \}.
\]
Since both $H_Q$ and $m_Q$ are continuous, $R$ is a closed subset in $G(A)^1$. 
Lemma 5. One has $Q(k)R K = R$ and $G(\mathbf{A})^1 = G(k) R$.

Proof. The first assertion is obvious by the definition of $H_Q$. To prove the second assertion, we choose a minimal point $x \in X_Q(g)$ for a given $g \in G(\mathbf{A})^1$. There is a $\gamma \in G(k)$ such that $x = \pi_X(\gamma)$. Then $H_Q(xg) = H_Q(\gamma g) = m_Q(g) = m_Q(\gamma g)$ since $m_Q$ is left $G(k)$-invariant. Therefore, we have $\gamma g \in R$.

Lemma 6. Let $C$ be an arbitrary subset of $G(\mathbf{A})^1$, and let $g \in G(\mathbf{A})^1$ and $\gamma \in G(k)$. Then we have the following:

1. $\gamma g \in R$ if and only if $\pi_X(\gamma) \in X_Q(g)$.
2. $X_Q(g) = \pi_X({\gamma \in G(k)} : \gamma g \in R)$.
3. $\gamma C \subset R$ if and only if $\pi_X(\gamma) \in \bigcap_{g \in C} X_Q(g)$.
4. $\bigcap_{g \in R} X_Q(g) = \{e\}$.
5. $\gamma R \subset R$ if and only if $\gamma \in Q(k)$.

Proof. By definition, $\gamma g \in R$ if and only if $m_Q(\gamma g) = H_Q(\gamma g)$. This is equivalent with $\pi_X(\gamma) \in X_Q(g)$ because of $m_Q(\gamma g) = m_Q(g)$. Both (2) and (3) follow from (1). For a point $x = \pi_X(\gamma) \in \bigcap_{g \in R} X_Q(g)$, we have $\gamma Q(k) R \subset R$, in other words, $x Q(k) \subset \bigcap_{g \in R} X_Q(g)$. Since $x Q(k)$ is an infinite set for $x \neq \pi$, by Bruhat decomposition, we must have $x = \pi$. This shows (4). (5) follows from (3) and (4).

Lemma 7. Let $g_0 \in R$ be an element such that $n_Q(g_0) > 1$ and $x_0$ be an arbitrary element in $X_Q(g_0)$. Then, any neighbourhood $U$ of $g_0$ in $G(\mathbf{A})^1$ contains a point $g$ such that $X_Q(g) \subset X_Q(g_0)$ and $x_0 \notin X_Q(g)$.

Proof. We may assume that $U$ satisfies $X_Q(g) \subset X_Q(g_0)$ for all $g \in U$ by Lemma 4. Since $n_Q(g_0) > 1$, there is an $x \in X_Q(g_0)$ such that $x \neq \pi$. This $x$ is of the form $\pi_X(w^\gamma)$ with $w \in W_G \setminus W_Q$ and $\gamma \in Q(k)$. By Lemma 2, there is a cocharacter $\xi = \xi_{w,e} \in X_*(S)_k$ such that $H_Q(w^\xi(\lambda)w^{-1}) > H_Q(\xi(\lambda))$ holds for all $\lambda \in \mathbf{A}_x^\times$. Let $\lambda \in \mathbf{A}_x^\times$ be an element sufficiently close to 1 so that $g_\lambda = \gamma^{-1} \xi(\lambda) \gamma g_0$ is contained in $U$. We have

$$H_Q(g_\lambda) = H_Q(\xi(\lambda) \gamma g_0) = H_Q(\xi(\lambda)) H_Q(\gamma g_0) = H_Q(\xi(\lambda)) H_Q(g_0) = H_Q(\xi(\lambda)) m_Q(g_0)$$

and

$$H_Q(x g_\lambda) = H_Q(w^\xi(\lambda) \gamma g_0) = H_Q(w^\xi(\lambda) w^{-1}) H_Q(\gamma g_0) = H_Q(w^\xi(\lambda) w^{-1}) m_Q(g_0).$$

If $x_0 = \pi$, then we choose $\lambda$ sufficiently close to 1 satisfying $\lambda^{-1} \in A_\lambda^\times$. Since $X_Q(g_\lambda) \subset X_Q(g_0)$ and $m_Q(g_\lambda) \leq H_Q(x g_\lambda) \leq H_Q(x g_0)$, $X_Q(g_\lambda)$ does not contain $\pi$. If $x_0 \neq \pi$, then we choose $x$ as $x_0$ and $\lambda \in A_\lambda^\times$ sufficiently close to 1. Since $m_Q(g_\lambda) \leq H_Q(g_\lambda) < H_Q(x_0 g_0)$, $X_Q(g_\lambda)$ does not contain $x_0$.

Lemma 8. $\min_{g \in G(\mathbf{A})^1} n_Q(g) = \min_{g \in R} n_Q(g) = 1$.

Proof. From Lemma 5 and the $G(k)$-invariance of $n_Q$, it follows that $\min_{g \in G(\mathbf{A})^1} n_Q(g)$ equals $\min_{g \in R} n_Q(g)$. If $g_0 \in R$ satisfies $\min_{g \in R} n_Q(g) = n_Q(g_0) > 1$, then, by Lemmas 5 and 7, there exists a point $g_1 \in G(\mathbf{A})^1$ and $\gamma_1 \in G(k)$ such that $n_Q(\gamma_1 g_1) = n_Q(g_1) < n_Q(g_0)$ and $\gamma_1 g_1 \in R$. This is a contradiction.

We define the subset $R_1$ of $R$ by

$$R_1 = \{g \in R : n_Q(g) = 1\} = \{g \in G(\mathbf{A})^1 : X_Q(g) = \{\pi\}\}.$$
Lemma 9. $R_1$ coincides with the interior $R^2$ of $R$ in $G(A)^1$.

Proof. For $g \in R_1$, we choose a neighbourhood $U$ of $g$ in $G(A)^1$ as in Lemma 4. Then $U \subset R_1$. Therefore, $R_1$ is open and is contained in $R^2$. If there exists an element $g_0 \in R^2$ such that $n_Q(g_0) > 1$, then, by Lemma 7, $R^2$ contains an element $g$ satisfying $\pi \notin X_Q(g)$. This contradicts to $g \in R$. □

It is obvious that $G(k)R_1 = \{ g \in G(A)^1 : n_Q(g) = 1 \}$.

Lemma 10. $G(k)R_1$ is open and dense in $G(A)^1$.

Proof. Since $R_1$ is open in $G(A)^1$, so is $G(k)R_1$. We assume that $G(A)^1 \setminus G(k)R_1$ has an interior point $g_0$. Let $U$ be a neighbourhood of $g_0$ in $G(A)^1$ so that $U \cap G(k)R_1 = \emptyset$. By Lemma 5, we can take $\gamma_0 \in G(k)$ such that $\gamma_0 g_0 \in R$. Since $n_Q(\gamma_0 g_0) = n_Q(g_0) > 1$, by Lemmas 5 and 7, there exists $\gamma_1 \in G(k)$ such that $n_Q(\gamma_1) < n_Q(g_0)$ and $\gamma_1 \gamma_0 \in R$. If $n_Q(\gamma_1) > 1$, then there exists $g_2 \in \gamma_1 \gamma_0 U$ and $\gamma_2 \in G(k)$ such that $n_Q(\gamma_2) < n_Q(\gamma_1)$ and $\gamma_2 g_2 \in R$. This process terminates at finite times. At the last step, we obtain an element $g_t \in \gamma_t \cdots \gamma_0$ such that $n_Q(g_t) = 1$. Then $(\gamma_{t-1} \cdots \gamma_0)^{-1}g_t$ is contained in $U \cap G(k)R_1$. This contradicts to $U \cap G(k)R_1 = \emptyset$. Therefore, $G(A)^1 \setminus G(k)R_1$ is nowhere dense in $G(A)^1$. □

Lemma 11. For $\gamma \in G(k)$, $R_1 \cap \gamma R \neq \emptyset$ if and only if $\gamma \in Q(k)$.

Proof. If $R_1 \cap \gamma R$ has an element $g$, then $\pi_X(\gamma^{-1}) \in X_Q(g) = \{ 0 \}$ by Lemma 6. □

Lemma 12. Let $R_1^\gamma$ be the closure of $R_1$. Then we have the following subdivision of $G(A)^1$:

$$G(A)^1 = \bigcup_{\gamma \in G(k)/Q(k)} \gamma R_1^\gamma.$$  

Proof. We fix an arbitrary $g \in G(A)^1$. By Lemma 10, there exists a sequence $\{ g_n \} \subset G(k)R_1$ such that $\lim_{n \to \infty} g_n = g$. We take a neighbourhood $U$ of $g$ as in Lemma 4 and may assume that $\{ g_n \} \subset U$. Since $g_n \in G(k)R_1$, $X_Q(g_n)$ consists of a single element $\pi_X(\gamma_n)$, where $\gamma_n \in G(k)$. From $g_n \in U$, it follows that $\pi_X(\gamma_n) \in X_Q(g)$ for all $n$. Since $X_Q(g)$ is a finite set, we can take a subsequence $\{ g_{n_j} \}$ such that $\pi_X(\gamma_{n_j}) = \pi_X(\gamma) \in X_Q(g)$ for all $n_j$. Then $\{ g_{n_j} \} \subset \gamma^{-1}R_1$, and $g$ is contained in the closure of $\gamma^{-1}R_1^\gamma$. □

For $g \in G(A)^1$, we put

$$S_Q(g) = \pi_X(\{ \gamma \in G(k) : \gamma g \in R_1^\gamma \}).$$

By Lemmas 6 and 12, $S_Q(g)$ is a non-empty subset of $X_Q(g)$.

Lemma 13. For $g_0 \in G(A)^1$, there is a neighbourhood $U$ of $g_0$ in $G(A)^1$ such that $S_Q(g) \subset S_Q(g_0)$ for all $g \in U$.

Proof. Let $U$ be a neighbourhood of $g_0$ such that $X_Q(g) \subset X_Q(g_0)$ for all $g \in U$. Since $g_0 \notin \gamma^{-1}R_1^\gamma$ for any $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$, we can take a sufficiently small $U$ so that $U \cap \gamma^{-1}R_1^\gamma = \emptyset$ for all $\pi_X(\gamma) \in X_Q(g_0) \setminus S_Q(g_0)$. Then, for any $g \in U$, $S_Q(g) \cap X_Q(g_0) \setminus S_Q(g_0)$ is empty, namely $S_Q(g)$ is contained in $S_Q(g_0)$. □

Remark. We do not know whether $R_1^\gamma = R$ holds or not in general. If $R_1^\gamma = R$ holds, then $S_Q(g) = X_Q(g)$ holds for all $g$.  

5. A fundamental domain of $G(A)^1$ with respect to $G(k)$

Definition. Let $T$ be a locally compact Hausdorff space and $\Gamma$ be a discrete group acting on $T$ from the left. Assume that the action of $\Gamma$ on $T$ is properly disconnected. An open subset $\Omega$ of $T$ is called an open fundamental domain of $T$ with respect to $\Gamma$ if $\Omega$ satisfies the following conditions:

(i) $T = \Gamma \Omega^-$, where $\Omega^-$ stands for the closure of $\Omega$ in $T$, and
(ii) $\Omega \cap \gamma \Omega^- = \emptyset$ if $\gamma \in \Gamma \setminus \{e\}$.

A subset $F$ of $T$ is called a fundamental domain of $T$ with respect to $\Gamma$ if there is an open fundamental domain $\Omega$ as above such that $\Omega \subset F \subset \Omega^-$.

By Baer and Levi’s theorem ([1], see also [12, §10]), an open fundamental domain of $T$ with respect to $\Gamma$ exists if the set of points stabilized by $\Gamma$ is discrete in $T$. Thus there exists an open fundamental domain $\Omega_Q$ of $R_1^-$ with respect to $Q(k)$. For a given subset $A$ of $R_1^-$, $A^o$ and $A^-$ denote the interior and the closure of $A$ in $G(A)^1$, respectively. Since $R_1^-$ is closed in $G(A)^1$, the closure of $A$ in $R_1^-$ coincides with $A^-$.

Lemma 14. We have $\Omega_Q^o = \Omega_Q \cap R_1$ and $\Omega_Q^- = (\Omega_Q \cap R_1)^-$.

Proof. Since $\Omega_Q$ is an open set in $R_1^-$ with respect to the relative topology, there is an open set $\mathcal{U}$ in $G(A)^1$ such that $\Omega_Q = \mathcal{U} \cap R_1$. Therefore, $\Omega_Q \cap R_1 = \mathcal{U} \cap R_1$ is open in $G(A)^1$, and hence $\Omega_Q^o = \Omega_Q \cap R_1$. Since $R_1$ is dense in $R_1^-$ and $\Omega_Q$ is relatively open in $R_1^-$, the closure of $\Omega_Q \cap R_1$ in $R_1^-$ contains $\Omega_Q$, i.e., $\Omega_Q \subset (\Omega_Q \cap R_1)^-$. Hence we have $\Omega_Q^- = (\Omega_Q \cap R_1)^-$.

Theorem 15. $\Omega_Q^o$ is an open fundamental domain of $G(A)^1$ with respect to $G(k)$.

Proof. From $R_1^- = Q(k)\Omega_Q$ and Lemma 12, it follows $G(A)^1 = G(k)\Omega_Q^-$. For $\gamma \in G(k)$, we assume $\Omega_Q^o \cap \gamma \Omega_Q^- \neq \emptyset$. By Lemma 11, $\gamma$ is contained in $Q(k)$. Since $\Omega_Q$ is an open fundamental domain of $R_1^-$ with respect to $Q(k)$, $\gamma$ must be equal to $e$.

For a given subset $A$ of $G(A)^1$, we denote by $\partial A$ the boundary of $A$.

Lemma 16. If $g_0 \in R_1^-$ attains a local maximum of $m_Q$, then $g_0$ is contained in $\partial R_1^-$. 

Proof. Suppose $g_0 \in R_1$. Since $R_1$ is open, $zg_0$ is contained in $R_1$ if $z \in Z_Q(A)$ is sufficiently close to $e$. Then we have

$$m_Q(zg_0) = H_Q(zg_0) = H_Q(z)H_Q(g_0) = H_Q(z)m_Q(g_0).$$

Since $H_Q(z)$ can vary on the interval $(1 - \epsilon, 1 + \epsilon)$ for a sufficiently small $\epsilon$, $m_Q(g_0)$ is not a local maximum of $m_Q$. 

Since $(\Omega_Q^-)^o = \Omega_Q^o \subset R_1$, the following theorem immediately follows from Lemma 16.

Theorem 17. If $g_0 \in \Omega_Q^o$ attains a local maximum of $m_Q$, then $g_0$ is contained in $\partial \Omega_Q^o \cap \partial R_1^-$. 

Remark. A point $g_0 \in G(A)^1$ is said to be extreme if $g_0$ attains a local maximum of $m_Q$. By Theorem 17, any extreme point is contained in $G(k)(\partial \Omega_Q^o \cap \partial R_1^-)$. A candidate of the notion analogous to perfect quadratic forms is the following: A point $g \in G(A)^1$ is said to be $Q$-perfect if there is a neighbourhood $\mathcal{U}$ of $g$ such that

$$\mathcal{U} \cap \bigcap_{\pi \chi(\delta) \in S_Q(g)} \delta^{-1}R_1^- = \{g\}.$$
6. The case when $G$ is of class number one

We put $K_f = \prod_{\sigma \in J_f} K_{\sigma}, G_{A, \infty} = G(k) \times K_f, G_{A, \infty}^{1} = G_{A, \infty} \cap G(A)^{1}$ and $G_o = G(k) \cap G_{A, \infty}$. By identifying $G(k_{\infty})$ with the subgroup $\{ (g_{\sigma}) \in G(A) : g_{\sigma} = e \text{ for all } \sigma \in \mathfrak{p}_{f} \}$ of $G(A)$, we put $G(k_{\infty})^{1} = G(k) \cap G(A)^{1}$. The number $n_{k}(G)$ of double cosets in $G(A)$ modulo $G(k)$ and $G_{A, \infty}$ is called the class number of $G$. For example, $n_{k}(GL_{n})$ is equal to the class number of $k$. If $G$ is almost $k$-simple, $k$-isotropic and simply connected, then $n_{k}(G) = 1$ by the strong approximation theorem. In this section, we assume that $n_{k}(G) = 1$. Then we have $G(A)^{1} = G(k)G_{A, \infty}^{1}$. Let $h_{Q}$ be the number of double cosets of $G(k)$ modulo $Q(k)$ and $G_o$. By [2, Proposition 7.5], $h_{Q}$ is equal to the class number of $M_{Q}$. Let $\{ \xi_{1} = e, \xi_{2}, \ldots, \xi_{h_{Q}} \}$ be a complete system of representatives of $Q(k)\backslash G(k)/G_{o}$. For each $\xi_{i}$, we define the subset $R_{\xi_{i}, \infty}$ of $G(k_{\infty})^{1}$ as

$$\{ g_{\infty} \in G(k_{\infty})^{1} : m_{Q}(g_{\infty}) = H_{Q}(\xi_{i}, g_{\infty}) \}.$$ 

Since $G(k)$ is a disjoint union of $Q(k)\xi_{i}G_{o}, i = 1, \ldots, h_{Q}, m_{Q}(g_{\infty})$ is equal to

$$\min_{1 \leq i \leq h_{Q}} \min_{\delta \in G_{o}} H_{Q}(\xi_{i}\delta g_{\infty}).$$

Lemma 18. One has

$$R = \bigcup_{i=1}^{h_{Q}} Q(k)\xi_{i}(R_{\xi_{i}, \infty} \times K_f).$$

Proof. For each $i, Q(k)\xi_{i}(R_{\xi_{i}, \infty} \times K_f) \subset R$ is trivial. Since

$$G(A)^{1} = \bigcup_{i=1}^{h_{Q}} Q(k)\xi_{i}G_{A, \infty}^{1}$$

by [2, §7], a given $g \in R$ is represented as $g = \gamma \xi_{i}(g_{\infty} \times g_{f})$ by some $i, \gamma \in Q(k)$ and $g_{\infty} \times g_{f} \in G_{A, \infty}^{1}$. Then $m_{Q}(g) = H_{Q}(g)$ implies $m_{Q}(g_{\infty}) = H_{Q}(\xi_{i}g_{\infty})$. Therefore, $g_{\infty} \in R_{\xi_{i}, \infty}$. \hfill \Box

We write $Q_{i}$ for the conjugate $\xi_{i}^{-1}Q_{i} \xi_{i}$ of $Q$. This $Q_{i}$ is a maximal $k$-parabolic subgroup of $G$. We put $Q_{i, o} = Q_{i}(k) \cap G_{A, \infty}$.

Lemma 19. If $g(R_{\xi_{i}, \infty} \times K_f) \cap (R_{\xi_{i}, \infty} \times K_f)$ is non-empty for $g \in Q_{i}(k)$, then $g \in Q_{i, o}$.

Proof. If there is an $h \in R_{\xi_{i}, \infty} \times K_f$ such that $gh \in R_{\xi_{i}, \infty} \times K_f$, then $g \in (R_{\xi_{i}, \infty} \times K_f)h^{-1} \subset G_{A, \infty}$. \hfill \Box

It is easy to prove that the group $Q_{i, o}$ stabilizes $R_{\xi_{i}, \infty} \times K_f$ by left multiplications. We fix a complete system $\{ \gamma_{ij} \}$ of representatives of $Q_{i}(k)/Q_{i, o}$. It follows from Lemma 19 that $\gamma_{ij}(R_{\xi_{i}, \infty} \times K_f) \cap \gamma_{ik}(R_{\xi_{i}, \infty} \times K_f) = \emptyset$ if $j \neq k$. Therefore, we obtain the following subdivision of $R$:

$$R = \bigcup_{i=1}^{h_{Q}} \bigcup_{j} \xi_{i} \gamma_{ij} (R_{\xi_{i}, \infty} \times K_f).$$

Let $R_{\xi_{i}, \infty}^{1}$ be the interior of $R_{\xi_{i}, \infty}$ and $R_{\xi_{i}, \infty}^{1}$ be the closure of $R_{\xi_{i}, \infty}^{1}$ in $G(k_{\infty})^{1}$. Since the union of (1) is disjoint, it is obvious that

$$R_{\xi_{i}, \infty}^{1} = \bigcup_{i=1}^{h_{Q}} \bigcup_{j} \xi_{i} \gamma_{ij} (R_{\xi_{i}, \infty}^{1} \times K_f).$$
Proposition 20. Let $\Omega_{i,\infty}$ be an open fundamental domain of $\mathbb{R}^*_i,\infty$ with respect to $Q_{i,\circ}$ for $i = 1, \cdots, h_Q$. Then the set

$$\Omega = \bigcup_{i=1}^{h_Q} \xi_i(\Omega_{i,\infty} \times K_f)$$

gives an open fundamental domain of $\mathbb{R}^-_i$ with respect to $Q(k)$.

Proof. Let $\Omega_{i,\infty}^-$ denote the closure of $\Omega_{i,\infty}$ in $G(k_\infty)^1$. For $g \in Q(k)$, we assume $\Omega \cap g\Omega^- \neq \emptyset$. Then, for some $i, j$, we have

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap g\xi_j(\Omega_{j,\infty} \times K_f) \neq \emptyset.$$  \hspace{1cm} (3)

There exist $\gamma_{jk}$ and $\delta \in Q_{j,\circ}$ such that $\xi_j^{-1}g\xi_j = \gamma_{jk}\delta$. Then (3) is the same as

$$\xi_i(\Omega_{i,\infty} \times K_f) \cap \xi_j(\delta\Omega_{j,\infty}^- \times K_f) \neq \emptyset.$$  

By (1), we have $i = j$, $\gamma_{jk} = e$ and $\Omega_{j,\infty} \cap \delta\Omega_{j,\infty}^- \neq \emptyset$. Since $\Omega_{j,\infty}$ is an open fundamental domain of $\mathbb{R}^*_j,\infty$ with respect to $Q_{j,\circ}$, $\delta$ must be equal to $e$. Therefore, $\Omega \cap g\Omega^- \neq \emptyset$ implies $g = e$. Finally, $Q(k)\Omega^- = \mathbb{R}^-_i$ follows from (2) and $Q_{i,\circ}\Omega_{i,\infty}^- = \mathbb{R}^*_i,\infty$. \hfill \square

By Theorem 17, we obtain the following.

Corollary 21. If $g_0 \in \Omega^-$ attains a local maximum of $m_Q$, then $g_0$ is contained in the set

$$\bigcup_{i=1}^{h_Q} \xi_i((\partial\Omega_{i,\infty}^- \cap \partial\mathbb{R}^*_i,\infty) \times K_f).$$

We consider the infinite part $\Omega_{\infty}$ of $\Omega$, i.e.,

$$\Omega_{\infty} = \bigcup_{i=1}^{h_Q} \xi_i\Omega_{i,\infty}.$$

Let $\Omega_{\infty}^\circ$ and $\Omega_{\infty}^\circ$ be the interior and the closure of $\Omega_{\infty}$ in $G(k_\infty)^1$, respectively. The projection from $G(A)^1 = G(k)G_{A,\infty}^1$ to the infinite component $G(k_\infty)^1$ gives an isomorphism $G(k)\backslash G(A)^1 = G_{\circ}\backslash G(k_\infty)^1$. Since $\Omega$ is a fundamental domain of $G(A)^1$ with respect to $G(k)$ by Theorem 16, we have $G_{\circ}\Omega_{\infty} = G(k_\infty)^1$.

Corollary 22. If $h_Q = 1$, then $\Omega_{\infty}$ is a fundamental domain of $G(k_\infty)^1$ with respect to $G_{\circ}$.

Proof. Since $\Omega_{\infty} = \Omega_{1,\infty}$ is a relatively open set in $\mathbb{R}^*_e,\infty$, we have $\Omega_{\infty}^\circ = \Omega_{\infty} \cap \mathbb{R}^*_e,\infty$. Thus the closure of $\Omega_{\infty}^\circ$ coincides with $\Omega_{\infty}$. If $\Omega_{\infty}^\circ \cap g\Omega_{\infty} \neq \emptyset$ for $g \in G_{\circ}$, then $(\Omega_{\infty}^\circ \times K_f) \cap g(\Omega_{\infty}^\circ \times K_f) \neq \emptyset$ because of $gK_f = K_f$. This implies $g = e$ since $\Omega_{\infty}^\circ \times K_f$ is an open fundamental domain of $G(A)^1$ with respect to $G(k)$. \hfill \square

7. Examples
Example 1. Let $G$ be a general linear group $\text{GL}_n$ defined over $\mathbb{Q}$, $P_0$ the group of upper triangular matrices in $G$ and $S$ the group of diagonal matrices in $G$. We fix an integer $k \in \{1, \cdots, n - 1\}$, and let

$$Q(\mathbb{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in \text{GL}_k(\mathbb{Q}), \ b \in M_{k,n-k}(\mathbb{Q}), \ d \in \text{GL}_{n-k}(\mathbb{Q}) \right\}.$$ 

Then $Q$ is a standard maximal $\mathbb{Q}$-parabolic subgroup of $G$. The rational character $\tilde{\alpha}_Q$ and the height $H_Q$ are given by

$$\tilde{\alpha}_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (\det a)^{n-k}/r (\det d)^{-k/r},$$

and

$$H_Q \left( \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = |\det a|^{-(n-k)/r} |\det d|^{k/r},$$

where $r$ denotes the greatest common divisor of $k$ and $n - k$. Since $h_Q = 1$, we have $\xi_1 = e$ and $Q_1 = Q$.

Let $P_n$ be the cone of positive definite $n$ by $n$ real symmetric matrices, and let $P_n^i$ be the intersection of $P_n$ and $\text{SL}_n(\mathbb{R})$. The group $G(\mathbb{Q}_\infty) = \text{GL}_n(\mathbb{R})$ acts on $P_n$ from the right by $(A, g) \mapsto A[g] = ^t g A g$ for $(A, g) \in P_n \times G(\mathbb{Q}_\infty)$. We choose a maximal compact subgroup $K_\infty$ of $G(\mathbb{Q}_\infty)$ as the stabilizer subgroup of the identity matrix $I_n \in P_n$. The map $\pi : g \mapsto ^t g^{-1} g^{-1}$ from $G(\mathbb{Q}_\infty)$ onto $P_n$ gives an isomorphism between $G(\mathbb{Q}_\infty)/K_\infty$ and $P_n$. Since $G(\mathbb{Q}_\infty) = \{ g \in G(\mathbb{Q}_\infty) : \det g = \pm 1 \}$, we have $G(\mathbb{Q}_\infty)/K_\infty \simeq \pi(G(\mathbb{Q}_\infty)^1) = P_n^1$.

An element $A \in P_n$ is written as

$$A = \begin{pmatrix} I_k & 0 \\ v u & I_{n-k} \end{pmatrix} \begin{pmatrix} w & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} I_k & u \\ 0 & I_{n-k} \end{pmatrix},$$

where $v \in P_k$, $w \in P_{n-k}$ and $u \in M_{k,n-k}(\mathbb{R})$. We write $u_A$, $A_{[k]}$ and $A_{[n-k]}$ for $u$, $v$ and $w$, respectively.

Let $K_f = \prod_{p \in P} \text{GL}_n(\mathbb{Z}_p)$. Then $G_Z = G(\mathbb{Q}) \cap G_{\mathbb{A}, \infty}$ and $Q_Z = Q(\mathbb{Q}) \cap G_{\mathbb{A}, \infty}$ are just groups $\text{GL}_n(\mathbb{Z})$ and $Q(\mathbb{Q}) \cap \text{GL}_n(\mathbb{Z})$ of integral matrices in $G(\mathbb{Q})$ and $Q(\mathbb{Q})$, respectively. For a given integral matrix $\gamma \in G_Z$, $X_\gamma$ stands for the $n$ by $k$ matrix consisting of the first $k$-columns of $\gamma$. Denote by $M_{n,k}(\mathbb{Z})^*$ the set of $X_\gamma$ for all $\gamma \in G_Z$. We define a closed subset $F_{n,k}$ of $P_n$ as follows:

$$F_{n,k} = \{ A \in P_n : \det A_{[k]} \leq \det (^t X A X) \text{ for all } X \in M_{n,k}(\mathbb{Z})^* \}.$$ 

As noted in [16, Example 4.1], we have

$$H_Q(\gamma g) = \det (^t X \gamma_{-1} \pi(g) X \gamma_{-1})^{n/2r}$$

for any $\gamma \in G_Z$ and $g \in G(\mathbb{Q}_\infty)$. Since $H_Q(g) = \det \pi(g)^{[k]}$, we obtain

$$R_{e, \infty}/K_\infty \simeq \pi(R_{e, \infty}) = F_{n,k} \cap \text{SL}_n(\mathbb{R}).$$

Therefore, $Q_Z \backslash R_{e, \infty}/K_\infty$ is isomorphic with $(F_{n,k} \cap \text{SL}_n(\mathbb{R}))/Q_Z$. If $\gamma \in Q_Z$ is of the form

$$\gamma = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$
with \( a \in \text{GL}_k(\mathbb{Z}) \), \( d \in \text{GL}_{n-k}(\mathbb{Z}) \) and \( b \in M_{k,n-k}(\mathbb{Z}) \), then components of \( {}^t\gamma A\gamma \) for \( A \in \mathbb{P}_n \) are given by
\[
\omega_{\gamma A\gamma} = a^{-1}(u_A + b), \quad ({}^t\gamma A\gamma)^{\bar{k}} = {}^t a A^{|k|} a, \quad ({}^t\gamma A\gamma)_{|n-k|} = {}^t d A_{|n-k|} d.
\]
Let \( \mathcal{D} \) and \( \mathcal{E} \) be arbitrary fundamental domains for \( P_k/\text{GL}_k(\mathbb{Z}) \) and \( P_{n-k}/\text{GL}_{n-k}(\mathbb{Z}) \), respectively. We define the subset \( F_{n,k}(\mathcal{D}, \mathcal{E}) \) of \( F_{n,k} \) as
\[
F_{n,k}(\mathcal{D}, \mathcal{E}) = \left\{ A \in F_{n,k} : A^{[k]} \in \mathcal{D}, A_{[n-k]} \in \mathcal{E}, u_A = (u_{ij}), -1/2 < u_{ij} < 1/2 \text{ for all } i, j, \text{ and } 0 < u_{11} \right\}.
\]
Since \( F_{n,k}(\mathcal{D}, \mathcal{E}) \) is a fundamental domain of \( F_{n,k} \) with respect to \( \mathbb{Q}_Z \), the inverse image \( \pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})) \) gives a fundamental domain of \( \mathbb{R}_{n,\infty} \) with respect to \( \mathbb{Q}_Z \). As a consequence of Theorem 15 and Proposition 20, the set
\[
\pi^{-1}(F_{n,k}(\mathcal{D}, \mathcal{E}) \cap \text{SL}_n(\mathbb{R})) \times K_f
\]
gives a fundamental domain of \( G(\mathbb{A})^1 \) with respect to \( G(\mathbb{Q}) \). Moreover, from Corollary 22, it follows that \( F_{n,k}(\mathcal{D}, \mathcal{E}) \) is a fundamental domain of \( P_n \) with respect to \( \text{GL}_n(\mathbb{Z}) \).

In the case of \( k = 1 \), this gives an inductive construction of a fundamental domain \( \Omega_n \) of \( P_n \) with respect to \( \text{GL}_n(\mathbb{Z}) \) as follows. First, put \( \Omega_2 = F_{2,1}(P_1, P_1) \). By definition, \( \Omega_2 \) is Minkowski’s fundamental domain of \( P_2 \). Then we define inductively \( \Omega_3 = F_{3,1}(P_1, \Omega_2), \cdots, \Omega_n = F_{n,1}(P_1, \Omega_{n-1}) \). The domain \( \Omega_n \) coincides with Grenier’s fundamental domain [5].

**Example 2.** Let \( k \) be a totally real number field of degree \( r \) and \( n = 2m \) be an even integer. We consider a symplectic group
\[
G(k) = \text{Sp}_n(k) = \left\{ g \in \text{GL}_{2m}(k) : {}^t g \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \end{array} \right) g = \left( \begin{array}{cc} 0 & -I_m \\ I_m & 0 \end{array} \right) \right\}.
\]
For a fixed \( k \in \{1, 2, \cdots, m\} \), \( Q \) denotes the maximal parabolic subgroup of \( G \) given as follows:
\[
Q(k) = U(k) M(k),
\]
\[
M(k) = \left\{ \delta(a, b) = \left( \begin{array}{ccc} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{array} \right) : a \in \text{GL}_k(k), \quad b = (b_{ij}) \in \text{Sp}_{2m-k}(k) \right\},
\]
\[
U(k) = \left\{ \left( \begin{array}{cccc} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{array} \right) : a \in \text{GL}_m(k) \right\}.
\]
The module of \( k \)-rational characters \( X_k^r(M) \) of \( M \) is a free \( \mathbb{Z} \)-module of rank 1 generated by the character
\[
\tilde{\alpha}_Q(\delta(a, b)) = \det a.
\]
The height \( H_Q : G(\mathbb{A}) \rightarrow \mathbb{R}_{\geq 0} \) is given by \( H_Q(g) = |\det a|_A^{-1} \) if \( g = u\delta(a, b)h \) with \( u \in U(\mathbb{A}), \delta(a, b) \in M(\mathbb{A}) \) and \( h \in K \).

We restrict ourselves to the case \( k = m \). An element of \( M(\mathbb{A}) \) is denoted by
\[
\delta(a) = \left( \begin{array}{cc} a & 0 \\ 0 & {}^t a^{-1} \end{array} \right), \quad (a \in \text{GL}_m(\mathbb{A})).
\]
Let
\[ H_m = \{ Z \in M_m(C) : \Im Z = Z, \ \Im Z \in \mathbb{P}_m \} \]
be the Siegel upper half space and \( H_m^r \) the direct product of \( r \) copies of \( H_m \). For \( Z = (Z_\sigma)_{\sigma \in \mathbb{P}_m} \in H_m^r \), \( \Re Z \), \( \Im Z \) and \( \det Z \) stand for \( (\Re Z_\sigma)_{\sigma \in \mathbb{P}_m} \), \( (\Im Z_\sigma)_{\sigma \in \mathbb{P}_m} \) and \( (\det Z_\sigma)_{\sigma \in \mathbb{P}_m} \), respectively. The group \( G(k_\infty) \) acts transitively on \( H_m^r \) by
\[ g(Z) = (a_\sigma Z_\sigma + b_\sigma)(c_\sigma Z_\sigma + d_\sigma)^{-1} \] \( \sigma \in \mathbb{P}_m \)
for \( Z = (Z_\sigma) \in H_m^r \) and
\[ g = (g_\sigma) = \left( \begin{array}{cc} a_\sigma & b_\sigma \\ c_\sigma & d_\sigma \end{array} \right) \in G(k_\infty). \]
The stabilizer \( K_\infty \) of \( Z_0 = (\sqrt{-1}I_m, \cdots, \sqrt{-1}I_m) \in H_m^r \) in \( G(k_\infty) \) is a maximal compact subgroup of \( G(k_\infty) \). We choose \( K \) as \( K_\infty \times \prod_{\sigma \in \mathbb{P}_m} \operatorname{Sp}_n(\mathcal{O}_\sigma) \). The map \( \pi : g_\infty \mapsto g(Z_0) \) from \( G(k_\infty) \) onto \( H_m^r \) give an isomorphism \( G(k_\infty)/K_\infty \cong H_m^r \), and hence \( G(k)\backslash G(A)/K \cong G_\infty \backslash H_m^r \). Since \( \{u(\sigma)h(Z_0)\} = d' a \) holds for \( u \in U(k_\infty), a \in \operatorname{GL}_m(k_\infty) \) and \( h \in K_\infty \), we have
\[ H_Q(g_\infty) = \operatorname{N}(\det \{g_\infty(Z_\sigma)\})^{-1/2} = \left( \prod_{\sigma \in \mathbb{P}_m} \det \{g_\sigma(\sqrt{-1}I_m)\} \right)^{-1/2} \]
for any \( g_\infty = (g_\sigma) \in G(k_\infty) \), where \( \operatorname{N}_{k_\infty}/R \) denotes the norm of \( k_\infty \) over \( R \).

The class number \( h_Q \) of \( M \cong \operatorname{GL}_m \) defined over \( k \) is equal to the class number \( h_k \) of \( k \).

We assume \( h_k = 1 \) for simplicity. Then we have \( G(k) = Q(k)G_\infty \) and \( G(A) = Q(k)G_\infty \), and hence
\[ m_Q(g_\infty) = \min_{\gamma \in G_\infty} H_Q(\gamma g_\infty). \]
Since
\[ \operatorname{N}_{k_\infty}/R(\det \{\gamma(Z)\}) = \prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_\sigma + \sigma(d))|^2 \operatorname{N}_{k_\infty}/R(\det Z) \]
for \( Z = (Z_\sigma) \in H_m^r \) and
\[ \gamma = \left( \begin{array}{cc} * & * \\ * & d \end{array} \right) \in G_\infty = \operatorname{Sp}_n(\mathcal{O}), \]
the condition \( m_Q(g_\infty) = H_Q(g_\infty) \) of \( g_\infty \) is equivalent with the following condition of \( Z = g_\infty(Z_0) \):
\[ \prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \quad \text{for all} \quad \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in G_\infty. \]

Therefore, the domain \( R_{e,\infty} \) modulo \( K_\infty \) is isomorphic with
\[ F = \left\{ (Z_\sigma) \in H_m^r : \prod_{\sigma \in \mathbb{P}_m} |\det(\sigma(c)Z_\sigma + \sigma(d))| \geq 1 \quad \text{for all} \quad \left( \begin{array}{cc} * & * \\ c & d \end{array} \right) \in G_\infty \right\}. \]

Let \( \mathcal{C} \) be an arbitrary fundamental domain of the additive group \( \mathbb{M}_m(k_\infty) \) with respect to \( \mathbb{M}_m(\mathcal{O}) \), and let \( \mathcal{D} \) be an arbitrary fundamental domains of \( \mathbb{P}_m^r \) with respect to \( \operatorname{GL}_m(\mathcal{O}) \). It is easy to see that
\[ F(\mathcal{C}, \mathcal{D}) = \{ Z \in F : \Re Z \in \mathcal{C}, \Im Z \in \mathcal{D} \}. \]
is a fundamental domain of $F$ with respect to $Q_\circ$. By Corollary 22, $F(C, D)$ is a fundamental domain of $H^r_\circ$ with respect to $G_\circ$.

If $k = \mathbb{Q}$ and $D$ is Minkowski’s fundamental domain, then $F(C, D)$ coincides with Siegel’s fundamental domain [10].

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**References**


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