A SURVEY ON VORONOÏ'S THEOREM

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Let V_n be the vector space of real $n \times n$ symmetric matrices and P_n the open cone of positive definite symmetric matrices in V_n . By $m_1(a)$, we denote the arithmetical minimum $\inf_{x \in \mathbf{Z}^n \setminus \{0\}} {}^t x a x$ of $a \in P_n$. The Hermite invariant is the positive valued function γ on P_n defined by $\gamma(a) = m_1(a)/\det(a)^{1/n}$. Its maximum γ_n is called Hermite's constant. The determination of γ_n is one of main problems in lattice sphere packings or the arithmetic theory of quadratic forms. Voronoï's fundamental theorem [61] gives a characterization of local maxima of γ , i.e., which can be stated that γ attains a local maximum on $a \in P_n$ if and only if a is perfect and eutactic. In the last half of 20th century, various generalizations of Hermite's constant and Voronoï's theorem were studied by many authors. In this paper, we give an account of a recent development concerning Voronoï's theorem.

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Notation. The Euclidean space $\mathbf{R}^n$ is equipped with the inner product $(x, y) = {}^t xy$. For $a \in M_n(\mathbf{R})$, $\|a\|$ denotes the operator norm of a , i.e.,

$$\|a\| = \sup_{x \in \mathbf{R}^n \setminus \{0\}} \left(\frac{(ax, ax)}{(x, x)} \right)^{1/2}.$$

Throughout this paper, V_n denotes the vector space of real $n \times n$ symmetric matrices, P_n the open cone of positive definite symmetric matrices in V_n and P_n^{semi} the closure of P_n in V_n . The vector space V_n is equipped with the inner product $\langle a_1, a_2 \rangle = \text{tr}(a_1 a_2)$ for $a_1, a_2 \in V_n$. The unimodular group $GL_n(\mathbf{Z})$ acts on V_n by $(a, g) \mapsto {}^t gag$ for $a \in V_n$ and $g \in GL_n(\mathbf{Z})$. In general, for a given ring R , the set of all $m \times n$ matrices with coefficients in R is denoted by $M_{m,n}(R)$. We write $M_n(R)$ for $M_{n,n}(R)$ and R^n for $M_{n,1}(R)$. The unit group of the matrix ring $M_n(R)$ is denoted by $GL_n(R)$. The identity matrix in $GL_n(R)$ is denoted by I_n .

For a constant $c \in \mathbf{R}$, $\mathbf{R}_{>c}$ and $\mathbf{R}_{\geq c}$ stand for the open interval $(c, +\infty)$ and the closed interval $[c, +\infty)$, respectively.

1. TYPE ONE FUNCTIONS AND VORONOI'S THEOREM

There are several methods to prove Voronoi's theorem [61, Théorème 17], e.g., [6], [50], [53], see also [27, §29], [28, §39], [39, §3.4] and [59, §3.1.7]. Convexity of the domain P_n and the concavity of both functions m_1 and $\det^{1/n}$ play key roles in some proofs. Poor and Yuen [46] investigated a family of such kind functions as m_1 and $\det^{1/n}$. This family is called type one functions. Every type one function ϕ is completely characterized by the corresponding semikernel $K_1(\phi)$. In this section, we first discuss type one functions and semikernels, and then formulate Voronoi type theorem in terms of type one functions. The semikernel $K_1(m_1)$ associated with m_1 is called the Ryshkov polyhedron. In the second half of this section, we investigate a description of faces of $K_1(m_1)$. This gives a well-known geometric interpretation of perfection.

1.1. Type one functions and semikernels.

Definition 1.1. A function $\phi : P_n^{\text{semi}} \rightarrow \mathbf{R}_{\geq 0}$ is called a type one function if ϕ satisfies the following conditions:

- (TO₁) $\phi(\theta a) = \theta \phi(a)$ for all $a \in P_n^{\text{semi}}$ and $\theta \geq 0$,
- (TO₂) $\phi(a_1 + a_2) \geq \phi(a_1) + \phi(a_2)$ for all $a_1, a_2 \in P_n^{\text{semi}}$,
- (TO₃) $\phi(a) > 0$ for all $a \in P_n$.

A type one function ϕ is called a type one class function if $\phi({}^t gag) = \phi(a)$ holds for all $a \in P_n^{\text{semi}}$ and $g \in GL_n(\mathbf{Z})$.

Example 1.1. The trace tr and the smallest eigenvalue λ_1 are type one functions, but not type one class functions. The reduced determinant $\det^{1/n}$ and the arithmetical minimum

$$m_1(a) = \inf_{x \in \mathbf{Z} \setminus \{0\}} {}^t x a x$$

are type one class functions.

For a type one function ϕ , the dual type one function $\phi^\circ : P_n^{\text{semi}} \rightarrow \mathbf{R}_{\geq 0}$ is defined to be

$$\phi^\circ(a) = \inf_{b \in P_n} \frac{\langle a, b \rangle}{\phi(b)}.$$

If ϕ is a type one class function, then so is ϕ° . The dual type one class function of m_1 is denoted by w_1 , which is called the dyadic trace. The dual type one class function of $\det^{1/n}$ is $n \det^{1/n}$.

Any type one function is continuous on P_n , but not necessarily continuous on P_n^{semi} . For example, w_1 is not continuous on P_n^{semi} ; however w_1 is upper semicontinuous on P_n^{semi} . Here a type one function ϕ is said to be upper semicontinuous at $a \in P_n^{\text{semi}}$ if

$$\phi(a) = \limsup_{b \rightarrow a} \phi(b) = \lim_{\epsilon \downarrow 0} (\sup\{\phi(b) \mid \|a - b\| \leq \epsilon, b \in P_n^{\text{semi}}\}).$$

In general, the dual ϕ° of an arbitrary type one function ϕ is necessarily upper semicontinuous on P_n^{semi} ([57, Corollary 2.7]).

Definition 1.2. Let K be a convex subset of P_n^{semi} such that $0 \notin K$, $\mathbf{R}_{\geq 1} \cdot K = K$ and $\mathbf{R}_{>0} \cdot K \supset P_n$.

- (1) K is called a kernel if K is closed in P_n^{semi} .
- (2) K is called a semikernel if the following three conditions are satisfied:
 - (SK₁) $K \cap (P_n \cup \{0\})$ is closed in $P_n \cup \{0\}$,
 - (SK₂) $\{\theta \geq 0 \mid \theta a \in K\}$ is closed in $[0, \infty)$ for any $a \in K$,
 - (SK₃) $a + b \subset K$ for all $a \in K$ and $b \in P_n^{\text{semi}}$.

It is easy to see that a kernel is a semikernel. The dual $K^\sqcup$ of a semikernel K is defined to be

$$K^\sqcup = \{a \in V_n \mid \langle a, b \rangle \geq 1 \text{ for all } b \in K\}.$$

This $K^\sqcup$ is a kernel.

There is a natural correspondence between type one functions and semikernels. For a type one function ϕ , we set

$$K_1(\phi) = \{a \in P_n^{\text{semi}} \mid \phi(a) \geq 1\}.$$

Conversely, for a semikernel K , define the function $\psi(K, \cdot) : P_n^{\text{semi}} \rightarrow \mathbf{R}_{\geq 0}$ by

$$\psi(K, a) = \max(\{\theta > 0 \mid a \in \theta \cdot K\} \cup \{0\}).$$

The existence of this maximum follows from the condition (SK₂). The following results were proved in [57, §1]

Proposition 1.1. *The correspondence $\phi \mapsto K_1(\phi)$ gives a bijection between the set of type one functions (resp. upper semicontinuous type one functions) and the set of semikernels (resp. kernels). For any type one function ϕ and any semikernel K , one has*

$$\psi(K_1(\phi), \cdot) = \phi, \quad K_1(\psi(K, \cdot)) = K$$

and moreover

$$\psi(K, \cdot)^\circ = \psi(K^\sqcup, \cdot).$$

Proposition 1.2. *For any type one function ϕ , we have*

$$\begin{cases} \phi^{\circ\circ}(a) = \phi(a) & \text{if } a \in P_n \\ \phi^{\circ\circ}(a) \geq \phi(a) & \text{if } a \in P_n^{\text{semi}} \setminus P_n. \end{cases}$$

If ϕ is upper semicontinuous on P_n^{semi} , then $\phi^{\circ\circ} = \phi$ on P_n^{semi} .

1.2. Voronoï's theorem of m_1/ϕ . Voronoï's theorem characterizes local maxima of the Hermite invariant $F_{\det^{1/n}} = m_1/\det^{1/n}$. A point $a \in P_n$ is said to be extreme (resp. strict extreme) if $F_{\det^{1/n}}$ attains a local maximum (resp. a strict local maximum) on a up to the multiplication by an element of $\mathbf{R}_{>0}$. Indeed, we do not need to distinguish between extreme points and strictly extreme points since any extreme point is strictly extreme ([39, Theorem 3.4.5]). For $a \in P_n$, $S(a)$ denotes the set of minimal integral vectors of a , i.e.,

$$S(a) = \{x \in \mathbf{Z}^n \setminus \{0\} : {}^t x a x = m_1(a)\}.$$

For any $y \in \mathbf{R}^n$, φ_y denotes the linear form $v \mapsto {}^t y v y$ on V_n .

Definition 1.3. Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t b b$. An element a is said to be perfect if the linear forms φ_{bx} ($x \in S(a)$) span the dual space V_n^* of V_n . An element a is said to be eutactic (resp. weakly eutactic) if there exist $\rho_x \in \mathbf{R}_{>0}$ (resp. $\rho_x \in \mathbf{R}$), $x \in S(a)$, such that

$$(1) \quad \text{tr} = \sum_{x \in S(a)} \rho_x \varphi_{bx}.$$

We note that these definitions of perfection, eutaxy and weakly eutaxy are independent of a choice of b . It follows from definition that a is perfect if and only if $\{x {}^t x \mid x \in S(a)\}$ spans V_n . If tr is represented as (1), then we have

$$\text{tr} = \sum_{x \in S(a)} \rho_x \varphi_{hbx}$$

for any orthogonal matrix h . The coefficients ρ_x are independent of h .

Any perfect element a is uniquely determined by $m_1(a)$ and $S(a)$, i.e., a is a unique solution of the system of linear equations in the unknown $v = (v_{ij})$: $\langle v, x {}^t x \rangle = m_1(a)$, $x \in S(a)$. If $m_1(a) \in \mathbf{Q}$, then its solution is contained in $V_n \cap M_n(\mathbf{Q})$ by Cramer's formula. This is none other than the rationality of perfect elements ([35, p.252, 5°]).

Theorem 1.3 (Korkine–Zorotareff). *If $a \in P_n$ is perfect and $m_1(a) \in \mathbf{Q}$, then $a \in P_n \cap M_n(\mathbf{Q})$.*

Voronoï's theorem [61, Théorème 17] is stated as follows.

Theorem 1.4 (Voronoï). *$a \in P_n$ is extreme if and only if a is perfect and eutactic.*

We fix a type one function ϕ . It is natural to ask whether the same kind of Voronoï's theorem holds for the function $F_\phi = m_1/\phi$ on P_n . An element $a \in P_n$ is said to be ϕ -extreme (resp. strictly ϕ -extreme) if F_ϕ attains a local maximum (resp. a strictly local maximum) on a up to the multiplication by an element of $\mathbf{R}_{>0}$. Assume ϕ is differentiable on P_n . Then

$$(\partial \log \phi)_b(v) = \lim_{t \rightarrow 0} \frac{\log \phi({}^t b(I_n + tv)b) - \log \phi({}^t b b)}{t}$$

exists for $b \in GL_n(\mathbf{R})$ and $v \in V_n$. We define ϕ -eutaxy as follows:

Definition 1.4. Let $a \in P_n$, and fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t b b$. An element a is said to be ϕ -eutactic if there exist $\rho_x > 0$ ($x \in S(a)$) such that $(\partial \log \phi)_b = \sum_{x \in S(a)} \rho_x \varphi_{bx}$.

In a similar fashion as eutaxy, this definition is independent of a choice of b . If $\phi = \det^{1/n}$, then $(\partial \log \phi)_b = \text{tr}$ and ϕ -eutaxy is the same as Definition 1.3.

It follows from (TO₁) and (TO₂) that ϕ is log-concave, i.e.,

$$\log \phi((1 - \theta)a_1 + \theta a_2) \geq (1 - \theta) \log \phi(a_1) + \theta \log \phi(a_2)$$

holds for all $a_1, a_2 \in P_n$ and $0 < \theta < 1$. We say ϕ is strictly log-concave if this inequality is strict for $a_1 \neq a_2$.

Voronoi's theorem is generalized as follows ([57, §2]).

Theorem 1.5. *Let ϕ be a strictly log-concave and differentiable type one function. Then, $a \in P_n$ is ϕ -extreme if and only if a is perfect and ϕ -eutactic. Moreover, any ϕ -extreme point is strictly ϕ -extreme.*

The line of the proof of Theorem 1.5 is the same as Barnes' [6] and Martinet's [39, §3.4]. We sketch an outline of the proof. We use the following two lemmas: the first is the same as [39, Lemmas 3.4.2 and 3.4.3] and the second is a generalization of [39, Lemma 3.4.4].

Lemma 1.6. *Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t b b$.*

- (1) *There exists a neighborhood $\mathcal{U}$ of I_n in $GL_n(\mathbb{R})$ such that $S({}^t b u b) \subset S(a)$ for any $u \in \mathcal{U}$.*
- (2) *There exists a neighborhood $\mathcal{V}$ of 0 in V_n such that*

$$m_1({}^t b(I_n + v)b) = m_1(a) \iff \min_{x \in S(a)} \varphi_{bx}(v) = 0$$

for any $v \in \mathcal{V}$.

Lemma 1.7. *Let ϕ be a strictly log-concave and differentiable type one function. Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t b b$.*

- (1) *There exists a neighborhood $\mathcal{V} \subset V_n$ of 0 such that either $v = 0$ or $\phi({}^t b(I_n + v)b) < \phi(a)$ holds for any $v \in \mathcal{V}$ with $(\partial \log \phi)_b(v) \leq 0$ and $I_n + v \in P_n$.*
- (2) *Let $\mathcal{C}$ be a closed cone in V_n such that $(\partial \log \phi)_b(v) > 0$ for all $v \in \mathcal{C} \setminus \{0\}$. Then there exists $\alpha > 0$ such that $\phi({}^t b(I_n + v)b) > \phi(a)$ holds for any $v \in \mathcal{C}$ with $0 < \|v\| < \alpha$.*

We set $\mathcal{D}_a = \{v \in V_n : \min_{x \in S(a)} \varphi_{bx}(v) \geq 0 \text{ and } (\partial \log \phi)_b(v) \leq 0\}$. By these lemmas, we obtain the following generalization of Korkine and Zolotareff's equivalent condition (cf. [39, Theorem 3.4.5]).

Lemma 1.8. *Let ϕ be a strictly log-concave and differentiable type one function. Then $a \in P_n$ is ϕ -extreme if and only if $\mathcal{D}_a = \{0\}$.*

Lemma 1.8 leads us to Theorem 1.5 as follows: Let $a \in P_n$ be perfect and ϕ -eutactic. Fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t b b$. For $v \in \mathcal{D}_a$, ϕ -eutaxy concludes $\varphi_{bx}(v) = 0$ for all $x \in S(a)$, and then $v = 0$ by perfection. Thus $\mathcal{D}_a = \{0\}$ and a is ϕ -extreme. Conversely, let a be ϕ -extreme. If $\varphi_{bx}(v) = 0$ for all $x \in S(a)$, then either v or $-v$ is contained in $\mathcal{D}_a$. Since $\mathcal{D}_a = \{0\}$, we have $v = 0$. This implies that a is perfect. The linear forms $-(\partial \log \phi)_b$ and φ_{bx} , $x \in S(a)$, satisfy

$$\begin{aligned} & \{v \in V_n : \min_{x \in S(a)} \varphi_{bx}(v) \geq 0 \text{ and } -(\partial \log \phi)_b(v) \geq 0\} \\ &= \bigcap_{x \in S(a)} \text{Ker}(\varphi_{bx}) \cap \text{Ker}(-(\partial \log \phi)_b) = \{0\}. \end{aligned}$$

Then, by Stiemke's theorem, a must be ϕ -eutactic. Here Stiemke's theorem asserts that, for a family of linear forms $\varphi_1, \dots, \varphi_r$ on $\mathbf{R}^N$, there exists $\rho_1, \dots, \rho_r \in \mathbf{R}_{>0}$ such that $\rho_1\varphi_1 + \dots + \rho_r\varphi_r = 0$ if and only if $\{v \in \mathbf{R}^N : \min_{1 \leq i \leq r} \varphi_i(v) \geq 0\} = \text{Ker}(\varphi_1) \cap \dots \cap \text{Ker}(\varphi_r)$.

1.3. Geometric characterizations of perfect forms. The kernel $K_1(m_1)$ is called the Ryshkov polyhedron. Ryshkov closely investigated polyhedral geometric structure of $K_1(m_1)$ and its dual $K_1(m_1)^\cup$ ([53], [55, Chapter III]). Since m_1 equals zero on the boundary of P_n^{semi} , the Ryshkov polyhedron $K_1(m_1)$ is contained in P_n . For an integral vector $x \in \mathbf{Z} \setminus \{0\}$ and a constant $\lambda \in \mathbf{R}$, $H_{x,\lambda}^+$ denotes the affine half-space $\{a \in V_n \mid \langle a, x^t x \rangle \geq \lambda\}$ in V_n . Then $K_1(m_1)$ is the intersection of affine half-spaces $H_{x,1}^+$, ($x \in \mathbf{Z}^n \setminus \{0\}$). It is known that $K_1(m_1)$ is a locally finite polyhedron, i.e., the intersection of $K_1(m_1)$ and an arbitrary polytope is a polytope, (see e.g., [27, Proposition 29.5], [59, Theorem 3.1]). In particular, $K_1(m_1) \cap \{a \in V_n \mid \text{tr}(a) \leq \lambda\}$ is a polytope for any sufficiently large constant $\lambda > 0$. We denote by $\partial K_1(m_1)$ the boundary of $K_1(m_1)$. In what follows, we give a description of faces of $K_1(m_1)$.

Lemma 1.9. *Let $a_1, \dots, a_r \in \partial K_1(m_1)$ and S be a non-empty finite subset of $\mathbf{Z}^n \setminus \{0\}$ such that $S \subset S(a_i)$ for $i = 1, \dots, r$. Then, for any $\lambda_1, \dots, \lambda_r \in \mathbf{R}_{\geq 0}$ with $\lambda_1 + \dots + \lambda_r = 1$, one has $\lambda_1 a_1 + \dots + \lambda_r a_r \in \partial K_1(m_1)$ and $S \subset S(\lambda_1 a_1 + \dots + \lambda_r a_r)$.*

Proof. Since $K_1(m_1)$ is convex, $\lambda_1 a_1 + \dots + \lambda_r a_r$ is contained in $K_1(m_1)$. If $x \in S$, then

$$\langle \sum_{i=1}^r \lambda_i a_i, x^t x \rangle = \sum_{i=1}^r \lambda_i m_1(a_i) = \sum_{i=1}^r \lambda_i = 1.$$

This means $m_1(\lambda_1 a_1 + \dots + \lambda_r a_r) = 1$ and $S \subset S(\lambda_1 a_1 + \dots + \lambda_r a_r)$. $\square$

For a non-empty finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$, define the subset $\mathcal{F}_S$ of $\partial K_1(m_1)$ as

$$\mathcal{F}_S = \{a \in \partial K_1(m_1) \mid S \subset S(a)\}.$$

We denote by $\mathcal{H}_S$ the affine subspace of V_n generated by $\mathcal{F}_S$, i.e.,

$$\mathcal{H}_S = \{\lambda_1 a_1 + \dots + \lambda_r a_r \mid 1 \leq r \in \mathbf{Z}, a_i \in \mathcal{F}_S, \lambda_i \in \mathbf{R}, \lambda_1 + \dots + \lambda_r = 1\}$$

if $\mathcal{F}_S \neq \emptyset$, or $\mathcal{H}_S = \emptyset$ if $\mathcal{F}_S = \emptyset$. Since S is non-empty, $\mathcal{H}_S$ is a proper affine subspace of V_n .

Lemma 1.10. *One has $\mathcal{F}_S = \partial K_1(m_1) \cap \mathcal{H}_S$. In particular, $\mathcal{F}_S$ is a face of $K_1(m_1)$ if $\mathcal{F}_S \neq \emptyset$.*

Proof. We assume $\mathcal{F}_S \neq \emptyset$ and fix an $a_0 \in \mathcal{F}_S$. Let $r = \dim \mathcal{H}_S$. There exist r elements $a_1, \dots, a_r \in \mathcal{F}_S$ such that $\{a_1 - a_0, \dots, a_r - a_0\}$ is a basis of the subspace $\{a - a_0 \mid a \in \mathcal{H}_S\}$. Any element $b \in \partial K_1(m_1) \cap \mathcal{H}_S$ is represented as

$$b = a_0 + \lambda_1(a_1 - a_0) + \dots + \lambda_r(a_r - a_0), \quad \lambda_1, \dots, \lambda_r \in \mathbf{R}.$$

Since $S \subset S(a_i)$ for $i = 0, 1, \dots, r$, we have $\langle a_i - a_0, x^t x \rangle = 0$ for all $x \in S$, and hence $\langle b, x^t x \rangle = \langle a_0, x^t x \rangle = 1$ for all $x \in S$. This means $S \subset S(b)$. Therefore, $\partial K_1(m_1) \cap \mathcal{H}_S$ is a subset of $\mathcal{F}_S$. $\square$

Lemma 1.11. *Any face of $K_1(m_1)$ is of the form $\mathcal{F}_S$ for some non-empty finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$.*

Proof. Let $\mathcal{F}$ be a face of $K_1(m_1)$ of dimension r . First, we assume $\mathcal{F}$ is a facet, i.e., $r = \dim V_n - 1$. There exist $r + 1$ elements $a_0, a_1, \dots, a_r \in \mathcal{F}$ such that $a_1 - a_0, \dots, a_r - a_0$ are linearly independent. We fix constants $\lambda_0, \lambda_1, \dots, \lambda_r$ such that $\lambda_0 + \lambda_1 + \dots + \lambda_r = 1$ and $0 < \lambda_i < 1$ for $i = 0, 1, \dots, r$. Since $\mathcal{F}$ is convex, the element $a = \lambda_0 a_0 + \lambda_1 a_1 + \dots + \lambda_r a_r$ is also contained in $\mathcal{F}$. For any $x \in S(a)$, we have

$$1 = m_1(a) = \langle a, x^t x \rangle = \sum_{i=0}^r \lambda_i \langle a_i, x^t x \rangle \geq \sum_{i=0}^r \lambda_i = 1.$$

Thus, $\langle a_i, x^t x \rangle$ equals $m_1(a_i) = 1$ for all $i = 0, 1, \dots, r$. This implies $S(a) \subset S(a_i)$ for all $i = 0, 1, \dots, r$. Let S be the intersection of $S(a_i)$, $i = 0, 1, \dots, r$. Since $S \subset S(a)$ is obvious, one has $S = S(a)$. By definition, the face $\mathcal{F}_S$ contains $a_0, a_1, \dots, a_r$. Therefore, $\partial K_1(m_1) \cap \mathcal{H}_S = \mathcal{F}_S$ contains $\mathcal{F}$. Since $\mathcal{H}_S$ is a proper affine subspace and $\mathcal{F}$ is a facet, we obtain $\mathcal{F}_S = \mathcal{F}$.

In general case, $\mathcal{F}$ is an intersection of finite number of facets, say $\mathcal{F}_{S_1}, \dots, \mathcal{F}_{S_k}$. By definition, we have

$$\mathcal{F} = \bigcap_{i=1}^k \mathcal{F}_{S_i} = \mathcal{F}_{S_1 \cup \dots \cup S_k}.$$

□

We denote by $\partial^0 K_1(m_1)$ the set of all vertices (= 0 dimensional faces) of $K_1(m_1)$. The next theorem is well-known.

Theorem 1.12. *For $a \in \partial K_1(m_1)$, the following three conditions are equivalent each other.*

- (1) a is perfect.
- (2) $a \in \partial^0 K_1(m_1)$.
- (3) There exists a neighbourhood $\mathcal{O}$ of a in P_n such that $S(b) \subsetneq S(a)$ for any $b \in \mathcal{O} \setminus \mathbf{R}_{>0}a$.

Proof. First we show the contraposition of (1) $\implies$ (2). Let $a \in \partial K_1(m_1) \setminus \partial^0 K_1(m_1)$. Then, there exist $a_1, a_2 \in \partial K_1(m_1)$ and $0 < \lambda_0 < 1$ such that $a = \lambda_0 a_1 + (1 - \lambda_0) a_2$. Both $S(a_1)$ and $S(a_2)$ contain $S(a)$. Assume a is perfect. By Lemma 1.9, $S(\lambda a_1 + (1 - \lambda) a_2)$ also contains $S(a)$ for all positive $\lambda < 1$. Therefore, $\lambda a_1 + (1 - \lambda) a_2$ is perfect. This contradicts Theorem 1.3.

Next we show (2) $\implies$ (3). Let $a \in \partial^0 K_1(m_1)$. By Lemma 1.11, there exists a finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$ such that $\{a\} = \mathcal{F}_S$. By Lemma 1.6, there is a neighbourhood $\mathcal{O}$ of a in P_n such that $S(b) \subset S(a)$ for all $b \in \mathcal{O}$. If $b \in \partial K_1(m_1) \cap \mathcal{O}$ and $S(b) = S(a)$, then we have $b \in \mathcal{F}_S$, and hence $b = a$. This $\mathcal{O}$ satisfies (3).

We show the contraposition of (3) $\implies$ (1). Let $a \in \partial K_1(m_1)$ be a non-perfect point. Then, there exists $c \in V_n \setminus \{0\}$ such that $\langle c, x^t x \rangle = 0$ for all $x \in S(a)$. If $\epsilon > 0$ is sufficiently small, then $a + \epsilon c$ is contained in P_n and $S(a + \epsilon c)$ is a subset of $S(a)$. Since $\langle a + \epsilon c, x^t x \rangle = 1$ for all $x \in S(a)$, we have $m_1(a + \epsilon c) = 1$ and $S(a + \epsilon c) = S(a)$. This means that a does not satisfy the condition (3). □

Corollary 1.13. *The set of all perfect elements of P_n coincides with $\mathbf{R}_{>0} \cdot \partial^0 K_1(m_1)$.*

In the rest of this section, we show that $K_1(m_1)$ is the convex hull of $\partial^0 K_1(m_1)$ in V_n . For $a \in \partial^0 K_1(m_1)$, we set

$$\mathcal{C}_a = \{b \in V_n \mid \langle b, x^t x \rangle \geq 0 \text{ for all } x \in S(a)\} = \bigcup_{x \in S(a)} H_{x^t x, 0}^+,$$

which is a polyhedral cone in V_n of finite faces. For a non-zero $b \in \mathcal{C}_a$, the ray $\mathbf{R}_{\geq 0} \cdot b$ is called an extreme ray of $\mathcal{C}_a$ if for any $b_1, b_2 \in \mathcal{C}_a$, whenever $b = (b_1 + b_2)/2$, we must have $b_1, b_2 \in \mathbf{R}_{\geq 0} \cdot b$.

Lemma 1.14. *Let $a \in \partial^0 K_1(m_1)$. If $\mathbf{R}_{\geq 0} \cdot b$ is an extreme ray of $\mathcal{C}_a$, then $b \notin P_n^{\text{semi}}$.*

Proof. We prove that $b \in P_n^{\text{semi}}$ leads us to a contradiction. Since a is perfect, the set $\{x^t x \mid x \in S(a)\}$ spans V_n . We set

$$S' = \{x \in S(a) \mid \langle b, x^t x \rangle = 0\}$$

and

$$W = \{c \in V_n \mid \langle c, x^t x \rangle = 0 \text{ for all } x \in S'\}.$$

Since $b \neq 0$, S' is non-empty and W is a subspace of V_n containing the line $\mathbf{R} \cdot b$.

First we assume $\dim W \geq 2$. There is a $c \in W$ such that b and c are linearly independent. If we assume $b \in P_n^{\text{semi}}$, then we have $\langle b, x^t x \rangle > 0$ for all $x \in S(a) \setminus S'$. Thus, for sufficiently small $\lambda > 0$, we have $\langle b \pm \lambda c, x^t x \rangle > 0$ for all $x \in S(a) \setminus S'$. From $\langle b \pm \lambda c, x^t x \rangle = 0$ for all $x \in S'$, it follows $b \pm \lambda c \in \mathcal{C}_a$. Then one has

$$b = \frac{1}{2}(b + \lambda c) + \frac{1}{2}(b - \lambda c)$$

and $b \pm \lambda c \notin \mathbf{R}_{\geq 0} \cdot b$. This is a contradiction.

Next we assume $\dim W = 1$, i.e., $W = \mathbf{R} \cdot b$. Let $N = \dim V_n$. Since the subspace spanned by $\{x^t x \mid x \in S'\}$ is the orthogonal complement of W , there are $N-1$ linearly independent vectors $x_1^t x_1, \dots, x_{N-1}^t x_{N-1}$ in $\{x^t x \mid x \in S'\}$. By the perfection of a , there exists $x_N \in S(a) \setminus S'$ such that $x_1^t x_1, \dots, x_{N-1}^t x_{N-1}, x_N^t x_N$ are linearly independent. If we assume $b \in P_n^{\text{semi}}$, then there is a "square root" $\sqrt{b} \in P_n^{\text{semi}}$ such that $(\sqrt{b})^2 = b$. For each $i = 1, \dots, N-1$, one has

$$0 = \langle b, x_i^t x_i \rangle = {}^t(\sqrt{b} x_i)(\sqrt{b} x_i),$$

i.e., $x_1, \dots, x_{N-1}$ are contained in the nullspace of $\sqrt{b}$. Thus there is a non-zero $y \in \mathbf{R}^n$ such that ${}^t y x_i = 0$ for $i = 1, \dots, N-1$. We choose a non-zero $z \in \mathbf{R}^n$ which is orthogonal to x_N . Then the non-zero symmetric matrix $(y^t z + z^t y)/2 \in V_n$ is orthogonal to $x_1^t x_1, \dots, x_N^t x_N$. This contradicts that $x_1^t x_1, \dots, x_N^t x_N$ spans V_n . $\square$

Proposition 1.15. *Let L be an edge (= one dimensional face) of $K_1(m_1)$. Then there are $a_1, a_2 \in \partial^0 K_1(m_1)$ such that $L = \{\lambda a_1 + (1-\lambda)a_2 \mid 0 \leq \lambda \leq 1\}$.*

Proof. For a sufficiently large $\theta > 0$, we set

$$K_1(m_1)_\theta = K_1(m_1) \cap \{a \in V_n \mid \langle a, \mathbf{I}_n \rangle \leq \theta\} \quad \text{and} \quad L_\theta = L \cap K_1(m_1)_\theta.$$

Since L_θ is an edge of the polytope $K_1(m_1)_\theta$, there are vertices a_1, a'_1 of $K_1(m_1)_\theta$ such that L_θ is the line joining a_1 and a'_1 . Since L_θ is not contained in the affine hyperplane $\{a \in V_n \mid \langle a, \mathbf{I}_n \rangle = \theta\}$, at least one of a_1 and a'_1 must be a vertex of $K_1(m_1)$. Let $a_1 \in \partial^0 K_1(m_1)$ and $b \in V_n$ be a direction of L . Thus, any point of L is of the form $a_1 + \lambda b$ for some $\lambda \geq 0$.

We show $\mathbf{R}_{\geq 0} \cdot b$ is an extreme ray of $\mathcal{C}_{a_1}$. There is an open interval $(0, \lambda_0)$ such that $a_1 + \lambda b \in L$ for all $\lambda \in (0, \lambda_0)$. Since $a_1 + \lambda b \in \partial K_1(m_1)$ for $\lambda \in (0, \lambda_0)$, we have $m_1(a_1 + \lambda b) = 1$ and

$$1 \leq \langle a_1 + \lambda b, x^t x \rangle = 1 + \lambda \langle b, x^t x \rangle$$

for all $x \in S(a_1)$. This means $b \in \mathcal{C}_{a_1}$. If $\mathbf{R}_{\geq 0} \cdot b$ is not an extreme ray of $\mathcal{C}_{a_1}$, then there are $b_1, b_2 \in \mathcal{C}_{a_1} \setminus \mathbf{R}_{\geq 0} \cdot b$ such that $b = (b_1 + b_2)/2$. For $i = 1, 2$ and a sufficiently small $\lambda > 0$, we have $a_1 + \lambda b_i \in P_n$ and $S(a_1 + \lambda b_i) \subset S(a_1)$. From $b_i \in \mathcal{C}_{a_1}$, it follows that for $x \in S(a_1 + \lambda b_i)$,

$$m_1(a_1 + \lambda b_i) = \langle a_1 + \lambda b_i, x^t x \rangle = 1 + \lambda \langle b_i, x^t x \rangle \geq 1.$$

Namely, both $a_1 + \lambda b_1$ and $a_1 + \lambda b_2$ are contained in $K_1(m_1) \setminus L$ and $a_1 + \lambda b$ is the middle point of $a_1 + \lambda b_1$ and $a_1 + \lambda b_2$. This is impossible since L is an edge of $K_1(m_1)$. Therefore $\mathbf{R}_{\geq 0} \cdot b$ must be an extreme ray of $\mathcal{C}_a$.

Since b is an indefinite symmetric matrix by Lemma 1.14, the value $\lambda_1 = \sup\{\lambda \geq 0 \mid a_0 + \lambda b \in K_1(m_1)\}$ is finite. Thus L is written as $L = \{a_1 + \lambda b \mid 0 \leq \lambda \leq \lambda_1\}$. Finally we show the point $a_2 = a_1 + \lambda_1 b$ is a vertex of $K_1(m_1)$. If $a_2 \notin \partial^0 K_1(m_1)$, then there are $c_1, c_2 \in \partial K_1(m_1)$ such that $a_2 = (c_1 + c_2)/2$ and $\lambda c_1 + (1 - \lambda)c_2 \in \partial K_1(m_1)$ for $0 \leq \lambda \leq 1$. In this case, the triangle of vertices a_1, c_1 and c_2 is contained in $\partial K_1(m_1)$. This contradicts that L is an edge of $K_1(m_1)$. $\square$

Corollary 1.16. *The Ryshkov domain $K_1(m_1)$ is the convex hull of $\partial^0 K_1(m_1)$.*

Proof. We fix an arbitrary $a \in K_1(m_1)$. If $\theta > \text{tr}(a)$, then $a \in K_1(m_1)_\theta$. Let $\{b_1, \dots, b_r\}$ be the set of all vertices of the polytope $K_1(m_1)_\theta$. Since $K_1(m_1)_\theta$ is the convex hull of $\{b_1, \dots, b_r\}$, a is of the form $\lambda_1 b_1 + \dots + \lambda_r b_r$ with $\lambda_1 + \dots + \lambda_r = 1$ and $\lambda_i \geq 0$, $i = 1, \dots, r$. Each b_i is either a vertex of $K_1(m_1)$ or the intersection of an edge of $K_1(m_1)$ and the affine hyperplane $\{c \in V_n \mid \text{tr}(c) = \theta\}$. In any case b_i is a point on an edge of $K_1(m_1)$. By Proposition 1.15, all b_i are contained in the convex hull of $\partial^0 K_1(m_1)$, and hence a is also contained in the convex hull of $\partial^0 K_1(m_1)$. $\square$

Proposition 1.15 is regarded as the dual statement of [55, Theorem 12.1]. From Proposition 1.15, it follows that the convex cone $K_1(m_1)$ does not have any extreme direction. Thus, Corollary 1.16 is a consequence of more general theorem [52, Theorem 18.5].

All subsets $K_1(m_1)$, $\partial K_1(m_1)$ and $\partial^0 K_1(m_1)$ of P_n are invariant by the action of $GL_n(\mathbf{Z})$. The finiteness of $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$ is due to Voronoï [61, §7 Théorème].

Theorem 1.17 (Voronoi). *The cardinality of $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$ is finite.*

Proof. We follow the argument of [59, Theorem 3.4]. Let $a \in \partial^0 K_1(m_1)$. By the reduction theory of Hermite or Minkowski, there exists an equivalent $a' \in aGL_n(\mathbf{Z})$ such that $\lambda_1 \lambda_2 \dots \lambda_n \leq c_n \det a'$, where λ_i denotes the i -th diagonal component of a' and c_n is the constant depending only on n . Since a' is perfect, there are n linearly independent minimal vectors $x_1, \dots, x_n$ in $S(a')$. Since $m_1(a') = m_1(a) = 1$, Hadamard's inequality leads us to $\det a' \leq \langle a', x_1^t x_1 \rangle \dots \langle a', x_n^t x_n \rangle = 1$. Therefore, we have

$$\text{tr}(a') = \lambda_1 + \dots + \lambda_n \leq n \lambda_1 \dots \lambda_n \leq n c_n$$

because of $1 = m_1(a') \leq \lambda_i$ for $i = 1, \dots, n$. This shows that any perfect element of $\partial^0 K_1(m_1)$ is $GL_n(\mathbf{Z})$ -equivalent to a vertex of polytope $K_1(m_1) \cap \{a \in V_n \mid \text{tr}(a) \leq n c_n\}$. $\square$

More generally, it is known that the set of all faces of $\partial K_1(m_1)$ has only finitely many $GL_n(\mathbf{Z})$ -orbits ([54, Theorem 5'], see also Theorem 3.2 below). The actual value of the cardinality $\rho_n = \#(\partial^0 K_1(m_1)/GL_n(\mathbf{Z}))$ is known up to $n = 8$ (cf. [58, §3.1]): one has $\rho_2 = 1, \rho_3 = 1, \rho_4 = 2, \rho_5 = 3, \rho_6 = 7, \rho_7 = 33, \rho_8 = 10916$.

We note that the face $\mathcal{F}_S$ may not necessarily be compact in genral. It is known that the non-empty face $\mathcal{F}_S$ is compact if and only if S spans $\mathbf{R}^n$ ([54, Theorem 1], see also [39, Remark 9.1.12]). If $S(a)$ spans $\mathbf{R}^n$, then $a \in P_n$ is said to be well-rounded. Any perfect point is obviously well-rounded. Any weakly eutactic point is also well-rounded ([14, Théorème 2.3]). The finiteness and the algebraicity of weakly eutactic classes are verified by Bergé and Martinet [14, Théorèmes 3.5 et 4.1].

Theorem 1.18 (Bergé and Martinet). *Let $\partial^{we}K_1(m_1)$ be the set of all weakly eutactic points in $\partial K_1(m_1)$. Then the cardinality of $\partial^{we}K_1(m_1)/GL_n(\mathbf{Z})$ is finite. Any $a \in \partial^{we}K_1(m_1)$ is contained in $GL_n(\overline{\mathbf{Q}})$, where $\overline{\mathbf{Q}}$ stands for the algebraic closure of $\mathbf{Q}$.*

Let $\partial^{wr}K_1(m_1)$ be the set of all well-rounded points in $\partial K_1(m_1)$. The quotient $\partial^{wr}K_1(m_1)/GL_n(\mathbf{Z})$ is compact (cf. [39, Proposition 9.1.6]).

1.4. Hermite like constants. Let ϕ be a type one class function and S_p be a complete set of representatives for $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$. From $P_n \subset \mathbf{R}_{>0} \cdot K_1(m_1)$, it follows

$$\sup_{a \in P_n} F_\phi(a) = \sup_{a \in K_1(m_1)} F_\phi(a) = \sup_{a \in K_1(m_1)} \frac{1}{\phi(a)}.$$

By Corollary 1.16, any $a \in K_1(m_1)$ is represented as

$$a = \lambda_1 a_1 + \cdots + \lambda_r a_r$$

by some $a_1, \dots, a_r \in \partial^0 K_1(m_1)$ and $\lambda_1, \dots, \lambda_r \in \mathbf{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_r = 1$. Then, since $\phi(a) \geq \min\{\phi(a_1), \dots, \phi(a_r)\}$, one has

$$\sup_{a \in K_1(m_1)} \frac{1}{\phi(a)} = \sup_{a \in \partial^0 K_1(m_1)} \frac{1}{\phi(a)} = \max_{a \in S_p} \frac{1}{\phi(a)}.$$

Therefore, the Hermite like constant

$$\delta_\phi = \max_{a \in P_n} F_\phi(a)$$

of F_ϕ is well-defined. In the case of $\phi = \det^{1/n}$, δ_ϕ coincides with the Hermite constant γ_n .

Let ϕ° be the dual type one class function of ϕ . By definition, the inequality $m_1(a) \leq \delta_{\phi^\circ} \phi^\circ(a)$ holds for all $a \in P_n$. By passing to the dual, one has $\phi^{\circ\circ}(a) \leq \delta_{\phi^\circ} w_1(a)$ for $a \in P_n$, and by Proposition 1.2,

$$\sup_{a \in P_n} \frac{\phi(a)}{w_1(a)} \leq \delta_{\phi^\circ}.$$

Thus, we can define the "dual" constant

$$\widehat{\delta}_\phi = \sup_{a \in P_n} \frac{\phi(a)}{w_1(a)}.$$

Indeed, we can show $\widehat{\delta}_\phi = \delta_{\phi^\circ}$ for any type one class function ϕ . In particular, this gives

$$\gamma_n = \delta_{\det^{1/n}} = \widehat{\delta}_{(\det^{1/n})^\circ} = n \sup_{a \in P_n} \frac{\det(a)^{1/n}}{w_1(a)}.$$

We write ξ_ϕ for the product $\delta_\phi \cdot \widehat{\delta}_\phi$. This satisfies the invariance $\xi_{\phi^\circ} = \xi_{C\phi} = \xi_\phi$ for any constant $C > 0$. For example, $\xi_{w_1} = \xi_{m_1} = \delta_{w_1}$ and $\xi_{\det^{1/n}} = \gamma_n^2/n$. By definition, we have the following:

Proposition 1.19. *The inequality $\xi_{w_1} \leq \xi_\phi$ holds for any type one class function ϕ .*

See [57, Propositions 3.2 and 3.3] for details.

2. RANKIN'S CONSTANT AND VORONOÏ'S THEOREM

Rankin [51] defined the constant $\gamma_{n,k}$ as a generalization of Hermite's constant, and proved Rankin's inequality among $\gamma_{n,k}$. About 40 years later, Coulangéon [22] formulated Voronoï's theorem of this case in terms of k -perfection and k -eutaxy. It is an open problem to find a geometric characterization of k -perfect forms. Bergé and Martinet [13] introduced the constant $\gamma'_{n,k}$ and proved several inequalities among $\gamma_{n,k}$ and $\gamma'_{n,k}$. In this section, we will survey Voronoï's theorem for $\gamma_{n,k}$ and $\gamma'_{n,1}$.

2.1. Rankin's constant. Let k be a positive integer with $1 \leq k \leq n-1$. We denote by $M_{n,k}^*(\mathbf{Z})$ the subset $\{X = (x_1, \dots, x_k) \in M_{n,k}(\mathbf{Z}) \mid x_1 \wedge x_2 \wedge \dots \wedge x_k \neq 0\}$ of $M_{n,k}(\mathbf{Z})$. The unimodular group $GL_k(\mathbf{Z})$ (resp. $GL_n(\mathbf{Z})$) acts on $M_{n,k}^*(\mathbf{Z})$ by right (resp. left) multiplications. For $X \in M_{n,k}^*(\mathbf{Z})$, define the function $D_X : P_n^{\text{semi}} \rightarrow \mathbf{R}_{\geq 0}$ by

$$D_X(a) = \det({}^t X a X)^{1/k}$$

for $a \in P_n^{\text{semi}}$. It is obvious that D_X is a type one function. The function $m_k : P_n^{\text{semi}} \rightarrow \mathbf{R}_{\geq 0}$ defined by

$$m_k(a) = \inf_{X \in M_{n,k}^*(\mathbf{Z})} D_X(a)$$

is a type one class function. This is regarded as a generalization of the arithmetical minimum function m_1 . It is obvious that $m_k(a) > 0$ if $a \in P_n$, and $m_k(a) = 0$ otherwise. Rankin [51] defined the constant

$$\gamma_{n,k} = \left(\max_{a \in P_n} \frac{m_k(a)}{\det(a)^{1/n}} \right)^k,$$

and then proved the inequality

$$\gamma_{n,k} \leq \gamma_{j,k}(\gamma_{n,j})^{k/j}$$

for $1 \leq k < j \leq n-1$ as a generalization of Mordell's inequality. By using this inequality, Rankin determined the value $\gamma_{4,2} = 3/2$. See 2.4 for other explicit values of $\gamma_{n,k}$.

2.2. Voronoi's theorem of $m_k/\det^{1/n}$. A point $a \in P_n$ is said to be k -extreme (strictly k -extreme) if $m_k/\det^{1/n}$ attains a local maximum (resp. a strict local maximum) on a up to the multiplication by an element of $\mathbf{R}_{>0}$. A Voronoi type characterization of k -extreme points was studied by Coulangeon [22]. The subset

$$S_k^*(a) = \{X \in M_{n,k}^*(\mathbf{Z}) \mid D_X(a) = m_k(a)\}$$

corresponding to $a \in P_n$ plays a key role. Since $D_{Xh}(a) = D_X(a)$ for all $h \in GL_k(\mathbf{Z})$, the set $S_k^*(a)$ is invariant by the action of $GL_k(\mathbf{Z})$, and hence the quotient $S_k(a) = S_k^*(a)/GL_k(\mathbf{Z})$ exists. We write $[X]$ for the element $X \cdot GL_k(\mathbf{Z})$ in $S_k(a)$. The following was proved in [22, Proposition 2.7].

Proposition 2.1 (Coulangeon). *The cardinality of $S_k(a)$ is finite.*

We recall the notion of k -perfection and k -eutaxy. For each $i = 1, 2, \dots, k$, we define the map $*_i : V_n \times M_{n,k}(\mathbf{R}) \longrightarrow M_{n,k}(\mathbf{R})$ by

$$v *_i X = (x_1, \dots, x_{i-1}, vx_i, x_{i+1}, \dots, x_k)$$

for $v \in V_n$ and $X = (x_1, \dots, x_k) \in M_{n,k}(\mathbf{R})$. Note that $*_i$ is linear in X but not in v . Then, for each $X \in M_{n,k}^*(\mathbf{Z})$, the linear map $\varphi_X : V_n \longrightarrow \mathbf{R}$ is defined by

$$\varphi_X(v) = \sum_{i=1}^k \det({}^t X \cdot (v *_i X)).$$

It is obvious that φ_X depends only on the class $[X] = X \cdot GL_k(\mathbf{Z})$. Another definition of φ_X is given by

$$\varphi_X(v) = \det({}^t X \cdot X) \cdot \langle p_X, v \rangle,$$

where p_X denotes the matrix representation of the orthogonal projection from $\mathbf{R}^n$ onto the subspace spanned by $\{x_1, \dots, x_k\}$.

Definition 2.1. Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t b b$. An element a is said to be k -perfect if $\{\varphi_{bX}\}_{[X] \in S_k(a)}$ spans the dual space V_n^* of V_n . An element a is said to be k -eutactic if there exist $\rho_X > 0$ ($[X] \in S_k(a)$) such that

$$\text{tr} = \sum_{[X] \in S_k(a)} \rho_X \varphi_{bX}.$$

These definitions of k -perfection and k -eutaxy do not depend on a choice of b . Now the main theorem of [22] is stated as follows:

Theorem 2.2 (Coulangeon). *A point $a \in P_n$ is k -extreme if and only if a is k -perfect and k -eutactic. Any k -extreme point is strictly k -extreme.*

The line of the proof of Theorem 2.2 is parallel to that of Theorem 1.5. Namely, the following sufficient and necessary condition for k -extremeness is shown: $a = {}^t b b \in P_n$ is k -extreme if and only if the set

$$\{v \in V_n : \min_{[X] \in S_k(a)} \varphi_{bX}(v) \geq 0 \text{ and } \text{tr}(v) \leq 0\}$$

is reduced to $\{0\}$ ([22, Théorème 3.2.2]). Theorem 2.2 follows from this and Stiemke's theorem.

The finiteness of k -perfect points was proved in [22, Théorème 4.5].

Theorem 2.3 (Coulangeon). *The number of classes in $\mathbf{R}_{>0} \backslash P_n / GL_n(\mathbf{Z})$ of k -perfect points is finite.*

Example 2.1. Let $L \subset \mathbf{R}^n$ be a full lattice, which means a $\mathbf{Z}$ -module of rank n . The dual lattice L^* of L is defined by $L^* = \{y \in \mathbf{R}^n : {}^t xy \in \mathbf{Z} \text{ for all } x \in L\}$. If $x_1, \dots, x_n$ is a basis of L , then we denote by $[L]$ the class of the Gram matrix $({}^t x_i x_j)_{1 \leq i, j \leq n}$ in $\mathbf{R}_{>0} \backslash P_n / GL_n(\mathbf{Z})$. In dimension 4, there are at least 5 inequivalent 2-perfect points, i.e., $[A_4], [A_4^*], [D_4], [W_4], [W_4^*]$ ([22, §5.1]). Here we use standard notations of root lattices. By W_4 , we denote the Watson lattice of rank 4, i.e.,

$$[W_4] = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \pmod{\mathbf{R}_{>0} GL_4(\mathbf{Z})}.$$

The maximum of $m_2 / \det^{1/4}$ is attained on $[D_4]$. In general $n \geq 4$, the class of any irreducible root lattice of rank n is 2-extreme and k -eutactic for all $k < n$ ([22, Théorème 5.1.1]).

When $k = 1$, Theorem 1.3 shows that the Hermite constant $\gamma_n = \gamma_{n,1}$ is an algebraic number. The algebraicity of k -perfect points and $\gamma_{n,k}$ for $k \geq 2$ was verified by Bavard [9, Théorème 2.2], [11, §1.5]. It is based on the following general result [11, Lemme 1.11].

Lemma 2.4. *Let $W \subset \mathbf{R}^N$ be an algebraic subset defined by polynomials with coefficients in $\overline{\mathbf{Q}} \cap \mathbf{R}$. Then the set of isolated points in W is a finite subset contained in $W \cap (\overline{\mathbf{Q}} \cap \mathbf{R})^N$.*

If a is k -perfect, then the set

$$W = \{(b, \lambda) \in V_n \times \mathbf{R} : D_X(b) - \lambda = 0 \text{ for all } X \in S_k^*(a)\}$$

satisfies the assumption of Lemma 2.4. Since k -perfect points are isolated in W ([11, Proposition 1.8]), we obtain

Theorem 2.5 (Bavard). *Any k -perfect point is contained in $P_n \cap M_n(\overline{\mathbf{Q}})$. In particular, $\gamma_{n,k}$ is an algebraic number.*

2.3. Some problems on k -perfect forms. Since m_k is a continuous type one function and vanishes on the boundary of P_n^{semi} , $K_1(m_k)$ is a kernel contained in P_n . As we have seen in §1.3, $K_1(m_1)$ is a locally finite polyhedral convex cone and perfect points are characterized as vertices of $K_1(m_1)$. When $k \geq 2$, $K_1(m_k)$ is not polyhedral, and we have the following problem.

Problem 1. Determine locations of k -perfect points in $K_1(m_k)$.

Theorem 1.12 (3) gives another characterization of perfect points in P_n . On variations of the set $S_k(a)$, the next is elementary ([22, Lemme 2.9]).

Proposition 2.6 (Coulangeon). *For $a \in P_n$, there is a neighborhood $\mathcal{U}$ of a in P_n such that $S_k(a') \subset S_k(a)$ for $a' \in \mathcal{U}$.*

We say that $S_k(a)$ is locally maximal if there is a neighborhood $\mathcal{U}$ of a in P_n such that $S_k(a') \subsetneq S_k(a)$ for all $a' \in \mathcal{U} \setminus \mathbf{R}_{>0}a$. Let $P_n^{(k)}$ be the set of $a \in P_n$ such that $S_k(a)$ is locally maximal. If $k = 1$, $P_n^{(1)}$ coincides with the set of all perfect points in P_n .

Problem 2. Does $P_n^{(k)}$ coincide with the set of all k -perfect points in P_n for any $k \geq 2$?

The cardinality $\sharp(S(a)) = 2\sharp(S_1(a))$ is called the kissing number of a . Determination of the maximum $\max_{a \in P_n} 2\sharp(S_1(a))$ is known as the lattice kissing number problem ([20, Chapter 1, §2]). The actual value of $\max_{a \in P_n} \sharp(S_1(a))$ is known for $1 \leq n \leq 9$ and $n = 24$ (cf. [66]). One can prove the estimate $\max_{a \in P_n} \sharp(S_1(a)) \leq 2^n - 1$ for all n as follows: Let $a \in P_n$ and $x, y \in S(a)$. If $y - x = 2z \in 2\mathbf{Z}^n$, then $m_1(a) = {}^t(x+2z)a(x+2z) = {}^t xax$, and hence ${}^t zaz = -{}^t xaz$. From ${}^t xax \leq {}^t(x+z)a(x+z)$, it follows ${}^t zaz \leq 0$, i.e., $z = 0$. This means that the natural map $\mathbf{Z}^n \rightarrow \mathbf{Z}^n/2\mathbf{Z}^n$ is injective on $S_1(a)$. This proof is due to Voronoï [61, p.107, Lemme], see also [42, §31, p.80] for more general result. When $n \geq 10$, Watson [65, Theorem 1] proved $\max_{a \in P_n} \sharp(S_1(a)) \leq 2^{n-2} + 8$. We have a similar problem for $k \geq 2$.

Problem 3. Bound the maximum $\max_{a \in P_n} \sharp(S_k(a))$.

If ϕ is a type one function, then one can ask about Voronoï's theorem for m_k/ϕ .

Problem 4. Prove Voronoï type theorem for m_k/ϕ when $k \geq 2$.

Let w_k be the dual type one class function of m_k . When $k \geq 2$, it is not trivial that the Hermite like constant of m_k/ϕ exists for a given type one class function ϕ . We set

$$\delta_{\phi,k} = \sup_{a \in P_n} \frac{m_k(a)}{\phi(a)}, \quad \widehat{\delta}_{\phi,k} = \sup_{a \in P_n} \frac{\phi(a)}{w_k(a)}$$

for a type one class function ϕ .

Problem 5. When are both $\delta_{\phi,k}$ and $\widehat{\delta}_{\phi,k}$ finite ?

It is easy to see that $\delta_{\phi,k} = \widehat{\delta}_{\phi^\circ,k}$ provided that both $\delta_{\phi,k}$ and $\widehat{\delta}_{\phi^\circ,k}$ are finite.

2.4. The Bergé–Martinet constant. The constant

$$\gamma'_{n,k} = \left(\sup_{a \in P_n} m_k(a)m_k(a^{-1}) \right)^{k/2}$$

was first defined by Bergé and Martinet. In [13], Bergé and Martinet proved several inequalities among $\gamma_{n,k}$ and $\gamma'_{n,k}$.

Theorem 2.7 (Bergé and Martinet). *One has the following:*

- (1) $\gamma'_{n,k} \leq \gamma_{n,k} \leq (\gamma_n)^k$ for $1 \leq k \leq n-1$.
- (2) $(\gamma_{n,k})^n \leq (\gamma_{n-k,k})^{n-k} (\gamma'_{n,k})^{2k}$ for $1 \leq k \leq n/2$.
- (3) $\gamma'_{n,2k} \leq (\gamma'_{n-k,k})^2$ for $1 \leq k \leq n/2$.
- (4) $(\gamma_{n,k})^{n-2k} \leq (\gamma_{n-k,k})^{n-k}$ for $1 \leq k \leq n-1$.
- (5) $\gamma_{n,n/2} = \gamma'_{n,n/2}$ if n is even.

When $k = 1$, an analog of Voronoï's theorem holds for the Bergé–Martinet invariant $F_{\text{BM}}(a) = \sqrt{m_1(a)m_1(a^{-1})}$. A point $a \in P_n$ is said to be dual-extreme (strictly dual extreme) if F_{BM} attains a local maximum (resp. a strict local maximum) on a up to the multiplication by an element of $\mathbf{R}_{>0}$. To define the dual-perfection and the dual-eutaxy, we use the same notation as in 1.2.

Definition 2.2. Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^tbb$. An element a is said to be dual-perfect if $\{\varphi_{bx}\}_{x \in S(a)} \cup \{\varphi_{tb^{-1}y}\}_{y \in S(a^{-1})}$ spans the dual space V_n^* of V_n . An element a is said to be dual-eutactic if there exist $\rho_x > 0$ ($x \in S(a)$) and $\rho_y > 0$ ($y \in S(a^{-1})$) such that

$$\sum_{x \in S(a)} \rho_x \varphi_{bx} = \sum_{y \in S(a^{-1})} \rho_y \varphi_{tb^{-1}y}.$$

Then one has:

Theorem 2.8 (Bergé and Martinet). *A point $a \in P_n$ is dual-extreme if and only if a is dual-perfect and dual-eutactic. Any dual extreme point is a strict dual extreme.*

As noticed in [39, p.99], the number of dual-perfect points in P_n modulo $\mathbf{R}_{>0}GL_n(\mathbf{Z})$ is infinite in general. In [12], Bergé proved the following:

Theorem 2.9 (Bergé). *The number of dual-extreme points in P_n modulo $\mathbf{R}_{>0}GL_n(\mathbf{Z})$ is finite. If $a \in P_n$ is dual extreme, then there exists $\lambda \in \mathbf{R}_{>0}$ such that $\lambda a \in GL_n(\overline{\mathbf{Q}})$. In particular, $\gamma'_n = \gamma'_{n,1}$ is an algebraic number.*

As to the explicit value of γ'_n , $\gamma'_8 = \gamma_8 = 2$ immediately follows from the self-duality of the E_8 -lattice. Bergé and Martinet determined the values $\gamma'_2 = 2/\sqrt{3}$, $\gamma'_3 = \sqrt{3/2}$ and $\gamma'_4 = \sqrt{2}$. In [48], Poor and Yuen proved the inequality

$$(2) \quad \frac{n}{(\gamma'_n)^2} \leq \inf_{(a,b) \in P_n \times P_n} \frac{\langle a, b \rangle}{m_1(a)m_1(b)} \leq \frac{n}{(\gamma'_n)^2}$$

and, by using this, they determined the following values.

Theorem 2.10 (Poor and Yuen). $\gamma'_5 = \sqrt{2}$, $\gamma'_6 = \sqrt{8/3}$ and $\gamma'_7 = \sqrt{3}$.

All known values of γ'_n satisfy $(\gamma'_n)^2 \in \mathbf{Q}$. The following problem is due to Martinet [39, Questions 3.8.12].

Problem 6. Is $(\gamma'_n)^2$ rational for all n ?

Applying Theorem 2.7 to the explicit values of $\gamma'_5, \gamma'_7, \gamma_{4,2}$ and γ_n for $n = 2, \dots, 8$, we have

Theorem 2.11. $\gamma_{6,2} = 3^{2/3}$, $\gamma'_{6,2} = 2$, $\gamma_{8,2} = \gamma'_{8,2} = 3$ and $\gamma_{8,3} = \gamma'_{8,3} = \gamma_{8,4} = \gamma'_{8,4} = 4$.

See [56, §2] for details. Barnes and Cohn ([7]) also proved the first part of the inequality (2). Since one has

$$\frac{1}{\xi_{w_1}} = \inf_{(a,b) \in P_n \times P_n} \frac{\langle a, b \rangle}{m_1(a)m_1(b)},$$

the first part of (2) is a special case of Proposition 1.19. Furthermore, the inequality (2) is generalized to $\gamma_{n,k}$ and $\gamma'_{n,k}$ as follows:

$$\frac{n}{(\gamma_{n,k})^{2/k}} \leq \inf_{(a,b) \in P_n \times P_n} \frac{\langle a, b \rangle}{m_k(a)m_k(b)} \leq \frac{n}{(\gamma'_{n,k})^{2/k}},$$

(see [56, Theorem 1]). To find an analog of Voronoï's theorem for the k -th Bergé–Martinet invariant $F_{\text{BM}}^{(k)}(a) = (m_k(a)m_k(a^{-1}))^{k/2}$ ([39, Problem 10.6.10]) solved by Bavard, see Example 3.7 below.

3. GENERALIZATIONS OF VORONOI'S THEOREM

There are several directions to generalize Voronoi's theorem. A natural generalization of the domain P_n was considered by Ash [3]. An extension of the geometric framework was developed by Bavard [9], [11]. A change of a base field from $\mathbf{Q}$ to an algebraic number field was studied by several authors [22], [31], [37], [44], [45]. In this section, we will survey these theories. We do not exhaust all of generalizations. See e.g., [39, Chapter 13] for other variations.

3.1. Voronoi's theorem of packing functions on symmetric cones. A generalization of the domain P_n is given by the notion of symmetric cones. Let Ω be an open convex cone in the Euclidean space $\mathbf{R}^N$. The open dual cone Ω^* of Ω is defined to be

$$\Omega^* = \{a \in \mathbf{R}^N : (a, b) > 0 \text{ for all } b \in \overline{\Omega} \setminus \{0\}\},$$

where $\overline{\Omega}$ denotes the closure of Ω in $\mathbf{R}^N$. If $\Omega = \Omega^*$ holds, then Ω is called a self-dual cone. We denote by G_Ω the stabilizer of Ω in $GL_N(\mathbf{R})$, i.e.,

$$G_\Omega = \{g \in GL_N(\mathbf{R}) : g\Omega = \Omega\}.$$

If G_Ω acts transitively on Ω , then Ω is said to be homogeneous. By a symmetric cone, we mean a self-dual homogeneous cone. See the textbook [25] for details of symmetric cones.

We fix a symmetric cone Ω . Let G_Ω° be the connected component of the identity in G_Ω . Then G_Ω° also acts transitively on Ω . We denote by K_a the stabilizer of $a \in \Omega$ in G_Ω° . There exists a point $e \in \Omega$ such that $K_e = G_\Omega^\circ \cap O_N(\mathbf{R})$. The group K_e is connected and gives a maximal compact subgroup of G_Ω° . Thus Ω is identified with the Riemannian symmetric space G_Ω°/K_e .

Let $L \subset \mathbf{R}^N$ be a lattice of rank N which contains e . Now we define the packing function $F_{(\Omega, L)} : \Omega \rightarrow \mathbf{R}_{>0}$, of which we study local maxima. First, the characteristic function φ_Ω of Ω is defined by

$$\varphi_\Omega(a) = \int_\Omega e^{-(a, b)} db, \quad (a \in \Omega).$$

The Lebesgue measure db is normalized so that $\varphi_\Omega(e) = 1$. The defining integral is uniformly convergent on any compact subset in Ω . It follows from the definition that $\varphi_\Omega(ga) = |\det g|^{-1} \varphi_\Omega(a)$ for all $g \in G_\Omega$ and $a \in \Omega$. Next, the minimum function m_L on Ω is defined by

$$m_L(a) = \min\{(a, b) : b \in (L \setminus \{0\}) \cap \overline{\Omega}\}.$$

Since Ω is self-dual, the value $m_L(a)$ is positive. Then the packing function $F_{\Omega, L}$ is defined by

$$F_{(\Omega, L)}(a) = m_L(a)^N \varphi_\Omega(a).$$

A point $a \in \Omega$ is said to be extreme if $F_{(\Omega, L)}$ attains a local maximum on a up to the multiplication by an element of $\mathbf{R}_{>0}$.

To state Ash's definition of eutaxy, we need a Jordan algebra structure of $\mathbf{R}^N$ induced from Ω . Let $\mathfrak{g}$ be the Lie algebra of G_Ω° , i.e.,

$$\mathfrak{g} = \{X \in M_N(\mathbf{R}) : \exp(X) \in G_\Omega^\circ\}.$$

Since Ω is a symmetric cone, $\mathfrak{g}$ is invariant by the transpose $X \mapsto {}^tX$. We set $\mathfrak{g}_\pm = \{X \in \mathfrak{g} : {}^tX = \pm X\}$. Then $\mathfrak{g}_-$ coincides with the Lie algebra of K_e . Moreover,

the map $\psi : \mathfrak{g}_+ \longrightarrow \mathbf{R}^N$ defined by $\psi(X) = Xe$ gives a linear isomorphism. We define the binary product $*$: $\mathbf{R}^N \times \mathbf{R}^N \longrightarrow \mathbf{R}^N$ by

$$a * b = \psi^{-1}(a)b.$$

This product satisfies

- (J₁) $a * b = b * a$
- (J₂) $a * (a^2 * b) = a^2 * (a * b)$, where a^2 means $a * a$
- (J₃) $e * a = a * e = a$
- (J₄) $(a * c, b) = (a, c * b)$

for all $a, b, c \in \mathbf{R}^N$. Namely, $*$ gives $\mathbf{R}^N$ a formally real Jordan algebra structure with the identity e . We denote by J_Ω this formally real Jordan algebra. Let $J_\Omega^\times$ be the subset of all invertible elements in J_Ω . Then Ω coincides with the connected component of $J_\Omega^\times$ which contains e .

We assume $L \otimes_{\mathbf{Z}} \mathbf{Q}$ gives a $\mathbf{Q}$ -structure of J_Ω . For $a \in \Omega$, we set

$$S_{(\Omega, L)}(a) = \{b \in (L \setminus \{0\}) \cap \overline{\Omega} : (a, b) = m_L(a)\}.$$

Definition 3.1. Let $a \in \Omega$ and a^{-1} be the inverse of a in the Jordan algebra J_Ω . A point a is said to be perfect if $S_{(\Omega, L)}(a)$ spans $\mathbf{R}^N$. A point a is said to be eutactic if there exist $\lambda_b \in \mathbf{R}_{>0}$, $b \in S_{(\Omega, L)}$, such that

$$a^{-1} = \sum_{b \in S_{(\Omega, L)}} \lambda_b b.$$

Let $g \in \exp(\mathfrak{g}_+)$. The Taylor expansion of $1/\varphi_\Omega$ at the point $a = ge$ is given by

$$\frac{1}{\varphi_\Omega(a+v)} = \det g \cdot \{1 + (a^{-1}, v) + \frac{1}{2}((a^{-1}, v)^2 - (g^{-1}v, g^{-1}v)) + O((v, v)^{3/2})\}$$

for $v \in \mathbf{R}^N$ ([3, Corollary to Proposition 3]). By using this formula, Ash proved that the function $1/F_{(\Omega, L)}$ is a topological Morse function on $\Omega/\mathbf{R}_{>0}$. Then Voronoï's theorem of $F_{(\Omega, L)}$ is derived from this result.

Theorem 3.1 (Ash). *A point $a \in \Omega$ is extreme if and only if a is perfect and eutactic.*

Let $K_1(m_L) = \{a \in \overline{\Omega} : (a, b) \geq 1 \text{ for all } b \in (L \setminus \{0\}) \cap \overline{\Omega}\}$, which is a polyhedral cone and is regarded as a generalization of the Ryshkov domain $K_1(m_1)$. Any perfect point $a \in \Omega$ of $m_L(a) \geq 1$ is a vertex of $K_1(m_L)$. We have the following finiteness:

Theorem 3.2 (Ash). *The discrete group $\Gamma = \{g \in G_\Omega^\circ : {}^t g L = L\}$ of G_Ω° acts on $K_1(m_L)$. The set of faces of $K_1(m_L)$ has only finitely many Γ -orbits. In particular, the number of perfect points in Ω modulo $\mathbf{R}_{>0}\Gamma$ is finite. Moreover, the number of eutactic points in Ω modulo $\mathbf{R}_{>0}\Gamma$ is finite.*

It is proved that a point $a \in \Omega$ is eutactic if and only if a is critical non-degenerate for $F_{(\Omega, L)}$ ([3, Theorem 2]). The finiteness of eutactic points is derived from the finiteness of Γ -orbits of critical points of $F_{(\Omega, L)}$.

Example 3.1. The cone P_n of positive definite symmetric matrices is a symmetric cone in V_n . In this case, e is chosen as the identity matrix I_n and the product $*$ is defined by

$$a * b = \frac{1}{2}(ab + ba)$$

for $a, b \in V_n$. When $L = \{v = (v_{ij}) \in M_n(\mathbf{Q}) \cap V_n \mid 2v \in M_n(\mathbf{Z}), v_{11}, \dots, v_{nn} \in \mathbf{Z}\}$, the packing function $F_{(P_n, L)}$ is equal to $(m_1 / \det^{1/n})^{n(n+1)/2}$. Ash's definition of perfection and eutaxy is equivalent to Definition 1.3 ([3, Corollary to Proposition 2]).

Example 3.2. Let B be the non degenerate bilinear form on $\mathbf{R}^n$ defined by

$$B(x, y) = x_1 y_1 - x_2 y_2 - \dots - x_n y_n$$

for $x = {}^t(x_1, \dots, x_n)$ and $y = {}^t(y_1, \dots, y_n) \in \mathbf{R}^n$. Then the Lorentz cone $\Omega_n = \{x \in \mathbf{R}^n : B(x, x) > 0, x_1 > 0\}$ is a symmetric cone in $\mathbf{R}^n$. We choose e as the unit vector ${}^t(1, 0, \dots, 0) \in \mathbf{R}^n$. Let $\{e\}^\perp$ be the orthogonal complement of e with respect to the usual inner product $(\cdot, \cdot)$ of $\mathbf{R}^n$. The product of the Jordan algebra J_{Ω_n} is defined by

$$(\lambda e + u) * (\lambda' e + u') = (\lambda \lambda' - B(u, u'))e + \lambda u' + \lambda' u$$

for $\lambda, \lambda' \in \mathbf{R}$ and $u, u' \in \{e\}^\perp$. The packing function $F_{(\Omega_n, \mathbf{Z}^n)}$ is given as

$$F_{(\Omega_n, \mathbf{Z}^n)}(a) = \frac{m_{\mathbf{Z}^n}(a)^n}{B(a, a)^{n/2}}.$$

Since $G_{\Omega_n}^\circ = \mathbf{R}_{>0} \cdot SO_0(1, n-1)$ acts transitively on Ω_n , we have

$$F_{(\Omega_n, \mathbf{Z}^n)}(\lambda g e) = \frac{m_{\mathbf{Z}^n}(g e)^n}{B(g e, g e)^{n/2}} = m_{\mathbf{Z}^n}(g e)^n$$

for $\lambda \in \mathbf{R}_{>0}$ and $g \in SO_0(1, n-1)$, and moreover,

$$m_{\mathbf{Z}^n}(g e) = \min_{x \in (\mathbf{Z}^n \setminus \{0\}) \cap \overline{\Omega_n}} (e, {}^t g x) = \min_{x \in (\mathbf{Z}^n \setminus \{0\}) \cap \overline{\Omega_n}} \left(\frac{({}^t g x, {}^t g x) + B(x, x)}{2} \right)^{1/2}.$$

Therefore, we have

$$\max_{a \in \Omega_n} F_{(\Omega_n, \mathbf{Z}^n)}(a) = \max_{[g] \in K_e \setminus SO_0(1, n-1)/\Gamma} \min_{x \in (\mathbf{Z}^n \setminus \{0\}) \cap \overline{\Omega_n}} \left(\frac{(g x, g x) + B(x, x)}{2} \right)^{n/2},$$

where $\Gamma = SO_0(1, n-1) \cap SL_n(\mathbf{Z})$.

Problem 7. Let Ω be an arbitrary symmetric cone. Replacing P_n and P_n^{semi} with Ω and $\overline{\Omega}$, respectively, in Definitions 1.1 and 1.2, we can define type one functions on $\overline{\Omega}$ and semikernels in $\overline{\Omega}$. For example, both $\varphi_\Omega^{-1/N}$ and m_L are continuously extended to type one functions on $\overline{\Omega}$. Can Proposition 1.1 and Theorem 1.5 be generalized to this setting?

As stated in Theorem 3.2, the number of eutactic classes in $\mathbf{R}_{>0} \setminus \Omega / \Gamma$ is finite. When $\Omega = P_n$, Ash verified a "mass formula with signs" of eutactic classes ([4], see also [39, Theorem 9.5.3]). For $a \in \partial K_1(m_1)$, we set $\Gamma_a = \{g \in SL_n(\mathbf{Z}) : {}^t g a g = a\}$. Let $\mathcal{F}_{S(a)}$ be the face of $K_1(m_1)$ defined in 1.3, i.e., $\mathcal{F}_{S(a)} = \{b \in \partial K_1(m_1) : S(a) \subset S(b)\}$.

Theorem 3.3 (Ash). *The set $\partial^e K_1(m_1)$ of eutactic points in $\partial K_1(m_1)$ satisfies*

$$\sum_{[a] \in \partial^e K_1(m_1) / GL_n(\mathbf{Z})} \frac{(-1)^{\dim \mathcal{F}_{S(a)}}}{\#\Gamma_a} = \chi(SL_n(\mathbf{Z})) = \begin{cases} -1/12 & (n = 2) \\ 0 & (n \geq 3) \end{cases},$$

where $\chi(SL_n(\mathbf{Z}))$ stands for the Euler characteristic of $SL_n(\mathbf{Z})$.

The actual value of $\epsilon_n = \#(\partial^e K_1(m_1)/GL_n(\mathbf{Z}))$ is known up to $n = 5$ (cf. [8]): one has $\epsilon_2 = 2, \epsilon_3 = 5, \epsilon_4 = 16, \epsilon_5 = 118$.

3.2. Bavard's theory. Let V be a Riemannian manifold and Γ a discrete subgroup of the isometry group of V . Let C be a set endowed with a right action of Γ . We consider a family of C^1 -functions $f_s : V \longrightarrow \mathbf{R}$ parametrized by $s \in C$. We assume the following two conditions:

- (B₁) $f_s \circ \gamma = f_{s\gamma}$ for all $s \in C$ and $\gamma \in \Gamma$.
- (B₂) The cardinality of the subset $\{s \in C : f_s(v) \leq \lambda\}$ of C is finite for any $v \in V$ and $\lambda \in \mathbf{R}$.

Then each f_s is called a length function on V . We write $\mathcal{E}$ for the quadruplet $(V, \Gamma, C, \{f_s\})$. What we do is to characterize local maxima of the function $F_{\mathcal{E}}(v) = \min_{s \in C} f_s(v)$ in $v \in V$. A point $v \in V$ is said to be extreme (resp. strictly extremum) if v attains a local maximum (resp. a strictly local maximum) of $F_{\mathcal{E}}$.

For a given $v \in V$, $T_v V$ stands for the tangent space of V at v and $X_s(v)$ stands for the gradient vector of f_s at v . By the condition (B₂), $S_{\mathcal{E}}(v) = \{s \in C : f_s(v) = F_{\mathcal{E}}(v)\}$ is a finite subset of C . Let $\text{Conv}(v)$ be the convex hull of $\{X_s(v)\}_{s \in S_{\mathcal{E}}(v)}$ in $T_v V$ and $\text{Aff}(v)$ the affine subspace spanned by $\{X_s(v)\}_{s \in S_{\mathcal{E}}(v)}$ in $T_v V$.

Definition 3.2. A point $v \in V$ is said to be perfect if $T_v V = \text{Aff}(v)$ holds. A point $v \in V$ is said to be eutactic if the origin $0 \in T_v V$ is contained in the interior of $\text{Conv}(v)$.

We need the following condition for $\mathcal{E}$.

- (C) For any $v \in V$, any subset $S' \subset S_{\mathcal{E}}(v)$ and any non-zero vector $X \in T_v V$ orthogonal to $\{X_s(v)\}_{s \in S'}$, there exists a C^1 -curve $c : [0, \epsilon) \longrightarrow V$ for a sufficiently small $\epsilon > 0$ such that $c(0) = v$, $c'(0) = X$ and $f_s(v) < f_s(c(t))$ for all $t \in (0, \epsilon)$ and $s \in S'$.

Now Bavard's theorem is stated as:

Theorem 3.4 (Bavard). *Assume $\mathcal{E}$ satisfies the condition (C). Then any extreme point in V is strictly extreme, and a point $v \in V$ is extreme if and only if v is perfect and eutactic.*

A function $f : V \longrightarrow \mathbf{R}$ is said to be convex if f is convex on any geodesic line on V , i.e.,

$$f(\ell(\lambda\alpha + (1-\lambda)\beta)) \leq \lambda f(\ell(\alpha)) + (1-\lambda)f(\ell(\beta))$$

holds for any geodesic $\ell : [0, \epsilon) \longrightarrow V$, $\alpha, \beta \in (0, \epsilon)$, $\alpha \neq \beta$, and $0 < \lambda < 1$. If this inequality is strict, then f is said to be strictly convex. It is proved that $\{f_s\}_{s \in C}$ satisfies the condition (C) if f_s is strictly convex for all $s \in C$.

Theorem 3.5 (Bavard). *Assume f_s is convex for all $s \in C$. Then a point $v \in V$ is strictly extreme if and only if v is perfect and eutactic.*

We give several examples.

Example 3.3. We consider the subset $P_n^1 = \{a \in P_n : \det a = 1\}$ of P_n , which is identified with the Riemannian symmetric space $SL_n(\mathbf{R})/SO_n(\mathbf{R})$. For $x \in \mathbf{Z}^n \setminus \{0\}$, define the length function $f_x : P_n^1 \longrightarrow \mathbf{R}$ by $f_x(a) = {}^t x a x$. The family $\{f_x\}_{x \in \mathbf{Z}^n \setminus \{0\}}$ satisfies (B₁) for $\Gamma = SL_n(\mathbf{Z})$, (B₂) and the condition (C) ([9, Example 1]). Thus one can apply Theorem 3.4 to $\mathcal{E} = (P_n^1, SL_n(\mathbf{Z}), \mathbf{Z}^n \setminus \{0\}, \{f_x\})$.

Since the definition of perfection and eutaxy of Definition 3.2 is equivalent to that of Definition 1.3, this case verifies Voronoi's theorem. The length function f_x is convex on P_n^1 for all $x \in \mathbf{Z}^n \setminus \{0\}$.

Example 3.4. Let G be a connected Lie subgroup of $SL_n(\mathbf{R})$ and $G \cdot I_n$ be the G -orbit of the identity matrix I_n in P_n^1 , i.e., $G \cdot I_n = \{^t g g : g \in G\}$. Assume G is invariant by the transpose $g \mapsto ^t g$. Then $G \cdot I_n$ is totally geodesic, and hence the restriction $f_x|_{G \cdot I_n}$ of the length function f_x to $G \cdot I_n$ is convex for all $x \in \mathbf{Z}^n \setminus \{0\}$. Thus one can apply Theorem 3.5 to $\mathcal{E} = (G \cdot I_n, G \cap SL_n(\mathbf{Z}), \mathbf{Z}^n \setminus \{0\}, \{f_x|_{G \cdot I_n}\})$.

Example 3.5. Assume n is even. Let G be the symplectic group, i.e.,

$$G = Sp_n(\mathbf{R}) = \left\{ g \in SL_n(\mathbf{R}) : ^t g \begin{pmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{pmatrix} \right\}.$$

In this case, the family $\{f_x|_{G \cdot I_n}\}_{x \in \mathbf{Z}^n \setminus \{0\}}$ satisfies the condition (C). This is a particular case of more general family [1, Lemme 3.3]. See [1, Théorème 3.1] for other symmetric spaces of classical type.

Example 3.6. For $X \in M_{n,k}^*(\mathbf{Z})$, define the length function $f_X : P_n^1 \rightarrow \mathbf{R}$ by $f_X(a) = \det(^t X a X)$, i.e., $f_X = D_X^k$. Then the family $\{f_X\}_{X \in M_{n,k}^*(\mathbf{Z})}$ satisfies the condition (C) ([9, Proposition 2.8]). Theorem 2.2 is verified again by Theorem 3.4 specialized to $\mathcal{E} = (P_n^1, SL_n(\mathbf{Z}), M_{n,k}^*(\mathbf{Z}), \{f_X\})$.

Example 3.7. We define the subset $C_{n,k}$ of $M_{n,k}(\mathbf{Z}) \times M_{n,k}(\mathbf{Z})$ by

$$C_{n,k} = \{(X, 0), (0, Y) : X, Y \in M_{n,k}^*(\mathbf{Z})\}.$$

This set is stable by the action of $SL_n(\mathbf{Z})$ i.e., $(X, Y)g = (^t g X, g^{-1} Y)$. For $(X, Y) \in C_{n,k}$, define the length function $f_{(X,Y)} : P_n \rightarrow \mathbf{R}$ by $f_{(X,Y)}(a) = D_X(a)^k + D_Y(a^{-1})^k$ for $a \in P_n$. Then the quadruplet $\mathcal{E} = (P_n, SL_n(\mathbf{Z}), C_{n,k}, \{f_{(X,Y)}\})$ satisfies the condition (C) [11, Théorème 5]. Bavard proved the set of extrem points of k -th Bergé–Martinet invariant $F_{\text{BM}}^{(k)}$ coincides with that of $F_{\mathcal{E}}$ ([11, Proposition 2.21]). Thus Voronoi's theorem for $F_{\text{BM}}^{(k)}$ results in that of $F_{\mathcal{E}}$.

In some cases, the finiteness and the algebraicity of perfect points were also proved by Bavard [11, Corollaire 2.12 et Théorème 1]. We explain the simplest case of Bavard's result. Let $\mathfrak{f}$ be one of the real number field $\mathbf{R}$, the complex number field $\mathbf{C}$ or the Hamilton quaternion field $\mathbf{H}$. For an $n \times n$ matrix $(\lambda_{ij}) \in M_n(\mathfrak{f})$ with entries in $\mathfrak{f}$, we write $(\lambda_{ij})^*$ for ${}^t(\bar{\lambda}_{ij})$, where $\lambda \mapsto \bar{\lambda}$ stands for the main involution of $\mathfrak{f}$. The set $P_n^1(\mathfrak{f}) = \{g^* g : g \in SL_n(\mathfrak{f})\}$ is a Riemannian symmetric space. We fix a subring $\mathfrak{o}_{\mathfrak{f}}$ in $\mathfrak{f}$ as $\mathfrak{o}_{\mathfrak{f}} = \mathbf{Z}$ if $\mathfrak{f} = \mathbf{R}$, $\mathfrak{o}_{\mathfrak{f}} = \mathbf{Z}[\sqrt{-1}]$ if $\mathfrak{f} = \mathbf{C}$ and $\mathfrak{o}_{\mathfrak{f}} = \mathbf{Z}[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ if $\mathfrak{f} = \mathbf{H}$, where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denotes the usual quaternion basis of $\mathbf{H}$. For $x \in \mathfrak{o}_{\mathfrak{f}}^n \setminus \{0\}$, define the length function $f_x : P_n^1(\mathfrak{f}) \rightarrow \mathbf{R}$ by $f_x(a) = x^* a x$. Then $\mathcal{E} = (P_n^1(\mathfrak{f}), SL_n(\mathfrak{o}_{\mathfrak{f}}), \mathfrak{o}_{\mathfrak{f}}^n \setminus \{0\}, \{f_x\})$ satisfies the condition (C) ([1, Corollaire 3.1]).

Theorem 3.6 (Bavard). *For $\mathcal{E} = (P_n^1(\mathfrak{f}), SL_n(\mathfrak{o}_{\mathfrak{f}}), \mathfrak{o}_{\mathfrak{f}}^n \setminus \{0\}, \{f_x\})$, the number of perfect points in $P_n^1(\mathfrak{f})$ modulo $SL_n(\mathfrak{o}_{\mathfrak{f}})$ is finite. Any perfect point in $P_n^1(\mathfrak{f})$ is algebraic over $\mathbf{Q}$, i.e., which is contained in $M_n(\mathfrak{o}_{\mathfrak{f}} \otimes_{\mathbf{Z}} \overline{\mathbf{Q}})$.*

More generally, Bavard proved such result for some totally geodesic subvarieties in $P_n^1(\mathfrak{f})$. However the algebraicity of $\gamma'_{n,k}$ for $k \geq 2$ is still unknown.

In connection with Ash's mass formula (Theorem 3.3), we append Bavard's mass formula [10, Théorème 1]. We recall $\partial^{wr} K_1(m_1)$ denotes the set of all well-rounded points in $\partial K_1(m_1)$. If $a \in \partial^{wr} K_1(m_1)$, then $\mathcal{F}_{S(a)}$ is a compact face. The family $\{\mathcal{F}_{S(a)}\}_{a \in \partial^{wr} K_1(m_1)}$ of compact faces has only finitely many $SL_n(\mathbf{Z})$ -orbits by Theorem 3.2. Let $\{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ be the complete set of representatives of $SL_n(\mathbf{Z})$ -orbits in $\{\mathcal{F}_{S(a)}\}_{a \in \partial^{wr} K_1(m_1)}$.

Theorem 3.7 (Bavard). $\{\mathcal{F}_1, \dots, \mathcal{F}_r\}$ satisfies

$$\sum_{i=1}^r \frac{(-1)^{\dim \mathcal{F}_i}}{\#\Gamma_i} = \chi(SL_n(\mathbf{Z})) = \begin{cases} -1/12 & (n=2) \\ 0 & (n \geq 3) \end{cases},$$

where Γ_i stands for the stabilizer of $\mathcal{F}_i$ in $SL_n(\mathbf{Z})$.

For a further study of this mass formula, see [15].

3.3. Voronoï's theorem over an algebraic number field $\mathbf{I}$. There are two methods of an extension of the base field. One is the additive generalization ([34], [37], [44], [45]) and another is the multiplicative generalization ([22], [31]). Both methods give the original Voronoï's theorem if the base field is $\mathbf{Q}$. We first explain the additive generalization.

Let $\mathfrak{k}$ be an algebraic number field of degree r and $\mathfrak{o}$ the ring of integers of $\mathfrak{k}$. The set of all infinite (resp. real and imaginary) places of $\mathfrak{k}$ is denoted by $\mathfrak{p}_\infty$ (resp. $\mathfrak{p}_1$ and $\mathfrak{p}_2$). Let $\mathfrak{k}_\sigma$ be the completion of $\mathfrak{k}$ at $\sigma \in \mathfrak{p}_\infty$, i.e., $\mathfrak{k}_\sigma = \mathbf{R}$ if $\sigma \in \mathfrak{p}_1$ and $\mathfrak{k}_\sigma = \mathbf{C}$ if $\sigma \in \mathfrak{p}_2$. We use the étale $\mathbf{R}$ -algebra $\mathfrak{k}_{\mathbf{R}} = \mathfrak{k} \otimes_{\mathbf{Q}} \mathbf{R}$, which is identified with $\prod_{\sigma \in \mathfrak{p}_\infty} \mathfrak{k}_\sigma$. For $\mathbf{x} = (x_\sigma) \in \mathfrak{k}_{\mathbf{R}}$, the conjugate $\bar{\mathbf{x}}$ of $\mathbf{x}$ is defined to be $\bar{\mathbf{x}} = (\bar{x}_\sigma)$, where $\bar{x}_\sigma$ is the complex conjugate of x_σ . The trace and the norm of $\mathfrak{k}_{\mathbf{R}}$ are defined as

$$\mathrm{Tr}_{\mathfrak{k}_{\mathbf{R}}}(\mathbf{x}) = \sum_{\sigma \in \mathfrak{p}_\infty} \mathrm{Tr}_{\mathfrak{k}_\sigma/\mathbf{R}}(x_\sigma), \quad \mathrm{Nr}_{\mathfrak{k}_{\mathbf{R}}}(\mathbf{x}) = \prod_{\sigma \in \mathfrak{p}_\infty} \mathrm{Nr}_{\mathfrak{k}_\sigma/\mathbf{R}}(x_\sigma)$$

for $\mathbf{x} = (x_\sigma) \in \mathfrak{k}_{\mathbf{R}}$.

Let $\mathfrak{k}_{\mathbf{R}}^n = \mathfrak{k}^n \otimes_{\mathbf{Q}} \mathbf{R}$ be the $\mathfrak{k}_{\mathbf{R}}$ -module of rank n . An element of $\mathfrak{k}_{\mathbf{R}}^n$ is denoted by a column vector $X = {}^t(\mathbf{x}_1, \dots, \mathbf{x}_n)$ with $\mathbf{x}_i \in \mathfrak{k}_{\mathbf{R}}$, $i = 1, \dots, n$. The group consisting of $\mathfrak{k}_{\mathbf{R}}$ -linear automorphisms of $\mathfrak{k}_{\mathbf{R}}^n$ is denoted by $GL_n(\mathfrak{k}_{\mathbf{R}})$, which is identified with $\prod_{\sigma \in \mathfrak{p}_\infty} GL_n(\mathfrak{k}_\sigma)$. As an $\mathbf{R}$ -vector space, $\mathfrak{k}_{\mathbf{R}}^n$ is equipped with the inner product

$$(X, Y) = \mathrm{Tr}_{\mathfrak{k}_{\mathbf{R}}}({}^t \bar{X} Y) = \mathrm{Tr}_{\mathfrak{k}_{\mathbf{R}}}(\bar{\mathbf{x}}_1 \mathbf{y}_1 + \dots + \bar{\mathbf{x}}_n \mathbf{y}_n),$$

for $X = {}^t(\mathbf{x}_1, \dots, \mathbf{x}_n), Y = {}^t(\mathbf{y}_1, \dots, \mathbf{y}_n) \in \mathfrak{k}_{\mathbf{R}}^n$. The group of isometries

$$O_n(\mathfrak{k}_{\mathbf{R}}) = \{g \in GL_n(\mathfrak{k}_{\mathbf{R}}) : (gX, gY) = (X, Y) \text{ for all } X, Y \in \mathfrak{k}_{\mathbf{R}}^n\}$$

is a maximal compact subgroup of $GL_n(\mathfrak{k}_{\mathbf{R}})$. We define the subsets $V_n(\mathfrak{k}_{\mathbf{R}})$ and $P_n(\mathfrak{k}_{\mathbf{R}})$ of $M_n(\mathfrak{k}_{\mathbf{R}})$ as follows:

$$V_n(\mathfrak{k}_{\mathbf{R}}) = \{a \in M_n(\mathfrak{k}_{\mathbf{R}}) : (aX, Y) = (X, aY) \text{ for all } X, Y \in \mathfrak{k}_{\mathbf{R}}^n\},$$

$$P_n(\mathfrak{k}_{\mathbf{R}}) = \{a \in V_n(\mathfrak{k}_{\mathbf{R}}) : (aX, X) > 0 \text{ for all } X \in \mathfrak{k}_{\mathbf{R}}^n \setminus \{0\}\}.$$

The set $V_n(\mathfrak{k}_{\mathbf{R}})$ is an $\mathbf{R}$ -subspace of $M_n(\mathfrak{k}_{\mathbf{R}})$ of dimension $n(n+1)\#\mathfrak{p}_1/2 + n^2\#\mathfrak{p}_2$, and $P_n(\mathfrak{k}_{\mathbf{R}})$ is a symmetric cone in $V_n(\mathfrak{k}_{\mathbf{R}})$. Let $V_n(\mathfrak{k}_{\mathbf{R}})^*$ be the dual space of $V_n(\mathfrak{k}_{\mathbf{R}})$ as an $\mathbf{R}$ -vector space. The trace $\mathrm{Tr}_{M_n(\mathfrak{k}_{\mathbf{R}})} \in V_n(\mathfrak{k}_{\mathbf{R}})^*$ is defined to be the composition of the matrix trace tr and $\mathrm{Tr}_{\mathfrak{k}_{\mathbf{R}}}$, i.e., $\mathrm{Tr}_{M_n(\mathfrak{k}_{\mathbf{R}})} = \mathrm{Tr}_{\mathfrak{k}_{\mathbf{R}}} \circ \mathrm{tr}$. For $X \in \mathfrak{k}_{\mathbf{R}}^n$, define the linear form $\varphi_X \in V_n(\mathfrak{k}_{\mathbf{R}})^*$ by $\varphi_X(a) = (aX, X)$ for $a \in V_n(\mathfrak{k}_{\mathbf{R}})$.

An $\mathfrak{o}$ -submodule Λ in $\mathfrak{k}_{\mathbf{R}}^n$ is called an $\mathfrak{o}$ -lattice if Λ is discrete and $\Lambda \otimes_{\mathbf{Z}} \mathbf{R} = \mathfrak{k}_{\mathbf{R}}^n$. Any projective $\mathfrak{o}$ -module in $\mathfrak{k}^n$ of rank n is regarded as an $\mathfrak{o}$ -lattice in $\mathfrak{k}_{\mathbf{R}}^n$ by the natural inclusion $\mathfrak{k}^n \subset \mathfrak{k}_{\mathbf{R}}^n$. Conversely, for any $\mathfrak{o}$ -lattice Λ in $\mathfrak{k}_{\mathbf{R}}^n$, there exists $g \in GL_n(\mathfrak{k}_{\mathbf{R}})$ such that $g^{-1}\Lambda$ is a projective $\mathfrak{o}$ -module in $\mathfrak{k}^n$ ([36, Lemma 3.2]). Thus, by Steinitz's theorem, any $\mathfrak{o}$ -lattice is isomorphic with an $\mathfrak{o}$ -module of the form $\mathfrak{o}^{n-1} \oplus \mathfrak{a}$, where $\mathfrak{a}$ is an ideal of $\mathfrak{o}$. Let $\mathfrak{a}_1 = \mathfrak{o}, \mathfrak{a}_2, \dots, \mathfrak{a}_{h_{\mathfrak{k}}}$ be a complete system of representatives of the ideal class group of $\mathfrak{k}$. If $\mathcal{H}_{\mathfrak{a}_i}$ denotes the $GL_n(\mathfrak{k}_{\mathbf{R}})$ -orbit of the $\mathfrak{o}$ -lattice $\mathfrak{o}^{n-1} \oplus \mathfrak{a}_i$, then the set of all $\mathfrak{o}$ -lattices in $\mathfrak{k}_{\mathbf{R}}^n$ is given by the disjoint union

$$\mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n} = \bigsqcup_{i=1}^{h_{\mathfrak{k}}} \mathcal{H}_{\mathfrak{a}_i}.$$

Each component $\mathcal{H}_{\mathfrak{a}_i}$ is identified with $GL_n(\mathfrak{k}_{\mathbf{R}})/GL(\mathfrak{o}^{n-1} \oplus \mathfrak{a}_i)$, where $GL(\mathfrak{o}^{n-1} \oplus \mathfrak{a}_i)$ denotes the stabilizer of $\mathfrak{o}^{n-1} \oplus \mathfrak{a}_i$ in $GL_n(\mathfrak{k}_{\mathbf{R}})$.

For $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$, the minimum $m_+(\Lambda)$ and the discriminant $\text{disc}(\Lambda)$ of Λ are defined to be

$$m_+(\Lambda) = \min_{X \in \Lambda \setminus \{0\}} (X, X), \quad \text{disc}(\Lambda) = \left(\frac{\text{Vol}(\mathfrak{k}_{\mathbf{R}}^n/\Lambda)}{\text{Vol}(\mathfrak{k}_{\mathbf{R}}^n/\mathfrak{o}^n)} \right)^2.$$

We denote by $S_+(\Lambda)$ the set of shortest vectors in Λ , i.e.,

$$S_+(\Lambda) = \{X \in \Lambda : (X, X) = m_+(\Lambda)\}.$$

As an analog of the Hermite invariant, we consider the function $F_+ : \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n} \longrightarrow \mathbf{R}_{>0}$ defined by

$$F_+(\Lambda) = \frac{m_+(\Lambda)}{\text{disc}(\Lambda)^{1/(rn)}}.$$

Obviously, F_+ depends only on the similar isometry class $\mathbf{R}^{\times} O_n(\mathfrak{k}_{\mathbf{R}}) \Lambda$ of Λ , i.e., F_+ is a function on $\mathbf{R}^{\times} O_n(\mathfrak{k}_{\mathbf{R}}) \backslash \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$. An $\mathfrak{o}$ -lattice $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$ is said to be extreme if F_+ attains a local maximum on $\mathbf{R}^{\times} O_n(\mathfrak{k}_{\mathbf{R}}) \Lambda$.

Definition 3.3. An $\mathfrak{o}$ -lattice $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$ is said to be perfect if $\{\varphi_X\}_{X \in S_+(\Lambda)}$ spans $V_n(\mathfrak{k}_{\mathbf{R}})^*$, and Λ is said to be eutactic if there are $\rho_X \in \mathbf{R}_{>0}$, $X \in S_+(\Lambda)$, such that

$$\text{Tr}_{M_n(\mathfrak{k}_{\mathbf{R}})} = \sum_{X \in S_+(\Lambda)} \rho_X \varphi_X.$$

Leibak [37] proved a weak version of Voronoï's theorem for F_+ restricted to the component $\mathcal{H}_{\mathfrak{a}_1} = \mathcal{H}_{\mathfrak{o}}$ consisting of free $\mathfrak{o}$ -lattices. Leibak's definition of eutaxy is weaker than that of Definition 3.3. Okuda and Yano [44] found a suitable definition of eutaxy to complete Leibak's result.

Theorem 3.8 (Leibak, Okuda and Yano). *An $\mathfrak{o}$ -lattice $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$ is extreme if and only if Λ is perfect and eutactic.*

By Humbert's reduction theory ([30]) and Cramer's formula, one has the following finiteness and algebraicity of perfect $\mathfrak{o}$ -lattices.

Theorem 3.9 (Okuda and Yano). *The number of similar isometry classes of perfect $\mathfrak{o}$ -lattices in $\mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$ is finite. Let $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}^n}$ be a perfect $\mathfrak{o}$ -lattice with $m_+(\Lambda) = 1$. If $g \in GL_n(\mathfrak{k}_{\mathbf{R}})$ such that $g^{-1}\Lambda \subset \mathfrak{k}^n$, then ${}^t \bar{g}_{\sigma} g_{\sigma} \in M_n(\mathfrak{l})$ for all $\sigma \in \mathfrak{p}_{\infty}$, where $\mathfrak{l}$ is the Galois closure of $\mathfrak{k}$ over $\mathbf{Q}$.*

In the case that $\mathfrak{k}$ is a real quadratic field, a classification of some perfect $\mathfrak{o}$ -lattices of small rank was given by Ong [45] and Leibak [38].

Koecher [34] studied the function F_+ in connection with the reduction theory of $P_n(\mathfrak{k}_{\mathbf{R}})$. The bound $\sharp(S_+(\Lambda)) \leq 2(2^{r_n} - 1)$ for all $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}}^n$ is proved by the similar way as in 2.3 (see [34, Lemma 12]).

3.4. Voronoi's theorem over an algebraic number field II. We use the same notation as in the previous section. For $a \in P_n(\mathfrak{k}_{\mathbf{R}})$, the multiplicative minimum $m_*(a)$ and the discriminant $\text{disc}(a)$ are defined to be

$$m_*(a) = \min_{X \in \mathfrak{o}^n \setminus \{0\}} \text{Nr}_{\mathfrak{k}_{\mathbf{R}}}(^t \bar{X} a X), \quad \text{disc}(a) = \text{Nr}_{\mathfrak{k}_{\mathbf{R}}}(\det a).$$

We denote by $S_*(a)$ the set of minimal integral vectors, i.e.,

$$S_*(a) = \{X \in \mathfrak{o}^n : \text{Nr}_{\mathfrak{k}_{\mathbf{R}}}(^t \bar{X} a X) = m_*(a)\}.$$

The unit group $\mathfrak{o}^\times$ acts on $S_*(a)$ by multiplication. The set $S_*(a)/\mathfrak{o}^\times$ of classes is finite ([31, Lemma 1]). Define the function $F_* : P_n(\mathfrak{k}_{\mathbf{R}}) \rightarrow \mathbf{R}_{>0}$ by

$$F_*(a) = \frac{m_*(a)}{\text{disc}(a)^{1/n}}.$$

The group $GL_n(\mathfrak{k}_{\mathbf{R}})$ acts transitively on $P_n(\mathfrak{k}_{\mathbf{R}})$ by $(a, g) \mapsto {}^t \bar{g} a g$ for $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ and $g \in GL_n(\mathfrak{k}_{\mathbf{R}})$. By definition, F_* is invariant by the action of the discrete subgroup $GL_n(\mathfrak{o}) \subset GL_n(\mathfrak{k}_{\mathbf{R}})$, i.e., one has

$$F_*({}^t \bar{g} a g) = F_*(a)$$

for all $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ and $g \in GL_n(\mathfrak{o})$. Moreover, if we set

$$\mathfrak{k}_{\mathbf{R}}^+ = \{x = (x_\sigma)_{\sigma \in \mathfrak{p}_\infty} \in \mathfrak{k}_{\mathbf{R}}^\times : x_\sigma \in \mathbf{R}_{>0} \text{ for all } \sigma \in \mathfrak{p}_\infty\},$$

then $F_*(xa) = F_*(a)$ holds for all $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ and $x \in \mathfrak{k}_{\mathbf{R}}^+$. Therefore, F_* is considered as a function on $\mathfrak{k}_{\mathbf{R}}^+ \backslash P_n(\mathfrak{k}_{\mathbf{R}})/GL_n(\mathfrak{o})$. An element $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ is said to be extreme if F_* attains a local maximum on the class $\mathfrak{k}_{\mathbf{R}}^+ a GL_n(\mathfrak{o})$.

Let $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ and $X \in \mathfrak{k}_{\mathbf{R}}^+ \setminus \{0\}$. Then ${}^t \bar{X} a X$ is invertible in $\mathfrak{k}_{\mathbf{R}}^\times$. The map $b \mapsto (bX, ({}^t \bar{X} a X)^{-1} X)$ on $V_n(\mathfrak{k}_{\mathbf{R}})$ defines an $\mathbf{R}$ -linear form. We write φ_X^a for this linear form. For $x = (x_\sigma) \in \mathfrak{k}_{\mathbf{R}}^+$, we set $\log(x) = (\log(x_\sigma)) \in \mathfrak{k}_{\mathbf{R}}$. From the definition, it follows $\varphi_X^a(\log(x)a) = \text{Tr}_{\mathfrak{k}_{\mathbf{R}}}(\log(x))$ for $x \in \mathfrak{k}_{\mathbf{R}}^+$. We denote by $\mathfrak{k}_{\mathbf{R}}^1$ the subset of $x \in \mathfrak{k}_{\mathbf{R}}^+$ such that $\text{Tr}_{\mathfrak{k}_{\mathbf{R}}}(\log(x)) = 0$, i.e.,

$$\mathfrak{k}_{\mathbf{R}}^1 = \{x = (x_\sigma)_{\sigma \in \mathfrak{p}_\infty} \in \mathfrak{k}_{\mathbf{R}}^+ : \text{Nr}_{\mathfrak{k}_{\mathbf{R}}}(x) = 1\}.$$

Thus φ_X^a is null on the $(\sharp(\mathfrak{p}_1) + \sharp(\mathfrak{p}_2) - 1)$ -dimensional subspace $\log(\mathfrak{k}_{\mathbf{R}}^1) \cdot a \subset V_n(\mathfrak{k}_{\mathbf{R}})$.

Definition 3.4. An element $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ is said to be perfect if $\{\varphi_X^a\}_{[X] \in S_*(a)/\mathfrak{o}^\times}$ spans the dual space $(V_n(\mathfrak{k}_{\mathbf{R}})/\log(\mathfrak{k}_{\mathbf{R}}^1) \cdot a)^*$, and a is said to be euatctic if there are $\rho_X \in \mathbf{R}_{>0}$, $[X] \in S_*(a)/\mathfrak{o}^\times$, such that the linear form $b \mapsto \text{Tr}_{M_n(\mathfrak{k}_{\mathbf{R}})}(a^{-1}b)$ on $V_n(\mathfrak{k}_{\mathbf{R}})$ is represented as

$$(b \mapsto \text{Tr}_{M_n(\mathfrak{k}_{\mathbf{R}})}(a^{-1}b)) = \sum_{[X] \in S_*(a)/\mathfrak{o}^\times} \rho_X \varphi_X^a.$$

This definition is due to Coulangeon. Icaza [31, Proposition 3] first proved a weak version of Voronoi's theorem for F_* , and later Coulangeon [23] completed a full version.

Theorem 3.10 (Icaza, Coulangéon). *An element $a \in P_n(\mathfrak{k}_{\mathbf{R}})$ is extreme if and only if a is perfect and eutactic.*

The finiteness and the algebraicity of perfect points were also proved by Coulangéon [23, Proposition 4.1].

Theorem 3.11 (Coulangéon). *The number of classes in $\mathfrak{k}_{\mathbf{R}}^+ \backslash P_n(\mathfrak{k}_{\mathbf{R}}) / GL_n(\mathfrak{o})$ of perfect elements is finite. If $a = (a_\sigma) \in P_n(\mathfrak{k}_{\mathbf{R}})$ is perfect, then there exists $x = (x_\sigma) \in \mathfrak{k}_{\mathbf{R}}^+$ such that $x_\sigma a_\sigma \in M_n(\overline{\mathbf{Q}})$ for all $\sigma \in \mathfrak{p}_\infty$.*

Theorem 3.10 is also verified by Bavard's theory. We set $P_n(\mathfrak{k}_{\mathbf{R}})^1 = \{a \in P_n(\mathfrak{k}_{\mathbf{R}}) : \text{disc}(a) = 1\}$, and for $X \in \mathfrak{o}^n \setminus \{0\}$ define the length function $f_X : P_n(\mathfrak{k}_{\mathbf{R}})^1 \rightarrow \mathbf{R}$ by $f_X(a) = \log(\text{Nr}_{\mathfrak{k}_{\mathbf{R}}}({}^t \overline{X} a X))$. Let $\mathcal{E} = (P_n(\mathfrak{k}_{\mathbf{R}})^1, SL_n(\mathfrak{o}), \mathfrak{o}^n \setminus \{0\}, \{f_X\})$. Then Bavard [11, Proposition 2.22] proved that the definition of perfection and eutaxy in Definition 3.4 coincides with that in Definition 3.2 for $\mathcal{E}$, and $\mathcal{E}$ satisfies the condition (C). In contrast to Theorem 3.2, the number of classes in $\mathfrak{k}_{\mathbf{R}}^+ \backslash P_n(\mathfrak{k}_{\mathbf{R}}) / GL_n(\mathfrak{o})$ of eutactic elements is not finite in general if $\mathfrak{k} \neq \mathbf{Q}$, see [23, p.162]. To resolve this problem, Bavard introduced the notion of non-degenerate points in his framework ([11, Définition 1.5]), and proved that the number of classes of non-degenerate eutactic elements is finite ([11, Proposition 2.24]).

If $\mathfrak{k}$ is an imaginary quadratic field, then F_* is essentially the same as F_+ restricted to $\mathcal{H}_{\mathfrak{o}}$. More precisely, we have

$$4F_*({}^t \bar{g} g) = F_+(g \mathfrak{o}^n)^2$$

for all $g \in GL_n(\mathfrak{k}_{\mathbf{R}}) = GL_n(\mathbf{C})$. In particular, the number of $S_*(a)/\mathfrak{o}^\times$ is bounded by $2(4^n - 1)/\#(\mathfrak{o}^\times)$. In general, any estimate of the number of $S_*(a)/\mathfrak{o}^\times$ is unknown.

Problem 8. Bound the maximum $\max_{a \in P_n(\mathfrak{k}_{\mathbf{R}})} \#(S_*(a)/\mathfrak{o}^\times)$.

The assertion of Theorem 3.10 is true even if the free $\mathfrak{o}$ -lattice $\mathfrak{o}^n$ in the definitions of m_* and $S_*(a)$ is replaced with a general $\mathfrak{o}$ -lattice $\Lambda \in \mathcal{H}_{\mathfrak{k}_{\mathbf{R}}}^n$. This was verified by Meyer [40, Théorème 3.21] in more general setting.

4. GENERALIZED HERMITE CONSTANTS OF FLAG VARIETIES

A generalization of Hermite's constant to algebraic groups was studied in [62] and [63]. The main problem in this theory is to formulate and verify Voronoï type theorems. This problem was completely solved by Meyer [40], [41] in the case of GL_n . Some inner forms of GL_n were studied in [24]. It is likely that Bavard's theory applies to many cases, e.g., see Example 4.2 below. However, to approach adelic Voronoï theorems for the generalized Hermite constants involving positive characteristic cases, we will need a suitable definition of perfection for our adelic setting.

4.1. Generalized Hermite constants. Let G be a connected affine algebraic group defined over $\mathbf{Q}$. For any $\mathbf{Q}$ -algebra A , $G(A)$ stands for the group of A -rational points of G . In particular, $G(\mathbf{A})$ denotes the adèle group of G . Let $\mathbf{X}_{\mathbf{Q}}^*(G)$ be the module of $\mathbf{Q}$ -rational charcters of G . We denote by $G(\mathbf{A})^1$ the subgroup $\{g \in G(\mathbf{A}) : |\chi(g)|_{\mathbf{A}} = 1 \text{ for all } \chi \in \mathbf{X}_{\mathbf{Q}}^*(G)\}$, where $|\cdot|_{\mathbf{A}}$ denotes the usual idele norm of the idele group of $\mathbf{Q}$. By the product formula of the idele norm, $G(\mathbf{Q})$ is contained in $G(\mathbf{A})^1$.

In the following, let G be a connected reductive algebraic group defined over $\mathbf{Q}$. We fix a minimal $\mathbf{Q}$ -parabolic subgroup P of G and a Levi subgroup M_P of P . The maximal central $\mathbf{Q}$ -split torus Z_P of M_P is a maximal $\mathbf{Q}$ -split torus of G . We choose a maximal $\mathbf{Q}$ -parabolic subgroup Q of G and its Levi subgroup M_Q such that $P \subset Q$ and $M_P \subset M_Q$. Let Z_G be the maximal central $\mathbf{Q}$ -split torus of G . Since Q is maximal, the module $\mathbf{X}_{\mathbf{Q}}^*(M_Q/Z_G)$ is of rank one, and hence there is a unique generator $\hat{\alpha}_Q$ of $\mathbf{X}_{\mathbf{Q}}^*(M_Q/Z_G)$ such that the restriction of $\hat{\alpha}_Q$ to Z_P/Z_G is a positive scalar multiple of a positive simple root with respect to (P, Z_P) . Let U_Q be the unipotent radical of Q , and let K be a maximal compact subgroup of $G(\mathbf{A})$ such that $G(\mathbf{A}) = P(\mathbf{A})K$. Then the height function $H_Q : G(\mathbf{A}) \rightarrow \mathbf{R}_{>0}$ is defined by

$$H_Q(umh) = |\hat{\alpha}_Q(m)|_{\mathbf{A}}^{-1}$$

for $u \in U_Q(\mathbf{A})$, $m \in M_Q(\mathbf{A})$ and $h \in K$. Indeed H_Q is a function on the space $Z_G(\mathbf{A})Q(\mathbf{A})^1 \backslash G(\mathbf{A}) = Q(\mathbf{A})^1 \backslash G(\mathbf{A})^1$. Define the function $F_Q : G(\mathbf{A}) \rightarrow \mathbf{R}_{>0}$ by

$$F_Q(g) = \min_{[v] \in Q(\mathbf{Q}) \backslash G(\mathbf{Q})} H_Q(vg).$$

The generalized Hermite constant γ_Q of $Q \backslash G$ is defined to be the maximum

$$\gamma_Q = \max_{[g] \in Z_G(\mathbf{A})G(\mathbf{Q}) \backslash G(\mathbf{A})/K} F_Q(g) = \max_{[g] \in G(\mathbf{Q}) \backslash G(\mathbf{A})^1/K} F_Q(g).$$

We assume the following two conditions for G and Q :

- (C₁) $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})K$.
- (C₂) $G(\mathbf{Q}) = Q(\mathbf{Q})G(\mathbf{Z})$, where $G(\mathbf{Z}) = G(\mathbf{Q}) \cap G(\mathbf{R})K$.

The condition (C₁) means that G is of class number one. By [17, Proposition 7.5], the condition (C₂) is satisfied if M_Q is of class number one. Then γ_Q is represented as

$$\gamma_Q = \max_{[g] \in G(\mathbf{Z}) \backslash G(\mathbf{R})/K_{\infty}} F_Q(g) = \max_{[g] \in G(\mathbf{Z}) \backslash G(\mathbf{R})/K_{\infty}} \min_{\gamma \in G(\mathbf{Z})} H_Q^{\infty}(\gamma g),$$

where K_{∞} and H_Q^{∞} denote the infinite components of K and H_Q , respectively.

Example 4.1. For $k = 1, \dots, n-1$, let

$$R_k(\mathbf{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} : a \in GL_k(\mathbf{Q}), b \in M_{k, n-k}(\mathbf{Q}), d \in GL_{n-k}(\mathbf{Q}) \right\}.$$

Then R_k is a maximal $\mathbf{Q}$ -parabolic subgroup of GL_n . It is well-known that GL_n and R_k satisfy both conditions (C₁) and (C₂). The character $\hat{\alpha}_{R_k}$ is given by

$$\hat{\alpha}_{R_k} \left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right) = (\det a)^{(n-k)/\gcd(k, n-k)} (\det d)^{-k/\gcd(k, n-k)}.$$

For $\gamma \in GL_n(\mathbf{Z})$, X_{γ} denotes the n by k matrix consisting of the first k -columns of γ . It is an easy exercise to prove that

$$H_{R_k}^{\infty}(\gamma g)^{\gcd(k, n-k)/n} = \det({}^t X_{\gamma^{-1}} g^{-1} g^{-1} X_{\gamma^{-1}})^{1/2}$$

holds for any $\gamma \in GL_n(\mathbf{Z})$ and $g \in SL_n(\mathbf{R})$. From this relation, it follows that $(\gamma_{R_k})^{2\gcd(k, n-k)/n}$ equals the Rankin constant $\gamma_{n,k}$. Thus, Coulangeon's result in 2.2 is interpreted as Voronoi's theorem of the function F_{R_k} .

Example 4.2. Let B be the non degenerate bilinear form on $\mathbf{Q}^n$ defined by

$$B(x, y) = x_1 y_1 - x_2 y_2 - \cdots - x_n y_n$$

for $x = {}^t(x_1, \dots, x_n)$ and $y = {}^t(y_1, \dots, y_n) \in \mathbf{Q}^n$. We assume $n \geq 3$ and put $e_{11} = {}^t(1, 1, 0, \dots, 0) \in \mathbf{Q}^n$. Let $\mathcal{N}_B(\mathbf{Q})$ be the set of all non-zero isotropic vectors in $\mathbf{Q}^n$ with respect to B . The special orthogonal group $SO_B(\mathbf{Q})$ of B transitively acts on $\mathcal{N}_B(\mathbf{Q})$, i.e., one has $\mathcal{N}_B(\mathbf{Q}) = SO_B(\mathbf{Q})e_{11}$. Let $P(\mathbf{Q})$ be the stabilizer of the isotropic line $\mathbf{Q}e_{11}$ in $SO_B(\mathbf{Q})$. Then P is a unique proper $\mathbf{Q}$ -parabolic subgroup of the algebraic group SO_B up to $SO_B(\mathbf{Q})$ -conjugates. For any finite prime p , K_p denotes the stabilizer of the $\mathbf{Z}_p$ -lattice $\mathbf{Z}_p^n$ in $SO_B(\mathbf{Q}_p)$. Since $\mathbf{Z}_p^n$ is a unimodular maximal lattice with respect to B , K_p is a maximal compact subgroup of $SO_B(\mathbf{Q}_p)$. At the infinite place ∞ , the stabilizer K_∞ of the vector $e = {}^t(1, 0, \dots, 0)$ in $SO_B(\mathbf{R})$ gives a maximal compact subgroup of $SO_B(\mathbf{R})$. The intersection $SO_B(\mathbf{Q}) \cap (SO_B(\mathbf{R}) \times \prod_{p < \infty} K_p)$ is the stabilizer $SO_B(\mathbf{Z})$ of the lattice $\mathbf{Z}^n$ in $SO_B(\mathbf{Q})$. The adèle group $SO_B(\mathbf{A})$ has the Iwasawa decomposition: $SO_B(\mathbf{A}) = P(\mathbf{A})(\prod_{p \leq \infty} K_p)$. The height function $H_P : SO_B(\mathbf{A}) \rightarrow \mathbf{R}_{>0}$ is given by

$$H_P(g) = \|g^{-1}e_{11}\|_{\mathbf{A}^n} = \prod_{p \leq \infty} \|g_p^{-1}e_{11}\|_p$$

for $g = (g_p)_{p \leq \infty} \in SO_B(\mathbf{A})$. Here, the local height $\|\cdot\|_p$ is defined by

$$\|x\|_p = \begin{cases} \sqrt{x_1^2 + \cdots + x_n^2} & (p = \infty) \\ \max(|x_1|_p, \dots, |x_n|_p) & (p < \infty) \end{cases}$$

for $x = {}^t(x_1, \dots, x_n) \in \mathbf{Q}_p^n$. If $g = umh$ with $u \in U_P(\mathbf{A})$, $m \in M_P(\mathbf{A})$ and $h \in \prod_{p \leq \infty} K_p$, then $H_P(g)$ equals the idele norm of the first component of the vector $m^{-1}e_{11}$. Since the class number of the indefinite lattice $(B, \mathbf{Z}^n)$ equals one, SO_B satisfies the condition (C_1) , and hence the generalized Hermite constant γ_P is written as

$$\gamma_P = \max_{[g] \in SO_B(\mathbf{Z}) \backslash SO_B(\mathbf{R}) / K_\infty} \min_{[v] \in P(\mathbf{Q}) \backslash SO_B(\mathbf{Q})} \|g^{-1}v^{-1}e_{11}\|_{\mathbf{A}^n}.$$

Let $\mathcal{N}_B^*(\mathbf{Z})$ denote the set of primitive vectors in $\mathcal{N}_B(\mathbf{Q}) \cap \mathbf{Z}^n$, i.e.,

$$\mathcal{N}_B^*(\mathbf{Z}) = \{x = {}^t(x_1, \dots, x_n) \in \mathbf{Z}^n \setminus \{0\} : B(x, x) = 0 \text{ and } \gcd(x_1, \dots, x_n) = 1\}.$$

From $SO_B(\mathbf{Q})e_{11} = \mathcal{N}_B(\mathbf{Q}) = \mathbf{Q}^\times \cdot \mathcal{N}_B^*(\mathbf{Z})$, it follows that

$$\min_{[v] \in P(\mathbf{Q}) \backslash SO_B(\mathbf{Q})} \|g^{-1}v^{-1}e_{11}\|_{\mathbf{A}^n} = \min_{x \in \mathcal{N}_B^*(\mathbf{Z})} \|g^{-1}x\|_{\mathbf{A}^n}.$$

Since x is a primitive isotropic vector and $g \in SO_B(\mathbf{R})$, we have

$$\|g^{-1}x\|_{\mathbf{A}^n} = \|g^{-1}x\|_\infty \times \prod_{p < \infty} \|x\|_p = \|g^{-1}x\|_\infty,$$

and hence

$$\gamma_P = \max_{[g] \in K_\infty \backslash SO_B(\mathbf{R}) / SO_B(\mathbf{Z})} \min_{x \in \mathcal{N}_B^*(\mathbf{Z})} \|gx\|_\infty.$$

We may compare this with Example 3.2. The group $SO_0(1, n-1)$ is the identity connected component of $SO_B(\mathbf{R})$. Since $K_\infty \backslash SO_B(\mathbf{R}) / SO_B(\mathbf{Z}) = K_e \backslash SO_0(1, n-1) / \Gamma$ and $\mathcal{N}_B^*(\mathbf{Z})$ equals the subset of primitive vectors in $\mathbf{Z}^n \setminus \{0\} \cap \partial\Omega_n$, one has

$$(3) \quad 2^{n/2} \max_{a \in \Omega_n} F_{(\Omega_n, \mathbf{Z}^n)}(a) \leq (\gamma_P)^n.$$

If $n = 3$, then $\gamma_P = \sqrt{\gamma_2} = \sqrt{2/\sqrt{3}}$ since SO_B is isomorphic with PGL_2 over $\mathbf{Q}$. In this case, the equality of (3) holds. In general n , Birch and Davenport's theorem ([16, Theorem A]) gives $\gamma_P \leq (\sqrt{2n}\gamma_{n-1})^{(n-1)/2}$. It is unknown whether the equality $2^{n/2} \max_{a \in \Omega_n} F_{(\Omega_n, \mathbf{Z}^n)}(a) = (\gamma_P)^n$ holds for all n or not. Bavard's theory applies to this example. By using the same notation as in Example 3.4, we can choose $\mathcal{E} = (SO_0(1, n-1) \cdot I_n, \Gamma, \mathcal{N}_B^*(\mathbf{Z}), \{f_x|_{SO_0(1, n-1)}\})$ as a quadruplet in question (cf. [9, Proposition 2.6]).

We should mention two remarks. Let G and P be the same as above, i.e., G is a connected reductive algebraic group defined over $\mathbf{Q}$ and P a minimal $\mathbf{Q}$ -parabolic subgroup. First, let R be a $\mathbf{Q}$ -parabolic subgroup of G such that $P \subset R$. Then the generalized Hermite constant γ_R is defined even if R is not maximal. However, in this case, there is no canonical height function on $R(\mathbf{A})^1 \backslash G(\mathbf{A})^1$. As a result, there are infinitely many multiplicatively independent generalized Hermite constants of $R \backslash G$. To define a height function on $R(\mathbf{A})^1 \backslash G(\mathbf{A})^1$, we choose a $\mathbf{Q}$ -rational embedding of the projective variety $R \backslash G$ into a projective space. Such an embedding is constructed by a strongly $\mathbf{Q}$ -rational irreducible representation $\pi : G \longrightarrow GL_N$ of G . By strongly $\mathbf{Q}$ -rational, we mean that π is a $\mathbf{Q}$ -rational morphism and the highest weight line l_π in $\overline{\mathbf{Q}}^N$ of π is defined over $\mathbf{Q}$. Assume $\pi(R)$ is the stabilizer of l_π in $\pi(G)$. Then the map $g \mapsto \pi(g^{-1})l_\pi$ gives rise to a $\mathbf{Q}$ -rational embedding of $R \backslash G$ into the projective space $\mathbf{P}^{N-1}$. If R_1 denotes the maximal parabolic subgroup of GL_N defined in a similar fashion as in Example 4.1, then $\mathbf{P}^{N-1}$ is identified with $R_1 \backslash GL_N$, and the height H_{R_1} is defined on $R_1(\mathbf{A})^1 \backslash GL_N(\mathbf{A})^1$. The composition of π and H_{R_1} gives a height function $H_{R, \pi}$ on $R(\mathbf{A})^1 \backslash G(\mathbf{A})^1$. Then the generalized Hermite constant $\gamma_{R, \pi}$ is defined to be

$$\gamma_{R, \pi} = \max_{[g] \in R(\mathbf{A})^1 \backslash G(\mathbf{A})^1} F_{R, \pi}(g), \quad \text{where } F_{R, \pi}(g) = \min_{[v] \in R(\mathbf{Q}) \backslash G(\mathbf{Q})} H_{R, \pi}(vg).$$

Second remark is a base change from $\mathbf{Q}$ to a number field $\mathfrak{k}$. The adelic definition of the generalized Hermite constants is immediately extended to reductive groups defined over $\mathfrak{k}$. Thus one can define $\gamma_{R, \pi}(\mathfrak{k})$ for a connected reductive group G defined over $\mathfrak{k}$, a parabolic $\mathfrak{k}$ -subgroup $R \subset G$ and a strongly $\mathfrak{k}$ -rational representation π . In this notation, the constant $\gamma_{R, \pi}(\mathbf{Q})$ means $\gamma_{R, \pi}$. We write $\gamma_n(\mathfrak{k})$ (resp. $\gamma_n(\mathfrak{k})_1$ and $\gamma_{n, k}(\mathfrak{k})$) for the generalized Hermite constant of $R_1 \backslash GL_n$ (resp. $(R_1 \cap SL_n) \backslash SL_n$ and $R_k \backslash GL_n$) defined over $\mathfrak{k}$. When $\mathfrak{k}$ is a totally real number field, $\gamma_2(\mathfrak{k})_1$ was implicitly occurred in Cohn's paper [19, §5, §9]. Newman [43, Chapter XI] defined $\gamma_n(\mathfrak{k})_1$ for imaginary quadratic fields $\mathfrak{k}$. For a general number field $\mathfrak{k}$, $\gamma_n(\mathfrak{k})_1$ was studied by Icaza [31] in terms of Humbert forms, i.e., $\gamma_n(\mathfrak{k})_1$ is relating with the function F_* as follows:

$$(\gamma_n(\mathfrak{k})_1)^{2n} = \max_{[a] \in \mathfrak{k}_{\mathbf{R}}^+ \backslash P_n(\mathfrak{k}_{\mathbf{R}}) / GL_n(\mathfrak{o})} F_*(a),$$

(cf. [64, §3]). Thunder [60] defined $\gamma_{n, k}(\mathfrak{k})$ as a generalization of Rankin's constant. The above definition of $\gamma_{R, \pi}(\mathfrak{k})$ was given in [62]. Meyer [40] investigated $\gamma_{R, \pi}(\mathfrak{k})$ when $G = GL_n$ and π is a Schur module realized as a polynomial representation, and established Voronoi's theorem for $F_{R, \pi}$ by using Bavard's theory. Among other results in [40], we present here only the algebraicity of $\gamma_{R, \pi}(\mathfrak{k})$:

Theorem 4.1 (Meyer). *Let $\mathfrak{k}$ be an arbitrary algebraic number field. When $G = GL_n$, all $\gamma_{R, \pi}(\mathfrak{k})$ are algebraic numbers.*

In the case that $\mathfrak{k}$ is a function field of one variable over a finite field, $\gamma_Q(\mathfrak{k})$ was studied in [63].

4.2. Generalized Hermite constants of Sp_n . Let $n = 2m$ be an even integer. We consider a symplectic group

$$G(\mathbf{Q}) = Sp_n(\mathbf{Q}) = \left\{ g \in GL_{2m}(\mathbf{Q}) : {}^t g \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -I_m \\ I_m & 0 \end{pmatrix} \right\}.$$

For $1 \leq k \leq m$, Q_k denotes the maximal parabolic subgroup of G given as follows:

$$\begin{aligned} Q_k(\mathbf{Q}) &= U_k(\mathbf{Q})L_k(\mathbf{Q}), \\ L_k(\mathbf{Q}) &= \left\{ \delta(a, b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} : \begin{array}{l} a \in GL_k(\mathbf{Q}) \\ b = (b_{ij}) \in Sp_{2(m-k)}(\mathbf{Q}) \end{array} \right\}, \\ U_k(\mathbf{Q}) &= \left\{ \begin{pmatrix} I_k & * & * & * \\ 0 & I_{m-k} & * & 0 \\ 0 & 0 & I_k & 0 \\ 0 & 0 & * & I_{m-k} \end{pmatrix} \in G(\mathbf{Q}) \right\}. \end{aligned}$$

The module of $\mathbf{Q}$ -rational characters $\mathbf{X}_{\mathbf{Q}}^*(L_k)$ of L_k is a free $\mathbf{Z}$ -module of rank 1 and its base is given by

$$\widehat{\alpha}_{Q_k}(\delta(a, b)) = \det a.$$

We fix a good maximal compact subgroup K of $G(\mathbf{A})$ so that $G(\mathbf{A})$ has the Iwasawa decomposition $G(\mathbf{A}) = Q_k(\mathbf{A})K$. Since G and Q_k satisfy both conditons (C₁) and (C₂), one has

$$(4) \quad \gamma_{Q_k} = \max_{[g] \in G(\mathbf{Z}) \backslash G(\mathbf{R})/K_{\infty}} \min_{\gamma \in G(\mathbf{Z})} H_{Q_k}^{\infty}(\gamma g).$$

We restrict ourselves to the case $k = m$. An element of $L_m(\mathbf{A})$ is denoted by

$$\delta(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \quad (a \in GL_m(\mathbf{A})).$$

By definition, we have

$$H_{Q_m}^{\infty}(u\delta(a)h) = |\det a|^{-1}, \quad (u \in U_m(\mathbf{R}), \delta(a) \in L_m(\mathbf{R}), h \in K_{\infty}).$$

Let

$$\mathcal{H}_m = \{X \in M_m(\mathbf{C}) : \operatorname{Re} Z \in V_m, \operatorname{Im} Z \in P_m\}$$

be the Siegel upper half space. The group $G(\mathbf{R})$ acts on $\mathcal{H}_m$ by

$$g\langle Z \rangle = (aZ + b)(cZ + d)^{-1}, \quad (g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{R}), Z \in \mathcal{H}_m).$$

Since it is possible to choose the maximal compact subgroup K_{∞} as the stabilizer of $Z_0 = \sqrt{-1}I_m \in \mathcal{H}_m$ in $G(\mathbf{R})$, we have $\operatorname{Im}\{(u\delta(a)h)\langle Z_0 \rangle\} = a^t a$, and hence

$$H_{Q_m}^{\infty}(g) = (\det \operatorname{Im}\{g\langle Z_0 \rangle\})^{-1/2}$$

for any $g \in G(\mathbf{R})$. Combining this with (4), we get

$$\gamma_{Q_m} = \max_{[g] \in G(\mathbf{Z}) \backslash G(\mathbf{R})/K_{\infty}} \min_{\gamma \in G(\mathbf{Z})} (\det \operatorname{Im}\{\gamma g\langle Z_0 \rangle\})^{-1/2}.$$

Since $g\langle Z_0 \rangle$ runs over a fundamental domain of $G(\mathbf{Z}) \backslash \mathcal{H}_m$, we have

$$\gamma_{Q_m} = \frac{1}{\min_{[Z] \in G(\mathbf{Z}) \backslash \mathcal{H}_m} \max_{\gamma \in G(\mathbf{Z})} (\det \operatorname{Im}\{\gamma\langle Z \rangle\})^{1/2}}.$$

Note that

$$\det \operatorname{Im}\{\gamma\langle Z \rangle\} = |\det(cZ+d)|^{-2} \det \operatorname{Im}\{Z\}, \quad (\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{R}), Z \in \mathcal{H}_m).$$

Siegel's fundamental domain $\mathfrak{S}_m$ of $G(\mathbf{Z}) \backslash \mathcal{H}_m$ is given as follows:

$$\mathfrak{S}_m = \left\{ Z = X + \sqrt{-1}Y \in \mathcal{H}_m \mid \begin{array}{l} \bullet |\det(cZ+d)| \geq 1, \quad \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G(\mathbf{Z}) \\ \bullet Y \in \mathfrak{M}_m, \quad |x_{ij}| \leq 1/2 \end{array} \right\}$$

where $\mathfrak{M}_m$ is Minkowski's domain, i.e.,

$$\mathfrak{M}_m = \left\{ Y \in P_m \mid \begin{array}{l} \bullet {}^t x Y x \geq y_{ii} \quad (\forall x \in \mathbf{Z}^m, \gcd(x_1, \dots, x_m) = 1), i = 1, \dots, m \\ \bullet y_{i,i+1} > 0, \quad i = 1, \dots, m-1 \end{array} \right\}.$$

From

$$Z \in \mathfrak{S}_m \implies \max_{\gamma \in G(\mathbf{Z})} \det \operatorname{Im}\{\gamma\langle Z \rangle\} = \det \operatorname{Im}\{Z\},$$

it follows that

$$\gamma_{Q_m} = \frac{1}{\min_{Z \in \mathfrak{S}_m} (\det \operatorname{Im}\{Z\})^{1/2}},$$

namely

$$\min_{Z \in \mathfrak{S}_m} \det \operatorname{Im}\{Z\} = \frac{1}{(\gamma_{Q_m})^2}.$$

If $m = 1$, we have $\min_{Z \in \mathfrak{S}_1} \det \operatorname{Im}\{Z\} = \sqrt{3}/2$, and hence $\gamma_{Q_1} = \sqrt{\gamma_2} = \sqrt{2/\sqrt{3}}$. Recently, Kawamura [32] determined the actual value of $\min_{Z \in \mathfrak{S}_2} \det \operatorname{Im}\{Z\}$ by using Gottschling's description of $\mathfrak{S}_2$ ([26]).

Theorem 4.2 (Kawamura). $\min_{Z \in \mathfrak{S}_2} \det \operatorname{Im}\{Z\} = 2/3$, and hence $\gamma_{Q_2} = \sqrt{3/2}$. This minimum is attained only when $Z = Z_8$ or $-\overline{Z}_8$, where

$$Z_8 = \frac{1}{3} \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \sqrt{-1} \frac{\sqrt{2}}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The domain $\mathfrak{S}_2$ is described by 28 polynomials in 6 real variables. Hayata [29] computed 0-dimensional cells of the boundary $\partial \mathfrak{S}_2$ of $\mathfrak{S}_2$. There are at least 170 0-dimensional cells of $\partial \mathfrak{S}_2$. The points Z_8 and $-\overline{Z}_8$ are contained in Hayata's list.

For general $n = 2m$, we have the following bound by [62, Example 3]:

$$\min_{Z \in \mathfrak{S}_m} \det \operatorname{Im}\{Z\} \leq \left\{ \frac{1}{m+1} \cdot \frac{\prod_{j=1}^{\lfloor \frac{m-1}{2} \rfloor} \xi(2j+1)}{\prod_{j=1}^m \xi(m+j)} \right\}^{\frac{2}{m+1}},$$

where $\xi(s)$ denotes the zeta function $\pi^{-s/2} \Gamma(s/2) \zeta(s)$.

Problem 9. Give a good lower bound of $\min_{Z \in \mathfrak{S}_m} \det \operatorname{Im}\{Z\}$.

Problem 10. Formulate a Voronoï type theorem for the function $Z \mapsto \det \operatorname{Im} Z$ on $\mathfrak{S}_m$.

We finish this example by some observation. Let

$$R_m(\mathbf{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, d \in GL_m(\mathbf{Q}), b \in M_m(\mathbf{Q}) \right\}.$$

be a maximal parabolic subgroup of GL_n . We have $\gamma_{R_m} = \gamma_{n,m}$ as seen in Example 4.1. It is obvious that $Sp_n(\mathbf{A}) \subset GL_n(\mathbf{A})$, $Sp_n(\mathbf{A}) \cap R_m(\mathbf{A}) = Q_m(\mathbf{A})$ and $H_{R_m}(g) = H_{Q_m}(g)^2$ for $g \in Sp_n(\mathbf{R})$. Since $Q_m(\mathbf{Q}) \backslash Sp_n(\mathbf{Q}) \subset R_m(\mathbf{Q}) \backslash GL_n(\mathbf{Q})$, the inequality

$$F_{R_m}(g) \leq F_{Q_m}(g)^2 \leq (\gamma_{Q_m})^2$$

holds for any $g \in Sp_n(\mathbf{A})$. If the maximum of F_{R_m} is attained on a point in $Sp_n(\mathbf{A})$, $\gamma_{n,m}$ would be bounded by $(\gamma_{Q_m})^2$. This happens when $m = 1$ or 2 and in fact we have $\gamma_{2,1} = (\gamma_{Q_1})^2$ and $\gamma_{4,2} = (\gamma_{Q_2})^2$.

4.3. Variation of the set of minimal points. We return to the general setting of 4.1. We write Y_Q and X_Q for $Q(\mathbf{A})^1 \backslash G(\mathbf{A})^1$ and $Q(\mathbf{Q}) \backslash G(\mathbf{Q})$, respectively. For $g \in G(\mathbf{A})^1$, the set $S_Q(g)$ of minimal points in X_Q is defined as

$$S_Q(g) = \{[x] \in X_Q : H_Q(xg) = F_Q(g)\}.$$

By Northcott's theorem, $S_Q(g)$ is a finite set. We prove the following:

Proposition 4.3. *For $g \in G(\mathbf{A})^1$, there is a neighbourhood $\mathcal{U}$ of g in $G(\mathbf{A})^1$ such that $S_Q(g') \subset S_Q(g)$ for all $g' \in \mathcal{U}$.*

Proof. We define the operator norm $\|g\|_Q$ of $g \in G(\mathbf{A})^1$ by

$$\|g\|_Q = \sup_{[y] \in Y_Q} \frac{H_Q(yg)}{H_Q(y)} = \max_{h \in K} H_Q(hg).$$

It is easy to see the following properties:

- $\|g_1 g_2\|_Q \leq \|g_1\|_Q \|g_2\|_Q$ for all $g_1, g_2 \in G(\mathbf{A})^1$.
- $\|h_1 g h_2\|_Q = \|g\|_Q$ for all $g \in G(\mathbf{A})^1$ and $h_1, h_2 \in K$.
- $\|h\|_Q = 1$ for all $h \in K$.
- $g \mapsto \|g\|_Q$ is continuous on $G(\mathbf{A})^1$.

We fix a $g \in G(\mathbf{A})^1$ and put $C = \min_{[x] \in X_Q \backslash S_Q(g)} H_Q(xg)$. Then $F_Q(g) < C$ and we can take a constant δ so that $1 < \delta < C/F_Q(g)$. Since $g \mapsto F_Q(g)$ is continuous on $G(\mathbf{A})^1$, the set

$$\mathcal{U} = \left\{ u \in G(\mathbf{A})^1 : \|u^{-1}\|_Q < \frac{C}{\delta F_Q(g)}, \frac{F_Q(gu)}{F_Q(g)} < \delta \right\}$$

is a neighbourhood of the identity in $G(\mathbf{A})^1$. Let $u \in \mathcal{U}$ and $[x] \in S_Q(gu)$. We have

$$\|u^{-1}\|_Q^{-1} H_Q(xg) \leq H_Q(xgu) = F_Q(gu).$$

If $[x] \notin S_Q(g)$, then $C \leq H_Q(xg)$ and

$$\delta F_Q(g) \leq \delta F_Q(g) H_Q(xg) / C < \|u^{-1}\|_Q^{-1} H_Q(xg) \leq F_Q(gu) < \delta F_Q(g).$$

This is a contradiction. Therefore, we have $[x] \in S_Q(g)$, and hence $S_Q(gu) \subset S_Q(g)$ for any $u \in \mathcal{U}$. $\square$

This proposition is a generalization of Proposition 2.6. As a consequence, one can define the local maximality of $S_Q(g)$. This leads us to similar problems as in 2.3, e.g.,

Problem 11. If $g \in G(\mathbf{A})^1$ is an extreme point of F_Q , is $S_Q(g)$ locally maximal?

Problem 12. Is $\sharp S_Q(g)$ bounded by a constant independent of g ?

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