# A survey on Voronoï's theorem

*Dedicated to Professor Takayuki Oda on his 60th birthday*

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In the last half of 20th century, various generalizations of Hermite's constant and Voronoï's theorem were studied by many authors. In this paper, we give an account of a recent development concerning Voronoï's theorem.

*Keywords*: Hermite constant, eutactic form, extreme form, perfect form

Let $V_n$ be the vector space of real $n \times n$ symmetric matrices and $P_n$ the open cone of positive definite symmetric matrices in $V_n$. By $m_1(a)$, we denote the arithmetical minimum $\inf_{x \in \mathbf{Z}^n \setminus \{0\}} {}^t xax$ of $a \in P_n$. The Hermite invariant is the positive valued function $\gamma$ on $P_n$ defined by $\gamma(a) = m_1(a)/\det(a)^{1/n}$. Its maximum $\gamma_n$ is called Hermite's constant. The determination of $\gamma_n$ is one of main problems in lattice sphere packings or the arithmetic theory of quadratic forms. Voronoï's fundamental theorem [62] gives a characterization of local maxima of $\gamma$, i.e., which can be stated that $\gamma$ attains a local maximum on $a \in P_n$ if and only if $a$ is perfect and eutactic. In the last half of 20th century, various generalizations of Hermite's constant and Voronoï's theorem were studied by many authors. In this paper, we give an account of a recent development concerning Voronoï's theorem.

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**Notation.** Throughout this paper, $V_n$ denotes the vector space of real $n \times n$ symmetric matrices, $P_n$ the open cone of positive definite symmetric matrices in $V_n$ and $P_n^{\text{semi}}$ the closure of $P_n$ in $V_n$. The vector space $V_n$ is equipped with the inner product $\langle a_1, a_2 \rangle = \mathrm{tr}(a_1 a_2)$ for $a_1, a_2 \in V_n$. The unimodular group $GL_n(\mathbf{Z})$ acts on $V_n$ by $(a, g) \mapsto {}^t gag$ for $a \in V_n$ and $g \in GL_n(\mathbf{Z})$. In general, for a given ring $R$, the set of all $m \times n$ matrices with coefficients in $R$ is denoted by $M_{m,n}(R)$. We write $M_n(R)$ for $M_{n,n}(R)$ and $R^n$ for $M_{n,1}(R)$. The unit group of the matrix ring $M_n(R)$ is denoted by $GL_n(R)$. The identity matrix in $GL_n(R)$ is denoted by $\mathrm{I}_n$.

A Euclidean space $\mathbf{R}^n$ is equipped with the inner product $(x, y) = {}^t xy$. For $a \in M_n(\mathbf{R})$, $\|a\|$ denotes the operator norm of $a$, i.e.,

$$\|a\| = \sup_{x \in \mathbf{R}^n \setminus \{0\}} \left( \frac{(ax, ax)}{(x, x)} \right)^{1/2}.$$

For a constant $c \in \mathbf{R}$, $\mathbf{R}_{>c}$ and $\mathbf{R}_{\geq c}$ stand for the open interval $(c, +\infty)$ and the closed interval $[c, +\infty)$, respectively.

## 1. Type one functions and Voronoï's theorem

There are several methods to prove Voronoï's theorem [62, Théorème 17], e.g., [6], [51], [54], see also [28, §29], [29, §39], [40, §3.4] and [60, §3.1.7].

Convexity of the domain $P_n$ and the concavity of both functions $m_1$ and $\det^{1/n}$ play key roles in some proofs. Poor and Yuen [47] investigated a family of such kind functions as $m_1$ and $\det^{1/n}$. This family is called type one functions. Every type one function $\phi$ is completely characterized by the corresponding semikernel $K_1(\phi)$. In this section, we first discuss type one functions and semikernels, and then formulate Voronoï type theorem in terms of type one functions. The semikernel $K_1(m_1)$ associated with $m_1$ is called the Ryshkov polyhedron. In the second half of this section, we investigate a description of faces of $K_1(m_1)$. This gives a well-known geometric interpretation of perfection.

## 1.1. *Type one functions and semikernels*

**Definition 1.1.** A function $\phi : P_n^{\mathrm{semi}} \to \mathbf{R}_{\geq 0}$ is called a type one function if $\phi$ satisfies the following conditions:
(TO$_1$) $\phi(\theta a) = \theta\phi(a)$ for all $a \in P_n^{\mathrm{semi}}$ and $\theta \geq 0$,
(TO$_2$) $\phi(a_1 + a_2) \geq \phi(a_1) + \phi(a_2)$ for all $a_1, a_2 \in P_n^{\mathrm{semi}}$,
(TO$_3$) $\phi(a) > 0$ for all $a \in P_n$.
A type one function $\phi$ is called a type one class function if $\phi({}^t gag) = \phi(a)$ holds for all $a \in P_n^{\mathrm{semi}}$ and $g \in GL_n(\mathbf{Z})$.

**Example 1.1.** The trace tr and the smallest eigenvalue $\lambda_1$ are type one functions, but not type one class functions. The reduced determinant $\det^{1/n}$ and the arithmetical minimum

$$m_1(a) = \inf_{x \in \mathbf{Z} \setminus \{0\}} {}^t xax$$

are type one class functions.

For a type one function $\phi$, the dual type one function $\phi^\circ : P_n^{\mathrm{semi}} \to \mathbf{R}_{\geq 0}$ is defined to be

$$\phi^\circ(a) = \inf_{b \in P_n} \frac{\langle a, b \rangle}{\phi(b)}.$$

If $\phi$ is a type one class function, then so is $\phi^\circ$. The dual type one class function of $m_1$ is denoted by $w_1$, which is called the dyadic trace. The dual type one class function of $\det^{1/n}$ is $n \det^{1/n}$.

Any type one function is continuous on $P_n$, but not necessarily continuous on $P_n^{\mathrm{semi}}$. For example, $w_1$ is not continuous on $P_n^{\mathrm{semi}}$; however $w_1$ is upper semicontinuous on $P_n^{\mathrm{semi}}$. Here a type one function $\phi$ is said to be upper semicontinuous at $a \in P_n^{\mathrm{semi}}$ if

$$\phi(a) = \limsup_{b \to a} \phi(b) = \lim_{\epsilon \downarrow 0} (\sup\{\phi(b) \ : \ \|a - b\| \leq \epsilon, \ b \in P_n^{\mathrm{semi}}\}) \,.$$

In general, the dual $\phi^\circ$ of an arbitrary type one function $\phi$ is necessarily upper semicontinuous on $P_n^{\rm semi}$ ( [58, Corollary 2.7]).

**Definition 1.2.** Let $K$ be a convex subset of $P_n^{\rm semi}$ such that $0 \notin K$, $\mathbf{R}_{\geq 1} \cdot K = K$ and $\mathbf{R}_{>0} \cdot K \supset P_n$.
(1) $K$ is called a kernel if $K$ is closed in $P_n^{\rm semi}$.
(2) $K$ is called a semikernel if the following three conditions are satisfied:
(SK$_1$) $K \cap (P_n \cup \{0\})$ is closed in $P_n \cup \{0\}$,
(SK$_2$) $\{ \theta \geq 0 \mid \theta a \in K \}$ is closed in $[0, \infty)$ for any $a \in K$,
(SK$_3$) $a + b \subset K$ for all $a \in K$ and $b \in P_n^{\rm semi}$.

It is easy to see that a kernel is a semikernel. The dual $K^\sqcup$ of a semikernel $K$ is defined to be

$$K^\sqcup = \{ a \in V_n \ : \ \langle a, b \rangle \geq 1 \ \text{ for all } b \in K \}.$$

This $K^\sqcup$ is a kernel.

There is a natural correspondence between type one functions and semikernels. For a type one function $\phi$, we set

$$K_1(\phi) = \{ a \in P_n^{\rm semi} \ : \ \phi(a) \geq 1 \}.$$

Conversely, for a semikernel $K$, define the function $\psi(K, \cdot) : P_n^{\rm semi} \to \mathbf{R}_{\geq 0}$ by

$$\psi(K, a) = \max\left(\{ \theta > 0 \ : \ a \in \theta \cdot K \} \cup \{0\}\right).$$

The existence of this maximum follows from the condition (SK$_2$). The following results were proved in [58, §1]

**Proposition 1.1.** *The correspondence $\phi \mapsto K_1(\phi)$ gives a bijection between the set of type one functions (resp. upper semicontinuous type one functions) and the set of semikernels (resp. kernels). For any type one function $\phi$ and any semikernel $K$, one has*

$$\psi(K_1(\phi), \cdot) = \phi\,, \qquad K_1(\psi(K, \cdot)) = K$$

*and moreover*

$$\psi(K, \cdot)^\circ = \psi(K^\sqcup, \cdot).$$

**Proposition 1.2.** *For any type one function $\phi$, we have*

$$\begin{cases} \phi^{\circ\circ}(a) = \phi(a) & \text{if } a \in P_n \\ \phi^{\circ\circ}(a) \geq \phi(a) & \text{if } a \in P_n^{\rm semi} \setminus P_n. \end{cases}$$

*If $\phi$ is upper semicontinuous on $P_n^{\rm semi}$, then $\phi^{\circ\circ} = \phi$ on $P_n^{\rm semi}$.*

### 1.2. *Voronoï's theorem of $m_1/\phi$*

Voronoï's theorem characterizes local maxima of the Hermite invariant $F_{\det^{1/n}} = m_1/\det^{1/n}$. A point $a \in P_n$ is said to be extreme (resp. strict extreme) if $F_{\det^{1/n}}$ attains a local maximum (resp. a strict local maximum) on $a$ up to the multiplication by an element of $\mathbf{R}_{>0}$. Indeed, we do not need to distinguish between extreme points and strictly extreme points since any extreme point is strictly extreme ( [40, Theorem 3.4.5]). For $a \in P_n$, $S(a)$ denotes the set of minimal integral vectors of $a$, i.e.,

$$S(a) = \{\, x \in \mathbf{Z}^n \setminus \{0\} \ : \ {}^t\!xax = m_1(a)\,\}.$$

For any $y \in \mathbf{R}^n$, $\varphi_y$ denotes the linear form $v \mapsto {}^t\!yvy$ on $V_n$.

**Definition 1.3.** Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t\!bb$. An element $a$ is said to be perfect if the linear forms $\varphi_{bx}$ ($x \in S(a)$) span the dual space $V_n^*$ of $V_n$. An element $a$ is said to be eutactic (resp. weakly eutactic) if there exist $\rho_x \in \mathbf{R}_{>0}$ (resp. $\rho_x \in \mathbf{R}$), $x \in S(a)$, such that

$$\mathrm{tr} = \sum_{x \in S(a)} \rho_x \varphi_{bx}. \tag{1}$$

We note that these definitions of perfection, eutaxy and weakly eutaxy are independent of a choice of $b$. It follows from definition that $a$ is perfect if and only if $\{x^tx \ : \ x \in S(a)\}$ spans $V_n$. If tr is represented as (1), then we have

$$\mathrm{tr} = \sum_{x \in S(a)} \rho_x \varphi_{hbx}$$

for any orthogonal matrix $h$. The coefficients $\rho_x$ are independent of $h$.

Any perfect element $a$ is uniquely determined by $m_1(a)$ and $S(a)$, i.e., $a$ is a unique solution of the system of linear equations in the unknown $v = {}^tv$: $\langle v, x^tx\rangle = m_1(a)$, $x \in S(a)$. If $m_1(a) \in \mathbf{Q}$, then its solution is contained in $V_n \cap M_n(\mathbf{Q})$ by Cramer's formula. This is none other than the rationality of perfect elements ( [36, p.252, 5°]).

**Theorem 1.1 (Korkine–Zorotareff).** *If $a \in P_n$ is perfect and $m_1(a) \in \mathbf{Q}$, then $a \in P_n \cap M_n(\mathbf{Q})$.*

Voronoï's theorem [62, Théorème 17] is stated as follows.

**Theorem 1.2 (Voronoï).** *$a \in P_n$ is extreme if and only if $a$ is perfect and eutactic.*

We fix a type one function $\phi$. It is natural to ask whether the same kind of Voronoï's theorem holds for the function $F_\phi = m_1/\phi$ on $P_n$. An element $a \in P_n$ is said to be $\phi$-extreme (resp. strictly $\phi$-extreme) if $F_\phi$ attains a local maximum (resp. a strictly local maximum) on $a$ up to the multiplication by an element of $\mathbf{R}_{>0}$. Assume $\phi$ is differentiable on $P_n$. Then

$$(\partial \log \phi)_b(v) = \lim_{t\to 0} \frac{\log \phi({}^t b(\mathrm{I}_n + tv)b) - \log \phi({}^t bb)}{t}$$

exists for $b \in GL_n(\mathbb{R})$ and $v \in V_n$. We define $\phi$-eutaxy as follows:

**Definition 1.4.** Let $a \in P_n$, and fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t bb$. An element $a$ is said to be $\phi$-eutactic if there exist $\rho_x > 0$ $(x \in S(a))$ such that $(\partial \log \phi)_b = \sum_{x\in S(a)} \rho_x \varphi_{bx}$.

In a similar fashion as eutaxy, this definition is independent of a choice of $b$. If $\phi = \det^{1/n}$, then $(\partial \log \phi)_b = \mathrm{tr}$, and hence $\det^{1/n}$-eutaxy is the same as Definition 1.3.

It follows from $(\mathrm{TO}_1)$ and $(\mathrm{TO}_2)$ that $\phi$ is log-concave, i.e,

$$\log \phi((1-\theta)a_1 + \theta a_2)) \geq (1-\theta)\log\phi(a_1) + \theta \log \phi(a_2)$$

holds for all $a_1, a_2 \in P_n$ and $0 < \theta < 1$. We say $\phi$ is strictly log-concave if this inequality is strict for $a_1 \neq a_2$.

In [58, §2], Voronoï's theorem is generalized as follows.

**Theorem 1.3.** *Let $\phi$ be a strictly log-concave and differentiable type one function. Then, $a \in P_n$ is $\phi$-extreme if and only if $a$ is perfect and $\phi$-eutactic. Moreover, any $\phi$-extreme point is strictly $\phi$-extreme.*

The line of the proof of Theorem 1.3 is the same as Barnes' [6] and Martinet's [40, §3.4] proof of Voronoï's theorem. We give an outline of the proof. We use the following two lemmas: the first is the same as [40, Lemmas 3.4.2 and 3.4.3] and the second is a generalization of [40, Lemma 3.4.4].

**Lemma 1.1.** *Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t bb$.*
*(1) There exists a neighborhood $\mathcal{U}$ of $\mathrm{I}_n$ in $GL_n(\mathbb{R})$ such that $S({}^t b {}^t uub) \subset S(a)$ for any $u \in \mathcal{U}$.*
*(2) There exists a neighborhood $\mathcal{V}$ of $0$ in $V_n$ such that*

$$m_1({}^t b(\mathrm{I}_n + v)b) = m_1(a) \iff \min_{x\in S(a)} \varphi_{bx}(v) = 0$$

*for any $v \in \mathcal{V}$.*

**Lemma 1.2.** *Let $\phi$ be a strictly log-concave and differentiable type one function. Let $a \in P_n$, and fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t\!bb$.*
*(1) There exists a neighborhood $\mathcal{V} \subset V_n$ of 0 such that either $v = 0$ or $\phi({}^t\!b(\mathrm{I}_n + v)b) < \phi(a)$ holds for any $v \in \mathcal{V}$ with $(\partial \log \phi)_b(v) \leq 0$ and $\mathrm{I}_n + v \in P_n$.*
*(2) Let $\mathcal{C}$ be a closed cone in $V_n$ such that $(\partial \log \phi)_b(v) > 0$ for all $v \in \mathcal{C} \backslash \{0\}$. Then there exists $\alpha > 0$ such that $\phi({}^t\!b(\mathrm{I}_n + v)b) > \phi(a)$ holds for any $v \in \mathcal{C}$ with $0 < \|v\| < \alpha$.*

We set $\mathcal{D}_a = \{v \in V_n \ : \ \min_{x \in S(a)} \varphi_{bx}(v) \geq 0 \ \text{ and } \ (\partial \log \phi)_b(v) \leq 0\}$. By these lemmas, we obtain the following generalization of Korkine and Zolotareff's equivalent condition (cf. [40, Theorem 3.4.5]).

**Lemma 1.3.** *Let $\phi$ be a strictly log-concave and differentiable type one function. Then $a \in P_n$ is $\phi$-extreme if and only if $\mathcal{D}_a = \{0\}$. Any $\phi$-extreme point is strictly $\phi$-extreme.*

Lemma 1.3 leads us to Theorem 1.3 as follows: Let $a \in P_n$ be perfect and $\phi$-eutactic. Fix an element $b \in GL_n(\mathbb{R})$ such that $a = {}^t\!bb$. For $v \in \mathcal{D}_a$, $\phi$-eutaxy concludes $\varphi_{bx}(v) = 0$ for all $x \in S(a)$, and then $v = 0$ by perfection. Thus $\mathcal{D}_a = \{0\}$ and $a$ is $\phi$-extreme. Conversely, let $a$ be $\phi$-extreme. If $\varphi_{bx}(v) = 0$ for all $x \in S(a)$, then either $v$ or $-v$ is contained in $\mathcal{D}_a$. Since $\mathcal{D}_a = \{0\}$, we have $v = 0$. This implies that $a$ is perfect. The linear forms $-(\partial \log \phi)_b$ and $\varphi_{bx}$, $x \in S(a)$, satisfy

$$\begin{aligned} &\{v \in V_n \ : \ \min_{x \in S(a)} \varphi_{bx}(v) \geq 0 \ \text{ and } \ -(\partial \log \phi)_b(v) \geq 0\} \\ &= \bigcap_{x \in S(a)} \mathrm{Ker}(\varphi_{bx}) \cap \mathrm{Ker}(-(\partial \log \phi)_b) = \{0\}\,. \end{aligned}$$

Then, by Stiemke's theorem, $a$ must be $\phi$-eutactic. Here Stiemke's theorem asserts that, for a family of linear forms $\varphi_1, \cdots, \varphi_r$ on $\mathbf{R}^N$, there exists $\rho_1, \cdots, \rho_r \in \mathbf{R}_{>0}$ such that $\rho_1\varphi_1 + \cdots + \rho_r\varphi_r = 0$ if and only if $\{v \in \mathbf{R}^N : \min_{1 \leq i \leq r} \varphi_i(v) \geq 0\} = \mathrm{Ker}(\varphi_1) \cap \cdots \cap \mathrm{Ker}(\varphi_r)$.

### 1.3. *Geometric characterizations of perfect forms*

The kernel $K_1(m_1)$ is called the Ryshkov polyhedron. Ryshkov [54], [56, Chapter III] closely investigated polyhedral geometric structure of $K_1(m_1)$ and its dual $K_1(m_1)^{\sqcup}$. Since $m_1$ equals zero on the boundary of $P_n^{\mathrm{semi}}$, the Ryshkov polyhedron $K_1(m_1)$ is contained in $P_n$. For an integral vector $x \in \mathbf{Z}^n \setminus \{0\}$ and a constant $\lambda \in \mathbf{R}$, $H^+_{x,\lambda}$ denotes the affine half-space

$\{a \in V_n \ : \ \langle a, x^t x\rangle \geq \lambda\}$ in $V_n$. Then $K_1(m_1)$ is the intersection of affine half-spaces $H^+_{x,1}$, ($x \in \mathbf{Z}^n \setminus \{0\}$). It is known that $K_1(m_1)$ is a locally finite polyhedron, i.e., the intersection of $K_1(m_1)$ and an arbitrary polytope is a polytope, (see e.g., [28, Proposition 29.5], [60, Theorem 3.1]). In particular, $K_1(m_1) \cap \{a \in V_n \ : \ \mathrm{tr}(a) \leq \lambda\}$ is a polytope for any sufficiently large constant $\lambda > 0$. We denote by $\partial K_1(m_1)$ the boundary of $K_1(m_1)$. In what follows, we give a description of faces of $K_1(m_1)$.

**Lemma 1.4.** *Let $a_1, \cdots, a_r \in \partial K_1(m_1)$ and $S$ be a non-empty finite subset of $\mathbf{Z}^n \setminus \{0\}$ such that $S \subset S(a_i)$ for $i = 1, \cdots, r$. Then, for any $\lambda_1, \cdots, \lambda_r \in \mathbf{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_r = 1$, one has $\lambda_1 a_1 + \cdots + \lambda_r a_r \in \partial K_1(m_1)$ and $S \subset S(\lambda_1 a_1 + \cdots + \lambda_r a_r)$.*

**Proof.** Since $K_1(m_1)$ is convex, $\lambda_1 a_1 + \cdots + \lambda_r a_r$ is contained in $K_1(m_1)$. If $x \in S$, then

$$\langle \sum_{i=1}^r \lambda_i a_i, x^t x\rangle = \sum_{i=1}^r \lambda_i m_1(a_i) = \sum_{i=1}^r \lambda_i = 1\,.$$

This means $m_1(\lambda_1 a_1 + \cdots + \lambda_r a_r) = 1$ and $S \subset S(\lambda_1 a_1 + \cdots + \lambda_r a_r)$. □

For a non-empty finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$, define the subset $\mathcal{F}_S$ of $\partial K_1(m_1)$ as

$$\mathcal{F}_S = \{a \in \partial K_1(m_1) \ : \ S \subset S(a)\}\,.$$

We denote by $\mathcal{H}_S$ the affine subspace of $V_n$ generated by $\mathcal{F}_S$, i.e.,

$$\mathcal{H}_S = \{\lambda_1 a_1 + \cdots + \lambda_r a_r \ : \ 1 \leq r \in \mathbf{Z},\ a_i \in \mathcal{F}_S,\ \lambda_i \in \mathbf{R},\ \lambda_1 + \cdots + \lambda_r = 1\}$$

if $\mathcal{F}_S \neq \emptyset$, or $\mathcal{H}_S = \emptyset$ if $\mathcal{F}_S = \emptyset$. Since $S$ is non-empty, $\mathcal{H}_S$ is a proper affine subspace of $V_n$.

**Lemma 1.5.** *One has $\mathcal{F}_S = \partial K_1(m_1) \cap \mathcal{H}_S$. In particular, $\mathcal{F}_S$ is a face of $K_1(m_1)$ if $\mathcal{F}_S \neq \emptyset$.*

**Proof.** We assume $\mathcal{F}_S \neq \emptyset$ and fix an $a_0 \in \mathcal{F}_S$. Let $r = \dim \mathcal{H}_S$. There exist $r$ elements $a_1, \cdots, a_r \in \mathcal{F}_S$ such that $\{a_1 - a_0, \cdots, a_r - a_0\}$ is a basis of the subspace $\{a - a_0 \ : \ a \in \mathcal{H}_S\}$. Any element $b \in \partial K_1(m_1) \cap \mathcal{H}_S$ is represented as

$$b = a_0 + \lambda_1(a_1 - a_0) + \cdots + \lambda_r(a_r - a_0), \qquad \lambda_1, \cdots, \lambda_r \in \mathbf{R}\,.$$

Since $S \subset S(a_i)$ for $i = 0, 1, \cdots, r$, we have $\langle a_i - a_0, x^t x\rangle = 0$ for all $x \in S$, and hence $\langle b, x^t x\rangle = \langle a_0, x^t x\rangle = 1$ for all $x \in S$. This means $S \subset S(b)$. Therefore, $\partial K_1(m_1) \cap \mathcal{H}_S$ is a subset of $\mathcal{F}_S$. □

**Lemma 1.6.** *Any face of $K_1(m_1)$ is of the form $\mathcal{F}_S$ for some non-empty finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$.*

**Proof.** Let $\mathcal{F}$ be a face of $K_1(m_1)$ of dimension $r$. First, we assume $\mathcal{F}$ is a facet, i.e., $r = \dim V_n - 1$. There exist $r+1$ elements $a_0, a_1, \cdots, a_r \in \mathcal{F}$ such that $a_1 - a_0, \cdots, a_r - a_0$ are linearly independent. We fix constants $\lambda_0, \lambda_1, \cdots, \lambda_r$ such that $\lambda_0 + \lambda_1 + \cdots + \lambda_r = 1$ and $0 < \lambda_i < 1$ for $i = 0, 1, \cdots, r$. Since $\mathcal{F}$ is convex, the element $a = \lambda_0 a_0 + \lambda_1 a_1 + \cdots + \lambda_r a_r$ is also contained in $\mathcal{F}$. For any $x \in S(a)$, we have

$$1 = m_1(a) = \langle a, x^t x \rangle = \sum_{i=0}^{r} \lambda_i \langle a_i, x^t x \rangle \geq \sum_{i=0}^{r} \lambda_i = 1\,.$$

Thus, $\langle a_i, x^t x \rangle$ equals $m_1(a_i) = 1$ for all $i = 0, 1, \cdots, r$. This implies $S(a) \subset S(a_i)$ for all $i = 0, 1, \cdots, r$. Let $S$ be the intersection of $S(a_i)$, $i = 0, 1, \cdots, r$. Since $S \subset S(a)$ is obvious, one has $S = S(a)$. By definition, the face $\mathcal{F}_S$ contains $a_0, a_1, \cdots, a_r$. Therefore, $\partial K_1(m_1) \cap \mathcal{H}_S = \mathcal{F}_S$ contains $\mathcal{F}$. Since $\mathcal{H}_S$ is a proper affine subspace and $\mathcal{F}$ is a facet, we obtain $\mathcal{F}_S = \mathcal{F}$.

In general case, $\mathcal{F}$ is an intersection of finite number of facets, say $\mathcal{F}_{S_1}, \cdots, \mathcal{F}_{S_k}$. By definition, we have

$$\mathcal{F} = \bigcap_{i=1}^{k} \mathcal{F}_{S_i} = \mathcal{F}_{S_1 \cup \cdots \cup S_k}\,.$$

□

We denote by $\partial^0 K_1(m_1)$ the set of all vertices (= 0 dimensional faces) of $K_1(m_1)$. The next theorem is well-known.

**Theorem 1.4.** *For $a \in \partial K_1(m_1)$, the following three conditions are equivalent each other.*
(1) *$a$ is perfect.*
(2) *$a \in \partial^0 K_1(m_1)$.*
(3) *There exists a neibourhood $\mathcal{O}$ of $a$ in $P_n$ such that $S(b) \subsetneqq S(a)$ for any $b \in \mathcal{O} \setminus \mathbf{R}_{>0} a$.*

**Proof.** First we show the contraposition of (1) $\Longrightarrow$ (2). Let $a \in \partial K_1(m_1) \setminus \partial^0 K_1(m_1)$. Then, there exist $a_1, a_2 \in \partial K_1(m_1)$ and $0 < \lambda_0 < 1$ such that $a = \lambda_0 a_1 + (1 - \lambda_0) a_2$. Both $S(a_1)$ and $S(a_2)$ contain $S(a)$. Assume $a$ is perfect. By Lemma 1.4, $S(\lambda a_1 + (1-\lambda) a_2)$ also contains $S(a)$ for all positive $\lambda < 1$. Therefore, $\lambda a_1 + (1-\lambda) a_2$ is perfect. This contradicts Theorem 1.1.

Next we show (2) $\Longrightarrow$ (3). Let $a \in \partial^0 K_1(m_1)$. By Lemma 1.6, there exists a finite subset $S \subset \mathbf{Z}^n \setminus \{0\}$ such that $\{a\} = \mathcal{F}_S$. By Lemma 1.1, there is a neighbourhood $\mathcal{O}$ of $a$ in $P_n$ such that $S(b) \subset S(a)$ for all $b \in \mathcal{O}$. If $b \in \partial K_1(m_1) \cap \mathcal{O}$ and $S(b) = S(a)$, then we have $b \in \mathcal{F}_S$, and hence $b = a$. This $\mathcal{O}$ satisfies (3).

We show the contraposition of (3) $\Longrightarrow$ (1). Let $a \in \partial K_1(m_1)$ be a non-perfect point. Then, there exists $c \in V_n \setminus \{0\}$ such that $\langle c, x^t x\rangle = 0$ for all $x \in S(a)$. If $\epsilon > 0$ is sufficiently small, then $a + \epsilon c$ is contained in $P_n$ and $S(a + \epsilon c)$ is a subset of $S(a)$. Since $\langle a + \epsilon c, x^t x\rangle = 1$ for all $x \in S(a)$, we have $m_1(a + \epsilon c) = 1$ and $S(a + \epsilon c) = S(a)$. This means that $a$ does not satisfy the condition (3). $\square$

**Corollary 1.1.** *The set of all perfect elements of $P_n$ coincides with $\mathbf{R}_{>0} \cdot \partial^0 K_1(m_1)$.*

In the rest of this section, we show that $K_1(m_1)$ is the convex hull of $\partial^0 K_1(m_1)$ in $V_n$. For $a \in \partial^0 K_1(m_1)$, we set

$$\mathcal{C}_a = \{b \in V_n \ : \ \langle b, x^t x\rangle \geq 0 \ \text{ for all } x \in S(a)\} = \bigcup_{x \in S(a)} H^+_{x^t x, 0} \, ,$$

which is a polyhedral cone in $V_n$ of finite faces. For a non-zero $b \in \mathcal{C}_a$, the ray $\mathbf{R}_{\geq 0} \cdot b$ is called an extreme ray of $\mathcal{C}_a$ if for any $b_1, b_2 \in \mathcal{C}_a$, whenever $b = (b_1 + b_2)/2$, we must have $b_1, b_2 \in \mathbf{R}_{\geq 0} \cdot b$.

**Lemma 1.7.** *Let $a \in \partial^0 K_1(m_1)$. If $\mathbf{R}_{\geq 0} \cdot b$ is an extreme ray of $\mathcal{C}_a$, then $b \notin P_n^{\text{semi}}$.*

**Proof.** We prove that $b \in P_n^{\text{semi}}$ leads us to a contradiction. Since $a$ is perfect, the set $\{x^t x \ : \ x \in S(a)\}$ spans $V_n$. We set

$$S' = \{x \in S(a) \ : \ \langle b, x^t x\rangle = 0\}$$

and

$$W = \{c \in V_n \ : \ \langle c, x^t x\rangle = 0 \ \text{ for all } x \in S'\} \, .$$

Since $b \neq 0$, $S'$ is non-empty and $W$ is a subspace of $V_n$ containing the line $\mathbf{R} \cdot b$.

First we assume $\dim W \geq 2$. There is a $c \in W$ such that $b$ and $c$ are linearly independent. If we assume $b \in P_n^{\text{semi}}$, then we have $\langle b, x^t x\rangle > 0$ for all $x \in S(a) \setminus S'$. Thus, for sufficiently small $\lambda > 0$, we have $\langle b \pm \lambda c, x^t x\rangle > 0$ for all $x \in S(a) \setminus S'$. From $\langle b \pm \lambda c, x^t x\rangle = 0$ for all $x \in S'$, it follows $b \pm \lambda c \in \mathcal{C}_a$. Then one has

$$b = \frac{1}{2}(b + \lambda c) + \frac{1}{2}(b - \lambda c)$$

and $b \pm \lambda c \notin \mathbf{R}_{\geq 0} \cdot b$. This is a contradiction.

Next we assume $\dim W = 1$, i.e., $W = \mathbf{R} \cdot b$. Let $N = \dim V_n$. Since the subspace spanned by $\{x^t x \ : \ x \in S'\}$ is the orthogonal complement of $W$, there are $N-1$ linearly independent vectors $x_1{}^t x_1, \cdots, x_{N-1}{}^t x_{N-1}$ in $\{x^t x \ : \ x \in S'\}$. By the perfection of $a$, there exists $x_N \in S(a) \setminus S'$ such that $x_1{}^t x_1, \cdots, x_{N-1}{}^t x_{N-1}, x_N{}^t x_N$ are linearly independent. If we assume $b \in P_n^{\mathrm{semi}}$, then there is a square root $\sqrt{b} \in P_n^{\mathrm{semi}}$ such that $(\sqrt{b})^2 = b$. For each $i = 1, \cdots, N-1$, one has

$$0 = \langle b, x_i{}^t x_i \rangle = {}^t(\sqrt{b} x_i)(\sqrt{b} x_i) \,,$$

i.e., $x_1, \cdots, x_{N-1}$ are contained in the nullspace of $\sqrt{b}$. Thus there is a non-zero $y \in \mathbf{R}^n$ such that ${}^t y x_i = 0$ for $i = 1, \cdots, N-1$. We choose a non-zero $z \in \mathbf{R}^n$ which is orthogonal to $x_N$. Then the non-zero symmetric matrix $(y^t z + z^t y)/2 \in V_n$ is orthogonal to $x_1{}^t x_1, \cdots, x_N{}^t x_N$. This contradicts that $x_1{}^t x_1, \cdots, x_N{}^t x_N$ spans $V_n$. $\square$

**Proposition 1.3.** *Let $L$ be an edge (= one dimensional face) of $K_1(m_1)$. Then there are $a_1, a_2 \in \partial^0 K_1(m_1)$ such that $L = \{\lambda a_1 + (1-\lambda) a_2 \ : \ 0 \leq \lambda \leq 1\}$.*

**Proof.** For a sufficiently large $\theta > 0$, we set

$$K_1(m_1)_\theta = K_1(m_1) \cap \{a \in V_n \ : \ \langle a, \mathrm{I}_n \rangle \leq \theta\} \quad \text{and} \quad L_\theta = L \cap K_1(m_1)_\theta \,.$$

Since $L_\theta$ is an edge of the polytope $K_1(m_1)_\theta$, there are vertices $a_1, a_1'$ of $K_1(m_1)_\theta$ such that $L_\theta$ is the line joining $a_1$ and $a_1'$. Since $L_\theta$ is not contained in the affine hyperplane $\{a \in V_n \ : \ \langle a, \mathrm{I}_n \rangle = \theta\}$, at least one of $a_1$ and $a_1'$ must be a vertex of $K_1(m_1)$. Let $a_1 \in \partial^0 K_1(m_1)$ and $b \in V_n$ be a direction of $L$. Thus, any point of $L$ is of the form $a_1 + \lambda b$ for some $\lambda \geq 0$.

We show $\mathbf{R}_{\geq 0} \cdot b$ is an extreme ray of $\mathcal{C}_{a_1}$. There is an open interval $(0, \lambda_0)$ such that $a_1 + \lambda b \in L$ for all $\lambda \in (0, \lambda_0)$. Since $a_1 + \lambda b \in \partial K_1(m_1)$ for $\lambda \in (0, \lambda_0)$, we have $m_1(a_1 + \lambda b) = 1$ and

$$1 \leq \langle a_1 + \lambda b, x^t x \rangle = 1 + \lambda \langle b, x^t x \rangle$$

for all $x \in S(a_1)$. This means $b \in \mathcal{C}_{a_1}$. If $\mathbf{R}_{\geq 0} \cdot b$ is not an extreme ray of $\mathcal{C}_{a_1}$, then there are $b_1, b_2 \in \mathcal{C}_{a_1} \setminus \mathbf{R}_{\geq 0} \cdot b$ such that $b = (b_1 + b_2)/2$. For $i = 1, 2$ and a sufficiently small $\lambda > 0$, we have $a_1 + \lambda b_i \in P_n$ and $S(a_1 + \lambda b_i) \subset S(a_1)$. From $b_i \in \mathcal{C}_{a_1}$, it follows that for $x \in S(a_1 + \lambda b_i)$,

$$m_1(a_1 + \lambda b_i) = \langle a_1 + \lambda b_i, x^t x \rangle = 1 + \lambda \langle b_i, x^t x \rangle \geq 1 \,.$$

Namely, both $a_1+\lambda b_1$ and $a_1+\lambda b_2$ are contained in $K_1(m_1)\setminus L$ and $a_1+\lambda b$ is the middle point of $a_1+\lambda b_1$ and $a_1+\lambda b_2$. This is impossible since $L$ is an edge of $K_1(m_1)$. Therefore $\mathbf{R}_{\geq 0}\cdot b$ must be an extreme ray of $\mathcal{C}_a$.

Since $b \notin P_n^{\mathrm{semi}}$ by Lemma 1.7, the value $\lambda_1 = \sup\{\lambda \geq 0 \ : \ a_0+\lambda b \in K_1(m_1)\}$ is finite. Thus $L$ is written as $L = \{a_1+\lambda b \ : \ 0 \leq \lambda \leq \lambda_1\}$. Finally we show the point $a_2 = a_1+\lambda_1 b$ is a vertex of $K_1(m_1)$. If $a_2 \notin \partial^0 K_1(m_1)$, then there are $c_1, c_2 \in \partial K_1(m_1)$ such that $a_2 = (c_1+c_2)/2$ and $\lambda c_1 + (1-\lambda)c_2 \in \partial K_1(m_1)$ for $0 \leq \lambda \leq 1$. In this case, the triangle of vertices $a_1, c_1$ and $c_2$ is contained in $\partial K_1(m_1)$. This contradicts that $L$ is an edge of $K_1(m_1)$. □

**Corollary 1.2.** *The Ryshkov domain $K_1(m_1)$ is the convex hull of $\partial^0 K_1(m_1)$.*

**Proof.** We fix an arbitrary $a \in K_1(m_1)$. If $\theta > \mathrm{tr}(a)$, then $a \in K_1(m_1)_\theta$. Let $\{b_1, \cdots, b_r\}$ be the set of all vertices of the polytope $K_1(m_1)_\theta$. Since $K_1(m_1)_\theta$ is the convex hull of $\{b_1,\cdots,b_r\}$, $a$ is of the form $\lambda_1 b_1+\cdots+\lambda_r b_r$ with $\lambda_1+\cdots+\lambda_r = 1$ and $\lambda_i \geq 0$, $i = 1,\cdots,r$. Each $b_i$ is either a vertex of $K_1(m_1)$ or the intersection of an edge of $K_1(m_1)$ and the affine hyperplane $\{c \in V_n \ : \ \mathrm{tr}(c) = \theta\}$. In any case $b_i$ is a point on an edge of $K_1(m_1)$. By Proposition 1.3, all $b_i$ are contained in the convex hull of $\partial^0 K_1(m_1)$, and hence $a$ is also contained in the convex hull of $\partial^0 K_1(m_1)$. □

Proposition 1.3 is regarded as the dual statement of [56, Theorem 12.1]. From Proposition 1.3, it follows that the convex cone $K_1(m_1)$ does not have any extreme direction. Thus, Corollary 1.2 is a consequence of more general theorem [53, Theorem 18.5].

All subsets $K_1(m_1)$, $\partial K_1(m_1)$ and $\partial^0 K_1(m_1)$ of $P_n$ are invariant by the action of $GL_n(\mathbf{Z})$. The finiteness of $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$ is due to Voronoï [62, §7 Théorèm]).

**Theorem 1.5 (Voronoï).** *The cardinality of $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$ is finite.*

**Proof.** We follows the argument of [60, Theorem 3.4]. Let $a \in \partial^0 K_1(m_1)$. By the reduction theory of Hermite or Minkowski, there exists an equivalent $a' \in aGL_n(\mathbf{Z})$ such that $\lambda_1\lambda_2\cdots\lambda_n \leq c_n \det a'$, where $\lambda_i$ denotes the $i$-th diagonal component of $a'$ and $c_n$ is the constant depending only on $n$. Since $a'$ is perfect, there are $n$ linearly independent minimal vectors $x_1,\cdots,x_n$ in $S(a')$. Since $m_1(a') = m_1(a) = 1$, Hadamard's inequality leads us to

$\det a' \leq \langle a', x_1{}^t x_1\rangle \cdots \langle a', x_n{}^t x_n\rangle = 1$. Therefore, we have

$$\mathrm{tr}(a') = \lambda_1 + \cdots + \lambda_n \leq n\lambda_1 \cdots \lambda_n \leq nc_n$$

because of $1 = m_1(a') \leq \lambda_i$ for $i = 1, \cdots, n$. This shows that any perfect element of $\partial^0 K_1(m_1)$ is $GL_n(\mathbf{Z})$-equivalent to a vertex of polytope $K_1(m_1) \cap \{a \in V_n \ : \ \mathrm{tr}(a) \leq nc_n\}$. □

More generally, it is known that the set of all faces of $\partial K_1(m_1)$ has only finitely many $GL_n(\mathbf{Z})$-orbits ( [55, Theorem 5'], see Theorem 3.2 below). The actual value of the cardinality $\rho_n = \sharp(\partial^0 K_1(m_1)/GL_n(\mathbf{Z}))$ is known up to $n = 8$ (cf. [59, §3.1]): one has $\rho_2 = \rho_3 = 1, \rho_4 = 2, \rho_5 = 3, \rho_6 = 7, \rho_7 = 33, \rho_8 = 10916$.

We note that the face $\mathcal{F}_S$ may not necessarily be compact in general. It is known that the non-empty face $\mathcal{F}_S$ is compact if and only if $S$ spans $\mathbf{R}^n$ ( [55, Theorem 1], see also [40, Remark 9.1.12]). If $S(a)$ spans $\mathbf{R}^n$, then $a \in P_n$ is said to be well-rounded. Any perfect point is obviously well-rounded. Any weakly eutactic point is also well-rounded ( [14, Théorème 2.3]). The finiteness and the algebraicity of weakly eutactic classes are verified by Bergé and Martinet [14, Théorèmes 3.5 et 4.1].

**Theorem 1.6 (Bergé and Martinet).** *Let $\partial^{we} K_1(m_1)$ be the set of all weakly eutactic points in $\partial K_1(m_1)$. Then the cardinality of $\partial^{we} K_1(m_1)/GL_n(\mathbf{Z})$ is finite. Any $a \in \partial^{we} K_1(m_1)$ is contained in $GL_n(\overline{\mathbf{Q}})$, where $\overline{\mathbf{Q}}$ stands for the algebraic closure of $\mathbf{Q}$.*

Let $\partial^{wr} K_1(m_1)$ be the set of all well-rounded points in $\partial K_1(m_1)$. The quotient $\partial^{wr} K_1(m_1)/GL_n(\mathbf{Z})$ is compact (cf. [40, Proposition 9.1.6]).

### 1.4. *Hermite like constants*

Let $\phi$ be a type one class function and $S_p$ be a complete set of representatives for $\partial^0 K_1(m_1)/GL_n(\mathbf{Z})$. From $P_n \subset \mathbf{R}_{>0} \cdot K_1(m_1)$, it follows

$$\sup_{a \in P_n} F_\phi(a) = \sup_{a \in K_1(m_1)} F_\phi(a) = \sup_{a \in K_1(m_1)} \frac{1}{\phi(a)} .$$

By Corollary 1.2, any $a \in K_1(m_1)$ is represented as

$$a = \lambda_1 a_1 + \cdots + \lambda_r a_r$$

by some $a_1, \cdots, a_r \in \partial^0 K_1(m_1)$ and $\lambda_1, \cdots, \lambda_r \in \mathbf{R}_{\geq 0}$ with $\lambda_1 + \cdots + \lambda_r = 1$. Then, since $\phi(a) \geq \min\{\phi(a_1), \cdots, \phi(a_r)\}$, one has

$$\sup_{a \in K_1(m_1)} \frac{1}{\phi(a)} = \sup_{a \in \partial^0 K_1(m_1)} \frac{1}{\phi(a)} = \max_{a \in S_p} \frac{1}{\phi(a)} .$$

Therefore, the Hermite like constant

$$\delta_\phi = \max_{a \in P_n} F_\phi(a)$$

of $F_\phi$ is well-defined. In the case of $\phi = \det^{1/n}$, $\delta_\phi$ coincides with the Hermite constant $\gamma_n$.

Let $\phi^\circ$ be the dual type one class function of $\phi$. By definition, the inequality $m_1(a) \leq \delta_{\phi^\circ} \phi^\circ(a)$ holds for all $a \in P_n$. By passing to the dual, one has $\phi^{\circ\circ}(a) \leq \delta_{\phi^\circ} w_1(a)$ for $a \in P_n$, and by Proposition 1.2,

$$\sup_{a \in P_n} \frac{\phi(a)}{w_1(a)} \leq \delta_{\phi^\circ} .$$

Thus, we can define the dual constant

$$\widehat{\delta}_\phi = \sup_{a \in P_n} \frac{\phi(a)}{w_1(a)} .$$

Indeed, we can show $\widehat{\delta}_\phi = \delta_{\phi^0}$ for any type one class function $\phi$. In particular, this gives

$$\gamma_n = \delta_{\det^{1/n}} = \widehat{\delta}_{(\det^{1/n})^\circ} = n \sup_{a \in P_n} \frac{\det(a)^{1/n}}{w_1(a)} .$$

We write $\xi_\phi$ for the product $\delta_\phi \cdot \widehat{\delta}_\phi$. This satisfies the invariance $\xi_{\phi^\circ} = \xi_{C\phi} = \xi_\phi$ for any constant $C > 0$. For example, $\xi_{w_1} = \xi_{m_1} = \delta_{w_1}$ and $\xi_{\det^{1/n}} = \gamma_n^2/n$. By definition, we have the following:

**Proposition 1.4.** *The inequality $\xi_{w_1} \leq \xi_\phi$ holds for any type one class function $\phi$.*

See [58, Propositions 3.2 and 3.3] for details.

## 2. Rankin's constant and Voronoï's theorem

Rankin [52] defined the constant $\gamma_{n,k}$ as a generalization of Hermite's constant, and proved Rankin's inequality among $\gamma_{n,k}$. About 40 years later, Coulangeon [22] formulated Voronoï's theorem of this case in terms of $k$-perfection and $k$-eutaxy. It is an open problem to find a geometric characterization of $k$-perfect forms. Bergé and Martinet [13] introduced the constant $\gamma'_{n,k}$ and proved several inequalities among $\gamma_{n,k}$ and $\gamma'_{n,k}$. In this section, we will survey Voronoï's theorem for $\gamma_{n,k}$ and $\gamma'_{n,1}$.

### 2.1. *Rankin's constant*

Let $k$ be a positive integer with $1 \leq k \leq n-1$. We denote by $M^*_{n,k}(\mathbf{Z})$ the subset $\{X = (x_1, \cdots, x_k) \in M_{n,k}(\mathbf{Z}) \ : \ x_1 \wedge x_2 \wedge \cdots \wedge x_k \neq 0\}$ of $M_{n,k}(\mathbf{Z})$. The unimodular group $GL_k(\mathbf{Z})$ (resp. $GL_n(\mathbf{Z})$) acts on $M^*_{n,k}(\mathbf{Z})$ by right (resp. left) multiplications. For $X \in M^*_{n,k}(\mathbf{Z})$, define the function $D_X : P_n^{\mathrm{semi}} \longrightarrow \mathbf{R}_{\geq 0}$ by

$$D_X(a) = \det({}^tXaX)^{1/k}$$

for $a \in P_n^{\mathrm{semi}}$. It is obvious that $D_X$ is a type one function. The function $m_k : P_n^{\mathrm{semi}} \longrightarrow \mathbf{R}_{\geq 0}$ defined by

$$m_k(a) = \inf_{X \in M^*_{n,k}(\mathbf{Z})} D_X(a)$$

is a type one class function. This is regarded as a generalization of the arithmetical minimum function $m_1$. It is obvious that $m_k(a) > 0$ if $a \in P_n$, and $m_k(a) = 0$ otherwise. Rankin [52] defined the constant

$$\gamma_{n,k} = \left( \max_{a \in P_n} \frac{m_k(a)}{\det(a)^{1/n}} \right)^k ,$$

and then proved the inequality

$$\gamma_{n,k} \leq \gamma_{j,k}(\gamma_{n,j})^{k/j}$$

for $1 \leq k < j \leq n-1$ as a generalization of Mordell's inequality. By using this inequality, Rankin determined the value $\gamma_{4,2} = 3/2$. See **2.4** for other explicit values of $\gamma_{n,k}$.

### 2.2. *Voronoï's theorem of* $\boldsymbol{m_k/\det^{1/n}}$

A point $a \in P_n$ is said to be $k$-extreme (strictly $k$-extreme) if $m_k/\det^{1/n}$ attains a local maximum (resp. a strict local maximum) on $a$ up to the multiplication by an element of $\mathbf{R}_{>0}$. A Voronoï type characterization of $k$-extreme points was studied by Coulangeon [22]. The subset

$$S_k^*(a) = \{X \in M^*_{n,k}(\mathbf{Z}) \ : \ D_X(a) = m_k(a)\}$$

corresponding to $a \in P_n$ plays a key role. Since $D_{Xh}(a) = D_X(a)$ for all $h \in GL_k(\mathbf{Z})$, the set $S_k^*(a)$ is invariant by the action of $GL_k(\mathbf{Z})$, and hence the quotient $S_k(a) = S_k^*(a)/GL_k(\mathbf{Z})$ exists. We write $[X]$ for the element $X \cdot GL_k(\mathbf{Z})$ in $S_k(a)$. The following was proved in [22, Proposition 2.7].

**Proposition 2.1 (Coulangeon).** *The cardinality of* $S_k(a)$ *is finite.*

We recall the notion of $k$-perfection and $k$-eutaxy. For each $i = 1, 2, \cdots, k$, we define the map $*_i : V_n \times M_{n,k}(\mathbf{R}) \longrightarrow M_{n,k}(\mathbf{R})$ by

$$v *_i X = (x_1, \cdots, x_{i-1}, vx_i, x_{i+1}, \cdots, x_k)$$

for $v \in V_n$ and $X = (x_1, \cdots, x_k) \in M_{n,k}(\mathbf{R})$. Note that $*_i$ is linear in $X$ but not in $v$. Then, for each $X \in M^*_{n,k}(\mathbf{Z})$, the linear map $\varphi_X : V_n \longrightarrow \mathbf{R}$ is defined by

$$\varphi_X(v) = \sum_{i=1}^{k} \det({}^t X \cdot (v *_i X)) .$$

It is obvious that $\varphi_X$ depends only on the class $[X] = X \cdot GL_k(\mathbf{Z})$. Another definition of $\varphi_X$ is given by

$$\varphi_X(v) = \det({}^t X \cdot X) \cdot < p_X, v > ,$$

where $p_X$ denotes the matrix representation of the orthogonal projection from $\mathbf{R}^n$ onto the subspace spanned by $\{x_1, \cdots, x_k\}$.

**Definition 2.1.** Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t bb$. An element $a$ is said to be $k$-perfect if $\{\varphi_{bX}\}_{[X] \in S_k(a)}$ spans the dual space $V_n^*$ of $V_n$. An element $a$ is said to be $k$-eutactic if there exist $\rho_X > 0$ ($[X] \in S_k(a)$) such that

$$\mathrm{tr} = \sum_{[X] \in S_k(a)} \rho_X \varphi_{bX} .$$

These definitions of $k$-perfection and $k$-eutaxy do not depend on a choice of $b$. Now the main theorem of [22] is stated as follows:

**Theorem 2.1 (Coulangeon).** *A point $a \in P_n$ is $k$-extreme if and only if $a$ is $k$-perfect and $k$-eutactic. Any $k$-extreme point is strictly $k$-extreme.*

The line of the proof of Theorem 2.1 is parallel to that of Theorem 1.3. Namely, the following sufficient and necessary condition for $k$-extremeness is shown: $a = {}^t bb \in P_n$ is $k$-extreme if and only if the set

$$\{v \in V_n \ : \ \min_{X \in S_k(a)} \varphi_{bX}(v) \geq 0 \ \text{ and } \ \mathrm{tr}(v) \leq 0\}$$

is reduced to $\{0\}$ ( [22, Théorème 3.2.2]). Theorem 2.1 follows from this and Stiemke's theorem.

The finiteness of $k$-perfect points was proved in [22, Théorèm 4.5].

**Theorem 2.2 (Coulangeon).** *The number of $k$-perfect points in $P_n$ modulo $\mathbf{R}_{>0} GL_n(\mathbf{Z})$ is finite.*

**Example 2.1.** Let $L \subset \mathbf{R}^n$ be a full lattice, which means a $\mathbf{Z}$-module of rank $n$. The dual lattice $L^*$ of $L$ is defined by $L^* = \{y \in \mathbf{R}^n : {}^t xy \in \mathbf{Z} \quad \text{for all } x \in L\}$. If $x_1, \cdots, x_n$ is a basis of $L$, then we denote by $[L]$ the class of the Gram matrix $({}^t x_i x_j)_{1 \le i,j \le n}$ in $\mathbf{R}_{>0} \backslash P_n / GL_n(\mathbf{Z})$. In dimension 4, there are at least 5 inequivalent 2-perfect points, i.e., $[A_4], [A_4^*], [D_4], [W_4], [W_4^*]$ ( [22, §5.1]). Here we use standard notations of root lattices. By $W_4$, we denote the Watson lattice of rank 4, i.e.,

$$[W_4] = \begin{pmatrix} 3 & -1 & -1 & -1 \\ -1 & 3 & -1 & -1 \\ -1 & -1 & 3 & -1 \\ -1 & -1 & -1 & 3 \end{pmatrix} \mod \mathbf{R}_{>0} GL_4(\mathbf{Z})\,.$$

The maximum of $m_2/\det^{1/4}$ is attained on $[D_4]$. In general $n \ge 4$, the class of any irreducible root lattice of rank $n$ is 2-extreme and $k$-eutactic for all $k < n$ ( [22, Théorème 5.1.1]).

When $k = 1$, Theorem 1.1 shows that the Hermite constant $\gamma_n = \gamma_{n,1}$ is an algebraic number. The algebraicity of $k$-perfect points and $\gamma_{n,k}$ for $k \ge 2$ was verified by Bavard [9, Théorèm 2.2], [11, §1.5]. It is based on the following general result [11, Lemme 1.11].

**Lemma 2.1.** *Let $W \subset \mathbf{R}^N$ be an algebraic subset defined by polynomials with coefficients in $\overline{\mathbf{Q}} \cap \mathbf{R}$. Then the set of isolated points in $W$ is a finite subset contained in $W \cap (\overline{\mathbf{Q}} \cap \mathbf{R})^N$.*

If $a$ is $k$-perfect, then the set

$$W = \{(b, \lambda) \in V_n \times \mathbf{R} \ : \ D_X(b) - \lambda = 0 \ \text{ for all } X \in S_k^*(a)\}$$

satisfies the assumption of Lemma 2.1. Since $k$-perfect points are isolated in $W$ ( [11, Proposition 1.8]), we obtain

**Theorem 2.3 (Bavard).** *Any $k$-perfect point is contained in $P_n \cap M_n(\overline{\mathbf{Q}})$. In particular, $\gamma_{n,k}$ is an algebraic number.*

### 2.3. *Some problems on k-perfect forms*

Since $m_k$ is a continuous type one function and vanishes on the boundary of $P_n^{\mathrm{semi}}$, $K_1(m_k)$ is a kernel contained in $P_n$. As we have seen in §1.3, $K_1(m_1)$ is a locally finite polyhedral convex cone and perfect points are characterized as vertices of $K_1(m_1)$. When $k \ge 2$, $K_1(m_k)$ is not polyhedral, and we have the following problem.

**Problem 2.1.** *Determine locations of $k$-perfect points in $K_1(m_k)$.*

Theorem 1.4 (3) gives another characterization of perfect points in $P_n$. On variations of the set $S_k(a)$, the next is elementary ( [22, Lemme 2.9]).

**Proposition 2.2 (Coulangeon).** *For $a \in P_n$, there is a neighborhood $\mathcal{U}$ of $a$ in $P_n$ such that $S_k(a') \subset S_k(a)$ for $a' \in \mathcal{U}$.*

We say that $S_k(a)$ is locally maximal if there is a neighborhood $\mathcal{U}$ of $a$ in $P_n$ such that $S_k(a') \subsetneqq S_k(a)$ for all $a' \in \mathcal{U} \setminus \mathbf{R}_{>0}a$. Let $P_n^{(k)}$ be the set of $a \in P_n$ such that $S_k(a)$ is locally maximal. If $k = 1$, $P_n^{(1)}$ coincides with the set of all perfect points in $P_n$.

**Problem 2.2.** *Does $P_n^{(k)}$ coincide with the set of all $k$-perfect points in $P_n$ for any $k \geq 2$ ?*

The cardinality $\sharp(S(a)) = 2\sharp(S_1(a))$ is called the kissing number of $a$. Determination of the maximum $\max_{a\in P_n} 2\sharp(S_1(a))$ is known as the lattice kissing number problem [20, Chapter 1, §2]. The actual value of $\max_{a\in P_n} \sharp(S_1(a))$ is known for $1 \leq n \leq 9$ and $n = 24$ (cf. [67]). One can prove the estimate $\max_{a\in P_n} \sharp(S_1(a)) \leq 2^n - 1$ for all $n$ as follows: Let $a \in P_n$ and $x, y \in S(a)$. If $y - x = 2z \in 2\mathbf{Z}^n$, then $m_1(a) = {}^t(x + 2z)a(x + 2z) = {}^txax$, and hence ${}^tzaz = -{}^txaz$. From ${}^txax \leq {}^t(x + z)a(x + z)$, it follows ${}^tzaz \leq 0$, i.e., $z = 0$. This means that the natural map $\mathbf{Z}^n \longrightarrow \mathbf{Z}^n/2\mathbf{Z}^n$ is injective on $S_1(a)$. This proof is due to Voronoï [62, p.107, Lemme], see also [43, §31, p.80] for more general result. When $n \geq 10$, Watson [66, Theorem 1] proved $\max_{a\in P_n} \sharp(S_1(a)) \leq 2^{n-2} + 8$. We have a similar problem for $k \geq 2$.

**Problem 2.3.** *Bound the maximum $\max_{a\in P_n} \sharp(S_k(a))$.*

If $\phi$ is a type one function, then one can ask about Voronoï's theorem for $m_k/\phi$.

**Problem 2.4.** *Prove Voronoï type theorem for $m_k/\phi$ when $k \geq 2$.*

Let $w_k$ be the dual type one class function of $m_k$. When $k \geq 2$, it is not trivial that the Hermite–Rankin like constant of $m_k/\phi$ exists for a given type one class function $\phi$. We set

$$\delta_{\phi,k} = \sup_{a\in P_n} \frac{m_k(a)}{\phi(a)}, \qquad \widehat{\delta}_{\phi,k} = \sup_{a\in P_n} \frac{\phi(a)}{w_k(a)}$$

for a type one class function $\phi$.

**Problem 2.5.** *When are both $\delta_{\phi,k}$ and $\widehat{\delta}_{\phi,k}$ finite ?*

It is easy to see that $\delta_{\phi,k} = \widehat{\delta}_{\phi^\circ,k}$ provided that both $\delta_{\phi,k}$ and $\widehat{\delta}_{\phi^\circ,k}$ are finite.

### 2.4. *The Bergé–Martinet constant*

The constant

$$\gamma'_{n,k} = \left(\sup_{a\in P_n} m_k(a)m_k(a^{-1})\right)^{k/2}$$

was first defined by Bergé and Martinet. In [13], they proved several inequalities among $\gamma_{n,k}$ and $\gamma'_{n,k}$.

**Theorem 2.4 (Bergé and Martinet).** *One has the following:*

(1) $\gamma'_{n,k} \le \gamma_{n,k} \le (\gamma_n)^k$ *for* $1 \le k \le n-1$.
(2) $(\gamma_{n,k})^n \le (\gamma_{n-k,k})^{n-k}(\gamma'_{n,k})^{2k}$ *for* $1 \le k \le n/2$.
(3) $\gamma'_{n,2k} \le (\gamma'_{n-k,k})^2$ *for* $1 \le k \le n/2$.
(4) $(\gamma_{n,k})^{n-2k} \le (\gamma_{n-k,k})^{n-k}$ *for* $1 \le k \le n-1$.
(5) $\gamma_{n,n/2} = \gamma'_{n,n/2}$ *if* $n$ *is even.*

When $k = 1$, an analog of Voronoï's theorem holds for the Bergé–Martinet invariant $F_{\mathrm{BM}}(a) = \sqrt{m_1(a)m_1(a^{-1})}$. A point $a \in P_n$ is said to be dual-extreme (strictly dual extreme) if $F_{\mathrm{BM}}$ attains a local maximum (resp. a strict local maximum) on $a$ up to the multiplication by an element of $\mathbf{R}_{>0}$. To define the dual-perfection and the dual-eutaxy, we use the same notation as in **1.2**.

**Definition 2.2.** Let $a \in P_n$. We fix an element $b \in GL_n(\mathbf{R})$ such that $a = {}^t bb$. An element $a$ is said to be dual-perfect if $\{\varphi_{bx}\}_{x\in S(a)} \cup \{\varphi_{{}^t b^{-1}y}\}_{y\in S(a^{-1})}$ spans the dual space $V_n^*$ of $V_n$. An element $a$ is said to be dual-eutactic if there exist $\rho_x > 0$ $(x \in S(a))$ and $\rho_y > 0$ $(y \in S(a^{-1}))$ such that

$$\sum_{x\in S(a)} \rho_x \varphi_{bx} = \sum_{y\in S(a^{-1})} \rho_y \varphi_{{}^t b^{-1}y}.$$

Then one has:

**Theorem 2.5 (Bergé and Martinet).** *A point* $a \in P_n$ *is dual-extreme if and only if* $a$ *is dual-perfect and dual-eutactic. Any dual extreme point is a strict dual extreme.*

As noticed in [40, p.99], the number of dual-perfect points in $P_n$ modulo $\mathbf{R}_{>0}GL_n(\mathbf{Z})$ is infinite in general. In [12], Bergé proved the following:

**Theorem 2.6 (Bergé).** *The number of dual-extreme points in $P_n$ modulo $\mathbf{R}_{>0}GL_n(\mathbf{Z})$ is finite. If $a \in P_n$ is dual extreme, then there exists $\lambda \in \mathbf{R}_{>0}$ such that $\lambda a \in GL_n(\overline{\mathbf{Q}})$. In particular , $\gamma'_n = \gamma'_{n,1}$ is an algebraic number.*

As to the explicit value of $\gamma'_n$, $\gamma'_8 = \gamma_8 = 2$ immediately follows from the self-duality of the $E_8$-lattice. Bergé and Martinet determined the values $\gamma'_2 = 2/\sqrt{3}$, $\gamma'_3 = \sqrt{3/2}$ and $\gamma'_4 = \sqrt{2}$. In [49], Poor and Yuen proved the inequality

$$\frac{n}{(\gamma_n)^2} \leq \inf_{(a,b)\in P_n\times P_n} \frac{\langle a,b\rangle}{m_1(a)m_1(b)} \leq \frac{n}{(\gamma'_n)^2} \tag{2}$$

and, by using this, they determined the following values.

**Theorem 2.7 (Poor and Yuen).** $\gamma'_5 = \sqrt{2}$, $\gamma'_6 = \sqrt{8/3}$ *and* $\gamma'_7 = \sqrt{3}$.

All known values of $\gamma'_n$ satisfy $(\gamma'_n)^2 \in \mathbf{Q}$. The following problem is due to Martinet [40, Questions 3.8.12].

**Problem 2.6.** *Is $(\gamma'_n)^2$ rational for all $n$ ?*

Applying Theorem 2.4 to the explicit values of $\gamma'_5$, $\gamma'_7$, $\gamma_{4,2}$ and $\gamma_n$ for $n = 2, \cdots, 8$, we have

**Theorem 2.8.** $\gamma_{6,2} = 3^{2/3}$, $\gamma'_{6,2} = 2$, $\gamma_{8,2} = \gamma'_{8,2} = 3$ *and* $\gamma_{8,3} = \gamma'_{8,3} = \gamma_{8,4} = \gamma'_{8,4} = 4$.

See [57, §2] for details. Barnes and Cohn [7] also proved the first part of the inequality (2). Since one has

$$\frac{1}{\xi_{w_1}} = \inf_{(a,b)\in P_n\times P_n} \frac{\langle a,b\rangle}{m_1(a)m_1(b)}\,,$$

the first part of (2) is a special case of Proposition 1.4. Furthermore, the inequality (2) is generalized to $\gamma_{n,k}$ and $\gamma'_{n,k}$ as follows:

$$\frac{n}{(\gamma_{n,k})^{2/k}} \leq \inf_{(a,b)\in P_n\times P_n} \frac{\langle a,b\rangle}{m_k(a)m_k(b)} \leq \frac{n}{(\gamma'_{n,k})^{2/k}}\,,$$

(see [57, Theorem 1]). To find an analog of Voronoï's theorem for the $k$-th Bergé–Martinet invariant $F^{(k)}_{\mathrm{BM}}(a) = (m_k(a)m_k(a^{-1}))^{k/2}$ ( [40, Problem 10.6.10]) solved by Bavard, see Example 3.7 below. See [26] for other Hermite like invariants.

## 3. Generalizations of Voronoï's theorem

There are several directions to generalize Voronoï's theorem. A natural generalization of the domain $P_n$ was considered by Koecher [35] and Ash [3]. An extension of the geometric framework was developed by Bavard [9], [11]. A change of a base field from $\mathbf{Q}$ to an algebraic number field was studied by several authors [22], [32], [38], [45], [46]. In this section, we will survey these theories. We do not exhaust all of generalizations. See e.g., [40, Chaprter 13] for other variations.

### 3.1. *Voronoï's theorem of packing functions on symmetric cones*

A generalization of the domain $P_n$ is given by the notion of symmetric cones. Let $\Omega$ be an open convex cone in the Euclidean space $\mathbf{R}^N$. The open dual cone $\Omega^*$ of $\Omega$ is defined to be

$$\Omega^* = \{a \in \mathbf{R}^N \ : \ (a,b) > 0 \ \text{ for all } b \in \overline{\Omega} \setminus \{0\}\}\,,$$

where $\overline{\Omega}$ denotes the closure of $\Omega$ in $\mathbf{R}^N$. If $\Omega = \Omega^*$ holds, then $\Omega$ is called a self-dual cone. We denote by $G_\Omega$ the stabilizer of $\Omega$ in $GL_N(\mathbf{R})$, i.e.,

$$G_\Omega = \{g \in GL_N(\mathbf{R}) \ : \ g\Omega = \Omega\}\,.$$

If $G_\Omega$ acts transitively on $\Omega$, then $\Omega$ is said to be homogeneous. By a symmetric cone, we mean a self-dual homogeneous cone. See the textbook [25] for details of symmetric cones.

We fix a symmetric cone $\Omega$. Let $G_\Omega^\circ$ be the connected component of the identity in $G_\Omega$. Then $G_\Omega^\circ$ also acts transitively on $\Omega$. We denote by $K_a$ the stabilizer of $a \in \Omega$ in $G_\Omega^\circ$. There exists a point $\boldsymbol{e} \in \Omega$ such that $K_{\boldsymbol{e}} = G_\Omega^\circ \cap O_N(\mathbf{R})$. The group $K_{\boldsymbol{e}}$ is connected and gives a maximal compact subgroup of $G_\Omega^\circ$. Thus $\Omega$ is identified with the Riemannian symmetric space $G_\Omega^\circ/K_{\boldsymbol{e}}$.

Let $L \subset \mathbf{R}^N$ be a lattice of rank $N$ which contains $e$. Now we define the packing function $F_{(\Omega,L)} : \Omega \longrightarrow \mathbf{R}_{>0}$, of which we study local maxima. First, the characteristic function $\varphi_\Omega$ of $\Omega$ is defined by

$$\varphi_\Omega(a) = \int_\Omega e^{-(a,b)} db\,, \qquad (a \in \Omega)\,.$$

The Lebesgue measure $db$ is normalized so that $\varphi_\Omega(\boldsymbol{e}) = 1$. The defining integral is uniformly convergent on any compact subset in $\Omega$. It follows from the definition that $\varphi_\Omega(ga) = |\det g|^{-1}\varphi_\Omega(a)$ for all $g \in G_\Omega$ and $a \in \Omega$.

Next, the minimum function $m_L$ on $\Omega$ is defined by

$$m_L(a) = \min\{(a,b) \ : \ b \in (L \setminus \{0\}) \cap \overline{\Omega}\}\,.$$

Since $\Omega$ is self-dual, the value $m_L(a)$ is positive. Then the packing function $F_{\Omega,L}$ is defined by

$$F_{(\Omega,L)}(a) = m_L(a)^N \varphi_\Omega(a)\,.$$

A point $a \in \Omega$ is said to be extreme if $F_{(\Omega.L)}$ attains a local maximum on $a$ up to the multiplication by an element of $\mathbf{R}_{>0}$.

To state Ash's definition of eutaxy, we need a Jordan algebra structure of $\mathbf{R}^N$ induced from $\Omega$. Let $\mathfrak{g}$ be the Lie algebra of $G_\Omega^\circ$, i.e.,

$$\mathfrak{g} = \{X \in M_N(\mathbf{R}) \ : \ \exp(X) \in G_\Omega^\circ\}\,.$$

Since $\Omega$ is a symmetric cone, $\mathfrak{g}$ is invariant by the transpose $X \mapsto {}^tX$. We set $\mathfrak{g}_\pm = \{X \in \mathfrak{g} \ : \ {}^tX = \pm X\}$. Then $\mathfrak{g}_-$ coincides with the Lie algebra of $K_{\boldsymbol{e}}$. Moreover, the map $\psi \,:\, \mathfrak{g}_+ \longrightarrow \mathbf{R}^N$ defined by $\psi(X) = X\boldsymbol{e}$ gives a linear isomorphism. We define the binary product $* \,:\, \mathbf{R}^N \times \mathbf{R}^N \longrightarrow \mathbf{R}^N$ by

$$a * b = \psi^{-1}(a)b\,.$$

This product satisfies

(J$_1$) $a * b = b * a$
(J$_2$) $a * (a^2 * b) = a^2 * (a * b)$, where $a^2$ means $a * a$
(J$_3$) $\boldsymbol{e} * a = a * \boldsymbol{e} = a$
(J$_4$) $(a * c, b) = (a, c * b)$

for all $a, b, c \in \mathbf{R}^N$. Namely, $*$ gives $\mathbf{R}^N$ a formally real Jordan algebra structure with the identity $\boldsymbol{e}$. We denote by $J_\Omega$ this formally real Jordan algebra. For $a \in J_\Omega$, the subalgebra $\mathbf{R}[a]$ of $J_\Omega$ generated by $a$ and $\boldsymbol{e}$ is an associative algebra. An element $a$ is said to be invertible if there exists an element $b \in \mathbf{R}[a]$ such that $a * b = \boldsymbol{e}$. This $b$ is unique and is denoted by $a^{-1}$. Let $J_\Omega^\times$ be the subset of all invertible elements in $J_\Omega$. Then $\Omega$ coincides with the connected component of $J_\Omega^\times$ which contains $e$.

We assume $L \otimes_{\mathbf{Z}} \mathbf{Q}$ gives a $\mathbf{Q}$-structure of $J_\Omega$. For $a \in \Omega$, we set

$$S_{(\Omega,L)}(a) = \{b \in (L \setminus \{0\}) \cap \overline{\Omega} \ : \ (a,b) = m_L(a)\}\,.$$

**Definition 3.1.** Let $a \in \Omega$ and $a^{-1}$ be the inverse of $a$ in the Jordan algebra $J_\Omega$. A point $a$ is said to be perfect if $S_{(\Omega.L)}(a)$ spanns $\mathbf{R}^N$. A point

$a$ is said to be eutactic if there exist $\lambda_b \in \mathbf{R}_{>0}$, $b \in S_{(\Omega,L)}$, such that

$$a^{-1} = \sum_{b \in S_{(\Omega,L)}} \lambda_b b\,.$$

Let $g \in \exp(\mathfrak{g}_+)$. The Taylor expansion of $1/\varphi_\Omega$ at the point $a = g\boldsymbol{e}$ is given by

$$\frac{1}{\varphi_\Omega(a+v)} = \det g \cdot \{1 + (a^{-1}, v) + \frac{1}{2}((a^{-1}, v)^2 - (g^{-1}v, g^{-1}v)) + O((v,v)^{3/2})\}$$

for $v \in \mathbf{R}^N$ ( [3, Corollary to Proposition 3]). By using this formula, Ash proved that the function $1/F_{(\Omega,L)}$ is a topological Morse function on $\Omega/\mathbf{R}_{>0}$. Voronoï's theorem of $F_{(\Omega,L)}$ follows from this fact.

**Theorem 3.1 (Ash).** *A point $a \in \Omega$ is extreme if and only if $a$ is perfect and eutactic.*

Let $K_1(m_L) = \{a \in \overline{\Omega} \ : \ (a,b) \geq 1 \ \text{ for all } b \in (L \setminus \{0\}) \cap \overline{\Omega}\}$, which is a polyhedral cone and is regarded as a generalization of the Ryshkov domain $K_1(m_1)$. Any perfect point $a \in \Omega$ of $m_L(a) = 1$ is a vertex of $K_1(m_L)$. We have the following finiteness:

**Theorem 3.2 (Ash).** *The discrete group $\Gamma = \{g \in G_\Omega^\circ \ : \ {}^t g L = L\}$ of $G_\Omega^\circ$ acts on $K_1(m_L)$. The set of faces of $K_1(m_L)$ has only finitely many $\Gamma$-orbits. In particular, the number of perfect points in $\Omega$ modulo $\mathbf{R}_{>0}\Gamma$ is finite. Moreover, the number of eutactic points in $\Omega$ modulo $\mathbf{R}_{>0}\Gamma$ is finite.*

It is proved in [3, Theorem 2] that a point $a \in \Omega$ is eutactic if and only if $a$ is critical non-degenerate for $F_{(\Omega,L)}$. The finiteness of eutactic points is derived from the finiteness of $\Gamma$-orbits of critical points of $F_{(\Omega,L)}$.

**Example 3.1.** The cone $P_n$ of positive definite symmetric matrices is a symmetric cone in $V_n$. In this case, $\boldsymbol{e}$ is chosen as the identity matrix $\mathrm{I}_n$ and the product $*$ is defined by

$$a * b = \frac{1}{2}(ab + ba)$$

for $a, b \in V_n$. When $L = \{v \in M_n(\mathbf{Q}) \cap V_n \ : \ 2v \in M_n(\mathbf{Z}),\ v_{11}, \cdots, v_{nn} \in \mathbf{Z}\}$, the packing function $F_{(P_n,L)}$ is equal to $(m_1/\det^{1/n})^{n(n+1)/2}$. Ash's definition of perfection and eutaxy is equivalent to Definition 1.3 ( [3, Corollary to Proposition 2]).

**Example 3.2.** Let $B$ be the non degenerate bilinear form on $\mathbf{R}^n$ defined by

$$B(x,y) = x_1y_1 - x_2y_2 - \cdots - x_ny_n$$

for $x = {}^t(x_1, \cdots, x_n)$ and $y = {}^t(y_1, \cdots, y_n) \in \mathbf{R}^n$. Then the Lorentz cone $\Omega_n = \{x \in \mathbf{R}^n \ : \ B(x,x) > 0, \ x_1 > 0\}$ is a symmetric cone in $\mathbf{R}^n$. We choose $\boldsymbol{e}$ as the unit vector ${}^t(1, 0, \cdots, 0) \in \mathbf{R}^n$. Let $\{\boldsymbol{e}\}^\perp$ be the orthogonal complement of $\boldsymbol{e}$ with respect to the usual inner product $(\cdot,\cdot)$ of $\mathbf{R}^n$. The product of the Jordan algebra $J_{\Omega_n}$ is defined by

$$(\lambda\boldsymbol{e} + u) * (\lambda'\boldsymbol{e} + u') = (\lambda\lambda' - B(u,u'))\boldsymbol{e} + \lambda u' + \lambda' u$$

for $\lambda, \lambda' \in \mathbf{R}$ and $u, u' \in \{\boldsymbol{e}\}^\perp$. The packing function $F_{(\Omega_n,\mathbf{Z}^n)}$ is given as

$$F_{(\Omega_n,\mathbf{Z}^n)}(a) = \frac{m_{\mathbf{Z}^n}(a)^n}{B(a,a)^{n/2}}.$$

Since $G^\circ_{\Omega_n} = \mathbf{R}_{>0} \cdot SO_0(1, n-1)$ acts transitively on $\Omega_n$, we have

$$F_{(\Omega_n,\mathbf{Z}^n)}(\lambda g\boldsymbol{e}) = \frac{m_{\mathbf{Z}^n}(g\boldsymbol{e})^n}{B(g\boldsymbol{e}, g\boldsymbol{e})^{n/2}} = m_{\mathbf{Z}^n}(g\boldsymbol{e})^n$$

for $\lambda \in \mathbf{R}_{>0}$ and $g \in SO_0(1, n-1)$, and moreover,

$$m_{\mathbf{Z}^n}(g\boldsymbol{e}) = \min_{x\in(\mathbf{Z}^n\setminus\{0\})\cap\overline{\Omega}_n} (\boldsymbol{e}, {}^tgx) = \min_{x\in(\mathbf{Z}^n\setminus\{0\})\cap\overline{\Omega}_n} \left(\frac{({}^tgx, {}^tgx) + B(x,x)}{2}\right)^{1/2}.$$

Therefore, we have

$$\max_{a\in\Omega_n} F_{(\Omega_n,\mathbf{Z}^n)}(a) = \max_{[g]\in K_e\setminus SO_0(1,n-1)/\Gamma} \ \min_{x\in(\mathbf{Z}^n\setminus\{0\})\cap\overline{\Omega}_n} \left(\frac{(gx, gx) + B(x,x)}{2}\right)^{n/2},$$

where $\Gamma = SO_0(1, n-1) \cap SL_n(\mathbf{Z})$.

**Problem 3.1.** *Let $\Omega$ be an arbitrary symmetric cone. Replacing $P_n$ and $P_n^{\mathrm{semi}}$ with $\Omega$ and $\overline{\Omega}$, respectively, in Definitions 1.1 and 1.2, we can define type one functions on $\overline{\Omega}$ and semikernels in $\overline{\Omega}$. For example, both $\varphi_\Omega^{-1/N}$ and $m_L$ are continuously extended to type one functions on $\overline{\Omega}$. Can Proposition 1.1 and Theorem 1.3 be generalized to this setting?*

As stated in Theorem 3.2, the number of eutactic classes in $\mathbf{R}_{>0}\backslash\Omega/\Gamma$ is finite. When $\Omega = P_n$, Ash verified a "mass formula with signs" of eutactic classes ([4], see also [40, Theorem 9.5.3]). For $a \in \partial K_1(m_1)$, we set $\Gamma_a = \{g \in SL_n(\mathbf{Z}) \ : \ {}^tgag = a\}$. Let $\mathcal{F}_{S(a)}$ be the face of $K_1(m_1)$ defined in **1.3**, i.e., $\mathcal{F}_{S(a)} = \{b \in \partial K_1(m_1) \ : \ S(a) \subset S(b)\}$.

**Theorem 3.3 (Ash).** *The set $\partial^e K_1(m_1)$ of eutactic points in $\partial K_1(m_1)$ satisfies*

$$\sum_{[a]\in\partial^e K_1(m_1)/GL_n(\mathbf{Z})} \frac{(-1)^{\dim \mathcal{F}_{S(a)}}}{\sharp\Gamma_a} = \chi(SL_n(\mathbf{Z})) = \begin{cases} -1/12 & (n=2) \\ 0 & (n\geq 3) \end{cases},$$

*where $\chi(SL_n(\mathbf{Z}))$ stands for the Euler characteristic of $SL_n(\mathbf{Z})$.*

The actual value of $\epsilon_n = \sharp(\partial^e K_1(m_1)/GL_n(\mathbf{Z}))$ is known up to $n=5$ (cf. [8]): one has $\epsilon_2 = 2, \epsilon_3 = 5, \epsilon_4 = 16, \epsilon_5 = 118$.

### 3.2. *Bavard's theory*

Let $V$ be a Riemannian manifold and $\Gamma$ a discrete subgroup of the isometry group of $V$. Let $C$ be a set endowed with a right action of $\Gamma$. We consider a family of $C^1$-functions $f_s : V \longrightarrow \mathbf{R}$ parameterized by $s \in C$. We assume the following two conditions:

(B$_1$) $f_s \circ \gamma = f_{s\gamma}$ for all $s \in C$ and $\gamma \in \Gamma$.
(B$_2$) The cardinality of the subset $\{s \in C : f_s(v) \leq \lambda\}$ of $C$ is finite for any $v \in V$ and $\lambda \in \mathbf{R}$.

Each $f_s$ is called a length function on $V$. We write $\mathcal{E}$ for the quadruplet $(V, \Gamma, C, \{f_s\})$. What we do is to characterize local maxima of the function $F_{\mathcal{E}}(v) = \min_{s\in C} f_s(v)$ in $v \in V$. A point $v \in V$ is said to be extreme (resp. strictly extreme) if $v$ attains a local maximum (resp. a strictly local maximum) of $F_{\mathcal{E}}$.

For a given $v \in V$, $T_vV$ stands for the tangent space of $V$ at $v$ and $X_s(v)$ stands for the gradient vector of $f_s$ at $v$. By the condition (B$_2$), $S_{\mathcal{E}}(v) = \{s \in C : f_s(v) = F_{\mathcal{E}}(v)\}$ is a finite subset of $C$. Let $\mathrm{Conv}(v)$ be the convex hull of $\{X_s(v)\}_{s\in S_{\mathcal{E}}(v)}$ in $T_vV$ and $\mathrm{Aff}(v)$ the affine subspace spanned by $\{X_s(v)\}_{s\in S_{\mathcal{E}}(v)}$ in $T_vV$.

**Definition 3.2.** A point $v \in V$ is said to be perfect if $T_vV = \mathrm{Aff}(v)$ holds. A point $v \in V$ is said to be eutactic if the origin $0 \in T_vV$ is contained in the interior of $\mathrm{Conv}(v)$.

We need the following condition for $\mathcal{E}$.

(C) For any $v \in V$, any subset $S' \subset S_{\mathcal{E}}(v)$ and any non-zero vector $X \in T_vV$ orthogonal to $\{X_s(v)\}_{s\in S'}$, there exists a $C^1$-curve $c : [0, \epsilon) \longrightarrow V$ for a sufficiently small $\epsilon > 0$ such that $c(0) = v$, $c'(0) = X$ and $f_s(v) < f_s(c(t))$ for all $t \in (0, \epsilon)$ and $s \in S'$.

Now Bavard's theorem is stated as:

**Theorem 3.4 (Bavard).** *Assume $\mathcal{E}$ satisfies the condition* (C). *Then any extreme point in $V$ is strictly extreme, and a point $v \in V$ is extreme if and only if $v$ is perfect and eutactic.*

A function $f : V \longrightarrow \mathbf{R}$ is said to be convex if $f$ is convex on any geodesic line on $V$, i.e.,

$$f(\ell(\lambda\alpha + (1-\lambda)\beta)) \leq \lambda f(\ell(\alpha)) + (1-\lambda) f(\ell(\beta))$$

holds for any geodesic $\ell : [0, \epsilon) \longrightarrow V$, $\alpha, \beta \in (0, \epsilon)$, $\alpha \neq \beta$, and $0 < \lambda < 1$. If this inequality is strict, then $f$ is said to be strictly convex. It is proved that $\{f_s\}_{s \in C}$ satisfies the condition (C) if $f_s$ is strictly convex for all $s \in C$.

**Theorem 3.5 (Bavard).** *Assume $f_s$ is convex for all $s \in C$. Then a point $v \in V$ is strictly extreme if and only if $v$ is perfect and eutactic.*

**Example 3.3.** We consider the subset $P_n^1 = \{a \in P_n : \det a = 1\}$ of $P_n$, which is identified with the Riemannian symmetric space $SL_n(\mathbf{R})/SO_n(\mathbf{R})$. For $x \in \mathbf{Z}^n \setminus \{0\}$, define the length function $f_x : P_n^1 \longrightarrow \mathbf{R}$ by $f_x(a) = {}^t x a x$. The family $\{f_x\}_{x \in \mathbf{Z}^n \setminus \{0\}}$ satisfies ($\mathrm{B}_1$) for $\Gamma = SL_n(\mathbf{Z})$, ($\mathrm{B}_2$) and the condition (C) ( [9, Example 1]). Thus one can apply Theorem 3.4 to $\mathcal{E} = (P_n^1, SL_n(\mathbf{Z}), \mathbf{Z}^n \setminus \{0\}, \{f_x\})$. Since the definition of perfection and eutaxy of Definition 3.2 is equivalent to that of Definition 1.3, this case verifies Voronoï's theorem. The length function $f_x$ is convex on $P_n^1$ for all $x \in \mathbf{Z}^n \setminus \{0\}$.

**Example 3.4.** Let $G$ be a connected Lie subgroup of $SL_n(\mathbf{R})$ and $G \cdot \mathrm{I}_n$ be the $G$-orbit of the identity matrix $\mathrm{I}_n$ in $P_n^1$, i.e., $G \cdot \mathrm{I}_n = \{{}^t g g : g \in G\}$. Assume $G$ is invariant by the transpose $g \mapsto {}^t g$. Then $G \cdot \mathrm{I}_n$ is totally geodesic, and hence the restriction $f_x|_{G \cdot \mathrm{I}_n}$ of the length function $f_x$ to $G \cdot \mathrm{I}_n$ is convex for all $x \in \mathbf{Z}^n \setminus \{0\}$. Thus one can apply Theorem 3.5 to $\mathcal{E} = (G \cdot \mathrm{I}_n, G \cap SL_n(\mathbf{Z}), \mathbf{Z}^n \setminus \{0\}, \{f_x|_{G \cdot \mathrm{I}_n}\})$.

**Example 3.5.** Assume $n$ is even. Let $G$ be the symplectic group, i.e.,

$$G = Sp_n(\mathbf{R}) = \left\{ g \in SL_n(\mathbf{R}) \ : \ {}^t g \begin{pmatrix} 0 & -\mathrm{I}_{n/2} \\ \mathrm{I}_{n/2} & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathrm{I}_{n/2} \\ \mathrm{I}_{n/2} & 0 \end{pmatrix} \right\}.$$

In this case, the family $\{f_x|_{G \cdot \mathrm{I}_n}\}_{x \in \mathbf{Z}^n \setminus \{0\}}$ satisfies the condition (C). This is a particular case of more general family [1, Lemme 3.3]. See [1, Théorème 3.1] for other symmetric spaces of classical type.

**Example 3.6.** For $X \in M^*_{n,k}(\mathbf{Z})$, define the length function $f_X : P^1_n \longrightarrow \mathbf{R}$ by $f_X(a) = \det({}^tXaX)$, i.e., $f_X = D^k_X$. Then the family $\{f_X\}_{X \in M^*_{n,k}(\mathbf{Z})}$ satisfies the condition (C) ( [9, Proposition 2.8]). Theorem 2.1 is verified again by Theorem 3.4 specialized to $\mathcal{E} = (P^1_n, SL_n(\mathbf{Z}), M^*_{n,k}(\mathbf{Z}), \{f_X\})$.

**Example 3.7.** We define the subset $C_{n,k}$ of $M_{n,k}(\mathbf{Z}) \times M_{n,k}(\mathbf{Z})$ by

$$C_{n,k} = \{(X, 0), (0, Y) \ : \ X, Y \in M^*_{n,k}(\mathbf{Z})\}\,.$$

This set is stable by the action of $SL_n(\mathbf{Z})$: $(X, Y)g = ({}^tgX, g^{-1}Y)$. For $(X, Y) \in C_{n,k}$, define the length function $f_{(X,Y)} : P_n \longrightarrow \mathbf{R}$ by $f_{(X,Y)}(a) = D_X(a)^k + D_Y(a^{-1})^k$ for $a \in P_n$. Then the quadruplet $\mathcal{E} = (P_n, SL_n(\mathbf{Z}), C_{n,k}, \{f_{(X,Y)}\})$ satisfies the condition (C) [11, Théorèm 5]. Bavard proved the set of extreme points of $k$-th Bergé–Martinet invariant $F^{(k)}_{\mathrm{BM}}$ coincides with that of $F_{\mathcal{E}}$ ( [11, Proposition 2.21]). Thus Voronoï's theorem for $F^{(k)}_{\mathrm{BM}}$ results in that of $F_{\mathcal{E}}$.

In some cases, the finiteness and the algebraicity of perfect points were also proved by Bavard [11, Corollaire 2.12 et Théorème 1]. We explain the simplest case of Bavard's result. Let $\mathsf{f}$ be one of the real number field $\mathbf{R}$, the complex number field $\mathbf{C}$ or the Hamilton quaternion field $\mathbf{H}$. For an $n \times n$ matrix $(\lambda_{ij}) \in M_n(\mathsf{f})$ with entries in $\mathsf{f}$, we write $(\lambda_{ij})^*$ for ${}^t(\overline{\lambda}_{ij})$, where $\lambda \mapsto \overline{\lambda}$ stands for the main involution of $\mathsf{f}$. The set $P^1_n(\mathsf{f}) = \{g^*g \ : g \in SL_n(\mathsf{f})\}$ is a Riemannian symmetric space. We fix a subring $\mathsf{o}_\mathsf{f}$ in $\mathsf{f}$ as $\mathsf{o}_\mathsf{f} = \mathbf{Z}$ if $\mathsf{f} = \mathbf{R}$, $\mathsf{o}_\mathsf{f} = \mathbf{Z}[\sqrt{-1}]$ if $\mathsf{f} = \mathbf{C}$ and $\mathsf{o}_\mathsf{f} = \mathbf{Z}[\mathbf{i}, \mathbf{j}, \mathbf{k}]$ if $\mathsf{f} = \mathbf{H}$, where $\{1, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ denotes the usual quaternion basis of $\mathbf{H}$. For $x \in \mathsf{o}^n_\mathsf{f} \setminus \{0\}$, define the length function $f_x : P^1_n(\mathsf{f}) \longrightarrow \mathbf{R}$ by $f_x(a) = x^*ax$. Then $\mathcal{E} = (P^1_n(\mathsf{f}), SL_n(\mathsf{o}_\mathsf{f}), \mathsf{o}^n_\mathsf{f} \setminus \{0\}, \{f_x\})$ satisfies the condition (C) ( [1, Corollaire 3.1]).

**Theorem 3.6 (Bavard).** *For $\mathcal{E} = (P^1_n(\mathsf{f}), SL_n(\mathsf{o}_\mathsf{f}), \mathsf{o}^n_\mathsf{f} \setminus \{0\}, \{f_x\})$, the number of perfect points in $P^1_n(\mathsf{f})$ modulo $SL_n(\mathsf{o}_\mathsf{f})$ is finite. Any perfect point in $P^1_n(\mathsf{f})$ is algebraic over $\mathbf{Q}$, i.e., which is contained in $M_n(\mathsf{o}_\mathsf{f} \otimes_\mathbf{Z} \overline{\mathbf{Q}})$.*

More generally, Bavard proved such result for some totally geodesic subvarieties in $P^1_n(\mathsf{f})$. However the algebraicity of $\gamma'_{n,k}$ for $k \geq 2$ is still unknown.

In connection with Ash's mass formula (Theorem 3.3), we append Bavard's mass formula [10, Théorème 1]. We recall $\partial^{wr}K_1(m_1)$ denotes the set of all well-rounded points in $\partial K_1(m_1)$. If $a \in \partial^{wr}K_1(m_1)$, then $\mathcal{F}_{S(a)}$ is a compact face. The family $\{\mathcal{F}_{S(a)}\}_{a \in \partial^{wr}K_1(m_1)}$ of compact faces has only finitely many $SL_n(\mathbf{Z})$-orbits by Theorem 3.2. Let $\{\mathcal{F}_1, \cdots, \mathcal{F}_r\}$ be the complete set of representatives of $SL_n(\mathbf{Z})$-orbits in $\{\mathcal{F}_{S(a)}\}_{a\in\partial^{w r} K_1(m_1)}$.

**Theorem 3.7 (Bavard).** *$\{\mathcal{F}_1, \cdots, \mathcal{F}_r\}$ satisfies*

$$\sum_{i=1}^{r} \frac{(-1)^{\dim \mathcal{F}_i}}{\sharp\Gamma_i} = \chi(SL_n(\mathbf{Z})) = \begin{cases} -1/12 & (n=2) \\ 0 & (n\geq 3) \end{cases},$$

*where $\Gamma_i$ stands for the stabilizer of $\mathcal{F}_i$ in $SL_n(\mathbf{Z})$.*

For a further study of this mass formula, see [15].

### 3.3. *Voronoï's theorem over an algebraic number field I*

There are two methods of an extension of the base field. One is the additive generalization (35], [38], [45], [46]) and another is the multiplicative generalization ([22], [32]). Both methods give the original Voronoï's theorem if the base field is $\mathbf{Q}$. We first explain the additive generalization.

Let $\mathsf{k}$ be an algebraic number field of degree $r$ and $\mathsf{o}_\mathsf{k}$ the ring of integers of $\mathsf{k}$. The set of all infinite (resp. real and imaginary) places of $\mathsf{k}$ is denoted by $\mathsf{p}_\infty$ (resp. $\mathsf{p}_1$ and $\mathsf{p}_2$). Let $\mathsf{k}_\sigma$ be the completion of $\mathsf{k}$ at $\sigma\in\mathsf{p}_\infty$, i.e., $\mathsf{k}_\sigma = \mathbf{R}$ if $\sigma\in\mathsf{p}_1$ and $\mathsf{k}_\sigma=\mathbf{C}$ if $\sigma\in\mathsf{p}_2$. We use the étale $\mathbf{R}$-algebra $\mathsf{k}_\mathbf{R} = \mathsf{k}\otimes_\mathbf{Q}\mathbf{R}$, which is identified with $\prod_{\sigma\in\mathsf{p}_\infty}\mathsf{k}_\sigma$. For $\boldsymbol{x}=(x_\sigma)\in\mathsf{k}_\mathbf{R}$, the conjugate $\overline{\boldsymbol{x}}$ of $\boldsymbol{x}$ is defined to be $\overline{\boldsymbol{x}}=(\overline{x}_\sigma)$, where $\overline{x}_\sigma$ is the complex conjugate of $x_\sigma$. The trace and the norm of $\mathsf{k}_\mathbf{R}$ are defined as

$$\mathrm{Tr}_{\mathsf{k}_\mathbf{R}}(\boldsymbol{x}) = \sum_{\sigma\in\mathsf{p}_\infty}\mathrm{Tr}_{\mathsf{k}_\sigma/\mathbf{R}}(x_\sigma), \qquad \mathrm{Nr}_{\mathsf{k}_\mathbf{R}}(\boldsymbol{x}) = \prod_{\sigma\in\mathsf{p}_\infty}\mathrm{Nr}_{\mathsf{k}_\sigma/\mathbf{R}}(x_\sigma)$$

for $\boldsymbol{x}=(x_\sigma)\in\mathsf{k}_\mathbf{R}$.

Let $\mathsf{k}_\mathbf{R}^n = \mathsf{k}^n\otimes_\mathbf{Q}\mathbf{R}$ be the $\mathsf{k}_\mathbf{R}$-module of rank $n$. An element of $\mathsf{k}_\mathbf{R}^n$ is denoted by a column vector $X={}^t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n)$ with $\boldsymbol{x}_i\in\mathsf{k}_\mathbf{R}$, $i=1,\cdots,n$. The group consisting of $\mathsf{k}_\mathbf{R}$-linear automorphisms of $\mathsf{k}_\mathbf{R}^n$ is denoted by $GL_n(\mathsf{k}_\mathbf{R})$, which is identified with $\prod_{\sigma\in\mathsf{p}_\infty}GL_n(\mathsf{k}_\sigma)$. As an $\mathbf{R}$-vector space, $\mathsf{k}_\mathbf{R}^n$ is equipped with the inner product

$$(X,Y) = \mathrm{Tr}_{\mathsf{k}_\mathbf{R}}({}^t\overline{X}Y) = \mathrm{Tr}_{\mathsf{k}_\mathbf{R}}(\overline{\boldsymbol{x}}_1\boldsymbol{y}_1+\cdots+\overline{\boldsymbol{x}}_n\boldsymbol{y}_n),$$

for $X={}^t(\boldsymbol{x}_1,\cdots,\boldsymbol{x}_n), Y={}^t(\boldsymbol{y}_1,\cdots,\boldsymbol{y}_n)\in\mathsf{k}_\mathbf{R}^n$. The group of isometries

$$O_n(\mathsf{k}_\mathbf{R}) = \{g\in GL_n(\mathsf{k}_\mathbf{R}) \ : \ (gX,gY)=(X,Y) \ \text{ for all } X,Y\in\mathsf{k}_\mathbf{R}^n\}$$

is a maximal compact subgroup of $GL_n(\mathsf{k}_{\mathbf{R}})$. We define the subsets $V_n(\mathsf{k}_{\mathbf{R}})$ and $P_n(\mathsf{k}_{\mathbf{R}})$ of $M_n(\mathsf{k}_{\mathbf{R}})$ as follows:

$$\begin{aligned} V_n(\mathsf{k}_{\mathbf{R}}) &= \{a \in M_n(\mathsf{k}_{\mathbf{R}}) \ : \ (aX, Y) = (X, aY) \ \text{ for all } X, Y \in \mathsf{k}_{\mathbf{R}}^n\}\,, \\ P_n(\mathsf{k}_{\mathbf{R}}) &= \{a \in V_n(\mathsf{k}_{\mathbf{R}}) \ : \ (aX, X) > 0 \ \text{ for all } X \in \mathsf{k}_{\mathbf{R}}^n \setminus \{0\}\}\,. \end{aligned}$$

The set $V_n(\mathsf{k}_{\mathbf{R}})$ is an $\mathbf{R}$-subspace of $M_n(\mathsf{k}_{\mathbf{R}})$ of dimension $n(n+1)\sharp\mathsf{p}_1/2 + n^2\sharp\mathsf{p}_2$, and $P_n(\mathsf{k}_{\mathbf{R}})$ is a symmetric cone in $V_n(\mathsf{k}_{\mathbf{R}})$. Let $V_n(\mathsf{k}_{\mathbf{R}})^*$ be the dual space of $V_n(\mathsf{k}_{\mathbf{R}})$ as an $\mathbf{R}$-vector space. The trace $\mathrm{Tr}_{M_n(\mathsf{k}_{\mathbf{R}})} \in V_n(\mathsf{k}_{\mathbf{R}})^*$ is defined to be the composition of the matrix trace tr and $\mathrm{Tr}_{\mathsf{k}_{\mathbf{R}}}$, i.e, $\mathrm{Tr}_{M_n(\mathsf{k}_{\mathbf{R}})} = \mathrm{Tr}_{\mathsf{k}_{\mathbf{R}}} \circ \mathrm{tr}$. For $X \in \mathsf{k}_{\mathbf{R}}^n$, define the linear form $\varphi_X \in V_n(\mathsf{k}_{\mathbf{R}})^*$ by $\varphi_X(a) = (aX, X)$ for $a \in V_n(\mathsf{k}_{\mathbf{R}})$.

An $\mathsf{o}_\mathsf{k}$-submodule $\Lambda$ in $\mathsf{k}_{\mathbf{R}}^n$ is called an $\mathsf{o}_\mathsf{k}$-lattice if $\Lambda$ is discrete and $\Lambda \otimes_{\mathbf{Z}} \mathbf{R} = \mathsf{k}_{\mathbf{R}}^n$. Any projective $\mathsf{o}_\mathsf{k}$-module in $\mathsf{k}^n$ of rank $n$ is regarded as an $\mathsf{o}_\mathsf{k}$-lattice in $\mathsf{k}_{\mathbf{R}}^n$ by the natural inclusion $\mathsf{k}^n \subset \mathsf{k}_{\mathbf{R}}^n$. Conversely, for any $\mathsf{o}_\mathsf{k}$-lattice $\Lambda$ in $\mathsf{k}_{\mathbf{R}}^n$, there exists $g \in GL_n(\mathsf{k}_{\mathbf{R}})$ such that $g^{-1}\Lambda$ is a projective $\mathsf{o}_\mathsf{k}$-module in $\mathsf{k}^n$ (see e.g., [37, Lemma 3.2]). Thus, by Steinitz's theorem, any $\mathsf{o}_\mathsf{k}$-lattice is isomorphic with an $\mathsf{o}_\mathsf{k}$-module of the form $\mathsf{o}_\mathsf{k}^{n-1} \oplus \mathsf{q}$, where $\mathsf{q}$ is an ideal of $\mathsf{o}_\mathsf{k}$. Let $\mathsf{q}_1 = \mathsf{o}_\mathsf{k}, \mathsf{q}_2, \cdots, \mathsf{q}_h$ be a complete system of representatives of the ideal class group of $\mathsf{k}$. If $\mathcal{H}_i$ denotes the $GL_n(\mathsf{k}_{\mathbf{R}})$-orbit of the $\mathsf{o}_\mathsf{k}$-lattice $\mathsf{o}_\mathsf{k}^{n-1} \oplus \mathsf{q}_i$, then the set $\mathcal{H}(\mathsf{k}_{\mathbf{R}}^n)$ of all $\mathsf{o}_\mathsf{k}$-lattices in $\mathsf{k}_{\mathbf{R}}^n$ is given by the disjoint union of $\mathcal{H}_1, \cdots, \mathcal{H}_h$:

$$\mathcal{H}(\mathsf{k}_{\mathbf{R}}^n) = \bigsqcup_{i=1}^{h} \mathcal{H}_i\,.$$

Each component $\mathcal{H}_i$ is identified with $GL_n(\mathsf{k}_{\mathbf{R}})/GL(\mathsf{o}_\mathsf{k}^{n-1} \oplus \mathsf{q}_i)$, where $GL(\mathsf{o}_\mathsf{k}^{n-1} \oplus \mathsf{q}_i)$ denotes the stabilizer of $\mathsf{o}_\mathsf{k}^{n-1} \oplus \mathsf{q}_i$ in $GL_n(\mathsf{k}_{\mathbf{R}})$.

For $\Lambda \in \mathcal{H}(\mathsf{k}_{\mathbf{R}}^n)$, the minimum $m_+(\Lambda)$ and the discriminant $\mathrm{disc}(\Lambda)$ of $\Lambda$ are defined to be

$$m_+(\Lambda) = \min_{X \in \Lambda \setminus \{0\}} (X, X)\,, \qquad \mathrm{disc}(\Lambda) = \left( \frac{\omega(\mathsf{k}_{\mathbf{R}}^n/\Lambda)}{\omega(\mathsf{k}_{\mathbf{R}}^n/\mathsf{o}_\mathsf{k}^n)} \right)^2\,,$$

where $\omega$ denotes an invariant measure on $\mathsf{k}_{\mathbf{R}}^n$. We denote by $S_+(\Lambda)$ the set of shortest vectors in $\Lambda$, i.e.,

$$S_+(\Lambda) = \{X \in \Lambda \ : \ (X, X) = m_+(\Lambda)\}\,.$$

As an analog of the Hermite invariant, we consider the function $F_+ : \mathcal{H}(\mathsf{k}_{\mathbf{R}}^n) \longrightarrow \mathbf{R}_{>0}$ defined by

$$F_+(\Lambda) = \frac{m_+(\Lambda)}{\mathrm{disc}(\Lambda)^{1/(rn)}}\,.$$

Obviously, $F_+$ depends only on the similar isometry class $\mathbf{R}^\times O_n(\mathsf{k}_\mathbf{R})\Lambda$ of $\Lambda$, i.e., $F_+$ is a function on $\mathbf{R}^\times O_n(\mathsf{k}_\mathbf{R})\backslash\mathcal{H}(\mathsf{k}_\mathbf{R}^n)$. An $\mathsf{o}_\mathsf{k}$-lattice $\Lambda \in \mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ is said to be extreme if $F_+$ attains a local maximum on $\mathbf{R}^\times O_n(\mathsf{k}_\mathbf{R})\Lambda$.

**Definition 3.3.** An $\mathsf{o}_\mathsf{k}$-lattice $\Lambda \in \mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ is said to be perfect if $\{\varphi_X\}_{X\in S_+(\Lambda)}$ spanns $V_n(\mathsf{k}_\mathbf{R})^*$, and $\Lambda$ is said to be eutactic if there are $\rho_X \in \mathbf{R}_{>0}$, $X \in S_+(\Lambda)$, such that

$$\mathrm{Tr}_{M_n(\mathsf{k}_\mathbf{R})} = \sum_{X\in S_+(\Lambda)} \rho_X \varphi_X \,.$$

Leibak [38] proved a weak version of Voronoï's theorem for $F_+$ restricted to the component $\mathcal{H}_1$ of free $\mathsf{o}_\mathsf{k}$-lattices. Leibak's definition of eutaxy is weaker than that of Definition 3.3. Okuda and Yano [45] found a suitable definition of eutaxy to complete Leibak's result.

**Theorem 3.8 (Leibak, Okuda and Yano).** *An $\mathsf{o}_\mathsf{k}$-lattice $\Lambda \in \mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ is extreme if and only if $\Lambda$ is perfect and eutactic.*

By Humbert's reduction theory and Cramer's formula, one has the following finiteness and algebraicity of perfect $\mathsf{o}_\mathsf{k}$-lattices.

**Theorem 3.9 (Okuda and Yano).** *The number of similar isometry classes of perfect $\mathsf{o}_\mathsf{k}$-lattices in $\mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ is finite. Let $\Lambda \in \mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ be a perfect $\mathsf{o}_\mathsf{k}$-lattice with $m_+(\Lambda) = 1$. If $g \in GL_n(\mathsf{k}_\mathbf{R})$ such that $g^{-1}\Lambda \subset \mathsf{k}^n$, then ${}^t\overline{g}_\sigma g_\sigma \in M_n(\mathsf{k}')$ for all $\sigma \in \mathsf{p}_\infty$, where $\mathsf{k}'$ is the Galois closure of $\mathsf{k}$ over $\mathbf{Q}$.*

In the case that $\mathsf{k}$ is a real quadratic field, a classification of some perfect $\mathsf{o}_\mathsf{k}$-lattices of small rank was given by Ong [46] and Leibak [39].

Koecher studied the function $F_+$ in connection with the reduction theory of $P_n(\mathsf{k}_\mathbf{R})$ and proved the finiteness of the number of similar isometry classes of perfect $\mathsf{o}_\mathsf{k}$-lattices in $\mathcal{H}_1$ ( [35, §9, 10]). The bound $\sharp(S_+(\Lambda)) \leq 2(2^{rn}-1)$ for all $\Lambda \in \mathcal{H}(\mathsf{k}_\mathbf{R}^n)$ is proved by the similar way as in **2.3** (see [35, Lemma 12]).

### 3.4. *Voronoï's theorem over an algebraic number field II*

We use the same notation as in the previous section. For $a \in P_n(\mathsf{k}_\mathbf{R})$, the multiplicative minimum $m_*(a)$ and the discriminant $\mathrm{disc}(a)$ are defined to be

$$m_*(a) = \min_{X\in\mathsf{o}_\mathsf{k}^n\backslash\{0\}} \mathrm{Nr}_{\mathsf{k}_\mathbf{R}}({}^t\overline{X}aX)\,, \qquad \mathrm{disc}(a) = \mathrm{Nr}_{\mathsf{k}_\mathbf{R}}(\det a)\,.$$

We denote by $S_*(a)$ the set of minimal integral vectors, i.e.,

$$S_*(a) = \{X \in \mathsf{o}_\mathsf{k}^n \ : \ \mathrm{Nr}_{\mathsf{k}_\mathbf{R}}({}^t\overline{X}aX) = m_*(a)\}\,.$$

The unit group $\mathsf{o}_\mathsf{k}^\times$ acts on $S_*(a)$ by multiplication. The set $S_*(a)/\mathsf{o}_\mathsf{k}^\times$ of classes is finite ( [32, Lemma 1]). Define the function $F_* \,:\, P_n(\mathsf{k}_\mathbf{R}) \longrightarrow \mathbf{R}_{>0}$ by

$$F_*(a) = \frac{m_*(a)}{\mathrm{disc}(a)^{1/n}}\,.$$

The group $GL_n(\mathsf{k}_\mathbf{R})$ acts transitively on $P_n(\mathsf{k}_\mathbf{R})$ by $(a,g) \mapsto {}^t\overline{g}ag$ for $a \in P_n(\mathsf{k}_\mathbf{R})$ and $g \in GL_n(\mathsf{k}_\mathbf{R})$. By definition, $F_*$ is invariant by the action of the discrete subgroup $GL_n(\mathsf{o}_\mathsf{k}) \subset GL_n(\mathsf{k}_\mathbf{R})$, i.e., one has

$$F_*({}^t\overline{g}ag) = F_*(a)$$

for all $a \in P_n(\mathsf{k}_\mathbf{R})$ and $g \in GL_n(\mathsf{o}_\mathsf{k})$. Moreover, if we set

$$\mathsf{k}_\mathbf{R}^+ = \{x = (x_\sigma)_{\sigma\in\mathsf{p}_\infty} \in \mathsf{k}_\mathbf{R}^\times \ : \ x_\sigma \in \mathbf{R}_{>0} \ \text{ for all } \sigma \in \mathsf{p}_\infty\}\,,$$

then $F_*(xa) = F_*(a)$ holds for all $a \in P_n(\mathsf{k}_\mathbf{R})$ and $x \in \mathsf{k}_\mathbf{R}^+$. Therefore, $F_*$ is considered as a function on $\mathsf{k}_\mathbf{R}^+\backslash P_n(\mathsf{k}_\mathbf{R})/GL_n(\mathsf{o}_\mathsf{k})$. An element $a \in P_n(\mathsf{k}_\mathbf{R})$ is said to be extreme if $F_*$ attains a local maximum on the class $\mathsf{k}_\mathbf{R}^+ aGL_n(\mathsf{o}_\mathsf{k})$.

Let $a \in P_n(\mathsf{k}_\mathbf{R})$ and $X \in \mathsf{k}^n \setminus \{0\}$. Then ${}^t\overline{X}aX$ is invertible in $\mathsf{k}_\mathbf{R}^\times$. The map $b \mapsto (bX, ({}^t\overline{X}aX)^{-1}X)$ on $V_n(\mathsf{k}_\mathbf{R})$ defines an $\mathbf{R}$-linear form. We write $\varphi_X^a$ for this linear form. For $x = (x_\sigma) \in \mathsf{k}_\mathbf{R}^+$, we set $\log(x) = (\log(x_\sigma)) \in \mathsf{k}_\mathbf{R}$. From the definition, it follows $\varphi_X^a(\log(x)a) = \mathrm{Tr}_{\mathsf{k}_\mathbf{R}}(\log(x))$ for $x \in \mathsf{k}_\mathbf{R}^+$. We denote by $\mathsf{k}_\mathbf{R}^1$ the subset of $x \in \mathsf{k}_\mathbf{R}^+$ such that $\mathrm{Tr}_{\mathsf{k}_\mathbf{R}}(\log(x)) = 0$, i.e.,

$$\mathsf{k}_\mathbf{R}^1 = \{x = (x_\sigma)_{\sigma\in\mathsf{p}_\infty} \in \mathsf{k}_\mathbf{R}^+ \ : \ \mathrm{Nr}_{\mathsf{k}_\mathbf{R}}(x) = 1\}\,.$$

Thus $\varphi_X^a$ is null on the $(\sharp(\mathsf{p}_1)+\sharp(\mathsf{p}_2)-1)$-dimensional subspace $\log(\mathsf{k}_\mathbf{R}^1)\cdot a \subset V_n(\mathsf{k}_\mathbf{R})$.

**Definition 3.4.** An element $a \in P_n(\mathsf{k}_\mathbf{R})$ is said to be perfect if $\{\varphi_X^a\}_{[X]\in S_*(a)/\mathsf{o}_\mathsf{k}^\times}$ spanns the dual space $(V_n(\mathsf{k}_\mathbf{R})/\log(\mathsf{k}_\mathbf{R}^1)\cdot a)^*$, and $a$ is said to be eutactic if there are $\rho_X \in \mathbf{R}_{>0}$, $[X] \in S_*(a)/\mathsf{o}_\mathsf{k}^\times$, such that the linear form $b \mapsto \mathrm{Tr}_{M_n(\mathsf{k}_\mathbf{R})}(a^{-1}b)$ on $V_n(\mathsf{k}_\mathbf{R})$ is represented as

$$(b \mapsto \mathrm{Tr}_{M_n(\mathsf{k}_\mathbf{R})}(a^{-1}b)) = \sum_{[X]\in S_*(a)/\mathsf{o}_\mathsf{k}^\times} \rho_X\varphi_X^a\,.$$

This definition is due to Coulangeon. Icaza [32, Proposition 3] first proved a weak version of Voronoï's theorem for $F_*$, and later Coulangeon [23] completed a full version.

**Theorem 3.10 (Icaza, Coulangeon).** *An element $a \in P_n(\mathsf{k}_{\mathbf{R}})$ is extreme if and only if $a$ is perfect and eutactic.*

The finiteness and the algebraicity of perfect points were also proved by Coulangeon [23, Proposition 4.1].

**Theorem 3.11 (Coulangeon).** *The number of perfect elements in $P_n(\mathsf{k}_{\mathbf{R}})$ modulo $\mathsf{k}_{\mathbf{R}}^{+} GL_n(\mathsf{o}_{\mathsf{k}})$ is finite. If $a = (a_\sigma) \in P_n(\mathsf{k}_{\mathbf{R}})$ is perfect, then there exists $x = (x_\sigma) \in \mathsf{k}_{\mathbf{R}}^{+}$ such that $x_\sigma a_\sigma \in M_n(\overline{\mathbf{Q}})$ for all $\sigma \in \mathsf{p}_\infty$.*

Theorem 3.10 is also verified by Bavard's theory. We set $P_n(\mathsf{k}_{\mathbf{R}})^1 = \{a \in P_n(\mathsf{k}_{\mathbf{R}}) \ : \ \mathrm{disc}(a) = 1\}$, and for $X \in \mathsf{o}_{\mathsf{k}}^n \setminus \{0\}$ define the length function $f_X \ : \ P_n(\mathsf{k}_{\mathbf{R}})^1 \longrightarrow \mathbf{R}$ by $f_X(a) = \log(\mathrm{Nr}_{\mathsf{k}_{\mathbf{R}}}({}^t\overline{X} a X))$. Let $\mathcal{E} = (P_n(\mathsf{k}_{\mathbf{R}})^1, SL_n(\mathsf{o}_{\mathsf{k}}), \mathsf{o}_{\mathsf{k}}^n \setminus \{0\}, \{f_X\})$. Then Bavard [11, Proposition 2.22] proved that the definition of perfection and eutaxy in Definition 3.4 coincides with that in Definition 3.2 for $\mathcal{E}$, and $\mathcal{E}$ satisfies the condition (C). In contrast to Theorem 3.2, the number of classes in $\mathsf{k}_{\mathbf{R}}^{+} \backslash P_n(\mathsf{k}_{\mathbf{R}})/GL_n(\mathsf{o}_{\mathsf{k}})$ of eutactic elements is not finite in general if $\mathsf{k} \neq \mathbf{Q}$, see [23, p.162]. To resolve this problem, Bavard introduced the notion of non-degenerate points in his framework ( [11, Définition 1.5]), and proved that the number of classes of non-degenerate eutactic elements is finite ( [11, Proposition 2.24]).

If $\mathsf{k}$ is an imaginary quadratic field, then $F_*$ is essentially the same as $F_+$ restricted to $\mathcal{H}_1$. More precisely, we have

$$4F_*({}^t\overline{g}g) = F_+(g\mathsf{o}_{\mathsf{k}}^n)^2$$

for all $g \in GL_n(\mathsf{k}_{\mathbf{R}}) = GL_n(\mathbf{C})$. In particular, the number of $S_*(a)/\mathsf{o}_{\mathsf{k}}^\times$ is bounded by $2(4^n - 1)/\sharp(\mathsf{o}_{\mathsf{k}}^\times)$. In general, any estimate of the number of $S_*(a)/\mathsf{o}_{\mathsf{k}}^\times$ is unknown.

**Problem 3.2.** *Bound the maximum* $\max_{a \in P_n(\mathsf{k}_{\mathbf{R}})} \sharp(S_*(a)/\mathsf{o}_{\mathsf{k}}^\times)$.

The assertion of Theorem 3.10 is true even if the free $\mathsf{o}_{\mathsf{k}}$-lattice $\mathsf{o}_{\mathsf{k}}^n$ in the definitions of $m_*$ and $S_*(a)$ is replaced with a general $\mathsf{o}_{\mathsf{k}}$-lattice $\Lambda \in \mathcal{H}(\mathsf{k}_{\mathbf{R}}^n)$. This was verified by Meyer [41, Théorème 3.21] in more general setting.

## 4. Generalized Hermite constants of flag varieties

A generalization of Hermite's constant to algebraic groups was studied in [63] and [64]. The main problem in this theory is to formulate and verify Voronoï type theorems. This problem was completely solved by Meyer [41], [42] in the case of $GL_n$. Some inner forms of $GL_n$ were studied in [24]. It is likely that Bavard's theory applies to many cases, e.g., see Example 4.2

below. However, to approach adelic Voronoï theorems for the generalized Hermite constants involving positive characteristic cases, we will need a suitable definition of perfection for our adelic setting.

### 4.1. *Generalized Hermite constants*

Let $G$ be a connected affine algebraic group defined over $\mathbf{Q}$. For any $\mathbf{Q}$-algebra $A$, $G(A)$ stands for the group of $A$-rational points of $G$. In particular, $G(\mathbf{A})$ denotes the adele group of $G$. Let $\mathbf{X}^*_{\mathbf{Q}}(G)$ be the module of $\mathbf{Q}$-rational characters of $G$. We denote by $G(\mathbf{A})^1$ the subgroup $\{g \in G(\mathbf{A}) \ : \ |\chi(g)|_{\mathbf{A}} = 1 \quad \text{for all } \chi \in \mathbf{X}^*_{\mathbf{Q}}(G)\}$, where $|\cdot|_{\mathbf{A}}$ denotes the usual idele norm of the idele group of $\mathbf{Q}$. By the product formula of the idele norm, $G(\mathbf{Q})$ is contained in $G(\mathbf{A})^1$.

In the following, let $G$ be a connected reductive algebraic group defined over $\mathbf{Q}$. We fix a minimal $\mathbf{Q}$-parabolic subgroup $P$ of $G$ and a Levi subgroup $M_P$ of $P$. The maximal central $\mathbf{Q}$-split torus $Z_P$ of $M_P$ is a maximal $\mathbf{Q}$-split torus of $G$. We choose a maximal $\mathbf{Q}$-parabolic subgroup $Q$ of $G$ and its Levi subgroup $M_Q$ such that $P \subset Q$ and $M_P \subset M_Q$. Let $Z_G$ be the maximal central $\mathbf{Q}$-split torus of $G$. Since $Q$ is maximal, the module $\mathbf{X}^*_{\mathbf{Q}}(M_Q/Z_G)$ is of rank one, and hence there is a unique generator $\widehat{\alpha}_Q$ of $\mathbf{X}^*_{\mathbf{Q}}(M_Q/Z_G)$ such that the restriction of $\widehat{\alpha}_Q$ to $Z_P/Z_G$ is a positive scalar multiple of a positive simple root with respect to $(P, Z_P)$. Let $U_Q$ be the unipotent radical of $Q$, and let $K$ be a maximal compact subgroup of $G(\mathbf{A})$ such that $G(\mathbf{A}) = P(\mathbf{A})K$. Then the height function $H_Q : G(\mathbf{A}) \longrightarrow \mathbf{R}_{>0}$ is defined by

$$H_Q(umh) = |\widehat{\alpha}_Q(m)|_{\mathbf{A}}^{-1}$$

for $u \in U_Q(\mathbf{A})$, $m \in M_Q(\mathbf{A})$ and $h \in K$. Indeed $H_Q$ is a function on the space $Z_G(\mathbf{A})Q(\mathbf{A})^1\backslash G(\mathbf{A}) = Q(\mathbf{A})^1\backslash G(\mathbf{A})^1$. Define the function $F_Q : G(\mathbf{A}) \longrightarrow \mathbf{R}_{>0}$ by

$$F_Q(g) = \min_{[v]\in Q(\mathbf{Q})\backslash G(\mathbf{Q})} H_Q(vg)\,.$$

The generalized Hermite constant $\gamma_Q$ of $Q\backslash G$ is defined to be the maximum

$$\gamma_Q = \max_{[g]\in Z_G(\mathbf{A})G(\mathbf{Q})\backslash G(\mathbf{A})/K} F_Q(g) = \max_{[g]\in G(\mathbf{Q})\backslash G(\mathbf{A})^1/K} F_Q(g)\,.$$

We assume the following two conditions for $G$ and $Q$:

($C_1$) $G(\mathbf{A}) = G(\mathbf{Q})G(\mathbf{R})K$.
($C_2$) $G(\mathbf{Q}) = Q(\mathbf{Q})G(\mathbf{Z})$, where $G(\mathbf{Z}) = G(\mathbf{Q}) \cap G(\mathbf{R})K$.

The condition ($C_1$) means that $G$ is of class number one. By [17, Proposition 7.5], the condition ($C_2$) is satisfied if $M_Q$ is of class number one. Then $\gamma_Q$ is represented as

$$\gamma_Q = \max_{[g]\in G(\mathbf{Z})\backslash G(\mathbf{R})/K_\infty} F_Q(g) = \max_{[g]\in G(\mathbf{Z})\backslash G(\mathbf{R})/K_\infty} \min_{\gamma\in G(\mathbf{Z})} H_Q^\infty(\gamma g)\,,$$

where $K_\infty$ and $H_Q^\infty$ denote the infinite components of $K$ and $H_Q$, respectively.

**Example 4.1.** For $k = 1, \cdots, n-1$, let

$$R_k(\mathbf{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;:\; a \in GL_k(\mathbf{Q}),\ b \in M_{k,n-k}(\mathbf{Q}),\ d \in GL_{n-k}(\mathbf{Q}) \right\}.$$

Then $R_k$ is a maximal $\mathbf{Q}$-parabolic subgroup of $GL_n$. It is well-known that $GL_n$ and $R_k$ satisfy both conditions ($C_1$) and ($C_2$). The character $\widehat{\alpha}_{R_k}$ is given by

$$\widehat{\alpha}_{R_k}\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = (\det a)^{(n-k)/\gcd(k,n-k)}(\det d)^{-k/\gcd(k,n-k)}\,.$$

For $\gamma \in GL_n(\mathbf{Z})$, $X_\gamma$ denotes the $n$ by $k$ matrix consisting of the first $k$-colums of $\gamma$. It is an easy exercise to prove that

$$H_{R_k}^\infty(\gamma g)^{\gcd(k,n-k)/n} = \det({}^t X_{\gamma^{-1}}\, {}^t g^{-1} g^{-1} X_{\gamma^{-1}})^{1/2}$$

holds for any $\gamma \in GL_n(\mathbf{Z})$ and $g \in SL_n(\mathbf{R})$. From this relation, it follows that $(\gamma_{R_k})^{2\gcd(k,n-k)/n}$ equals the Rankin constant $\gamma_{n,k}$. Thus, Coulangeon's result in **2.2** is interpreted as Voronoï's theorem of the function $F_{R_k}$.

**Example 4.2.** Let $B$ be a non degenerate bilinear form on $\mathbf{Q}^n$ defined by

$$B(x,y) = x_1y_1 - x_2y_2 - \cdots - x_ny_n$$

for $x = {}^t(x_1, \cdots, x_n)$ and $y = {}^t(y_1, \cdots, y_n) \in \mathbf{Q}^n$. We assume $n \geq 3$ and put $\boldsymbol{e}_{11} = {}^t(1,1,0,\cdots,0) \in \mathbf{Q}^n$. Let $\mathcal{N}_B(\mathbf{Q})$ be the set of all non-zero isotropic vectors in $\mathbf{Q}^n$ with respect to $B$. The special orthogonal group $SO_B(\mathbf{Q})$ of $B$ transitively acts on $\mathcal{N}_B(\mathbf{Q})$, i.e., one has $\mathcal{N}_B(\mathbf{Q}) = SO_B(\mathbf{Q})\boldsymbol{e}_{11}$. Let $P(\mathbf{Q})$ be the stabilizer of the isotropic line $\mathbf{Q}\boldsymbol{e}_{11}$ in $SO_B(\mathbf{Q})$. Then $P$ is a unique proper $\mathbf{Q}$-parabolic subgroup of the algebraic group $SO_B$ up to $SO_B(\mathbf{Q})$-conjugates. For any finite prime $p$, $K_p$ denotes the stabilizer of the $\mathbf{Z}_p$-lattice $\mathbf{Z}_p^n$ in $SO_B(\mathbf{Q}_p)$. Since $\mathbf{Z}_p^n$ is a unimodular maximal lattice with respect to $B$, $K_p$ is a maximal compact subgroup of $SO_B(\mathbf{Q}_p)$. At the infinite place $\infty$, the stabilizer $K_\infty$ of the vector $\boldsymbol{e} = {}^t(1,0,\cdots,0)$ in

$SO_B(\mathbf{R})$ gives a maximal compact subgroup of $SO_B(\mathbf{R})$. The intersection $SO_B(\mathbf{Q}) \cap (SO_B(\mathbf{R}) \times \prod_{p<\infty} K_p)$ is the stabilizer $SO_B(\mathbf{Z})$ of the lattice $\mathbf{Z}^n$ in $SO_B(\mathbf{Q})$. The adele group $SO_B(\mathbf{A})$ has the Iwasawa decomposition: $SO_B(\mathbf{A}) = P(\mathbf{A})(\prod_{p\leq\infty} K_p)$. The height function $H_P : SO_B(\mathbf{A}) \longrightarrow \mathbf{R}_{>0}$ is given by

$$H_P(g) = \|g^{-1}\boldsymbol{e}_{11}\|_{\mathbf{A}^n} = \prod_{p\leq\infty} \|g_p^{-1}\boldsymbol{e}_{11}\|_p$$

for $g = (g_p)_{p\leq\infty} \in SO_B(\mathbf{A})$. Here, the local height $\|\cdot\|_p$ is defined by

$$\|x\|_p = \begin{cases} \sqrt{x_1^2 + \cdots + x_n^2} & (p = \infty) \\ \max(|x_1|_p, \cdots, |x_n|_p) & (p < \infty) \end{cases}$$

for $x = {}^t(x_1, \cdots, x_n) \in \mathbf{Q}_p^n$. If $g = umh$ with $u \in U_P(\mathbf{A})$, $m \in M_P(\mathbf{A})$ and $h \in \prod_{p\leq\infty} K_p$, then $H_P(g)$ equals the idele norm of the first component of the vector $m^{-1}\boldsymbol{e}_{11}$. Since the class number of the indefinite lattice $(B, \mathbf{Z}^n)$ equals one, $SO_B$ satisfies the condition $(\mathrm{C}_1)$, and hence the generalized Hermite constant $\gamma_P$ is written as

$$\gamma_P = \max_{[g]\in SO_B(\mathbf{Z})\backslash SO_B(\mathbf{R})/K_\infty} \min_{[v]\in P(\mathbf{Q})\backslash SO_B(\mathbf{Q})} \|g^{-1}v^{-1}\boldsymbol{e}_{11}\|_{\mathbf{A}^n} .$$

Let $\mathcal{N}_B^*(\mathbf{Z})$ denote the set of primitive vectors in $\mathcal{N}_B(\mathbf{Q}) \cap \mathbf{Z}^n$, i.e.,

$$\mathcal{N}_B^*(\mathbf{Z}) = \{x \in \mathbf{Z}^n \setminus \{0\} \ : \ B(x, x) = 0 \text{ and } \gcd(x_1, \cdots, x_n) = 1\} .$$

From $SO_B(\mathbf{Q})\boldsymbol{e}_{11} = \mathcal{N}_B(\mathbf{Q}) = \mathbf{Q}^\times \cdot \mathcal{N}_B^*(\mathbf{Z})$, it follows that

$$\min_{[v]\in P(\mathbf{Q})\backslash SO_B(\mathbf{Q})} \|g^{-1}v^{-1}\boldsymbol{e}_{11}\|_{\mathbf{A}^n} = \min_{x\in\mathcal{N}_B^*(\mathbf{Z})} \|g^{-1}x\|_{\mathbf{A}^n} .$$

Since $x$ is a primitive isotropic vector and $g \in SO_B(\mathbf{R})$, we have

$$\|g^{-1}x\|_{\mathbf{A}^n} = \|g^{-1}x\|_\infty \times \prod_{p<\infty} \|x\|_p = \|g^{-1}x\|_\infty ,$$

and hence

$$\gamma_P = \max_{[g]\in K_\infty\backslash SO_B(\mathbf{R})/SO_B(\mathbf{Z})} \min_{x\in\mathcal{N}_B^*(\mathbf{Z})} \|gx\|_\infty .$$

We may compare this with Example 3.2. The group $SO_0(1, n-1)$ is the identity connected component of $SO_B(\mathbf{R})$. Since $K_\infty\backslash SO_B(\mathbf{R})/SO_B(\mathbf{Z}) = K_e\backslash SO_0(1, n-1)/\Gamma$ and $\mathcal{N}_B^*(\mathbf{Z})$ equals the subset of primitive vectors in $\mathbf{Z}^n \setminus \{0\} \cap \partial\overline{\Omega}_n$, one has

$$2^{n/2} \max_{a\in\Omega_n} F_{(\Omega_n,\mathbf{Z}^n)}(a) \leq (\gamma_P)^n . \tag{3}$$

If $n = 3$, then $\gamma_P = \sqrt{\gamma_2} = \sqrt{2/\sqrt{3}}$ since $SO_B$ is isomorphic with $PGL_2$ over $\mathbf{Q}$. In this case, the equality of (3) holds. In general $n$, Birch and Davenport's theorem [16, Theorem A] gives $\gamma_P \leq (\sqrt{2n}\gamma_{n-1})^{(n-1)/2}$. It is unknown whether the equality $2^{n/2}\max_{a\in\Omega_n} F_{(\Omega_n,\mathbf{Z}^n)}(a) = (\gamma_P)^n$ holds for all $n$ or not. Bavard's theory applies to this example. By using the same notation as in Example 3.4, we can choose $\mathcal{E} = (SO_0(1, n-1)\cdot \mathrm{I}_n, \Gamma, \mathcal{N}^*_B(\mathbf{Z}), \{f_x|_{SO_0(1,n-1)}\})$ as a quadruplet in question (cf. [9, Proposition 2.6]).

We should mention two remarks. Let $G$ and $P$ be the same as above, i.e., $G$ is a connected reductive algebraic group defined over $\mathbf{Q}$ and $P$ a minimal $\mathbf{Q}$-parabolic subgroup. First, let $R$ be a $\mathbf{Q}$-parabolic subgroup of $G$ such that $P \subset R$. Then the generalized Hermite constant $\gamma_R$ is defined even if $R$ is not maximal. However, in this case, there is no canonical height function on $R(\mathbf{A})^1\backslash G(\mathbf{A})^1$. As a result, there are infinitely many multiplicatively independent generalized Hermite constants of $R\backslash G$. To define a height function on $R(\mathbf{A})^1\backslash G(\mathbf{A})^1$, we choose a $\mathbf{Q}$-rational embedding of the projective variety $R\backslash G$ into a projective space. Such an embedding is constructed by a strongly $\mathbf{Q}$-rational irreducible representation $\pi : G \longrightarrow GL_N$ of $G$. By strongly $\mathbf{Q}$-rational, we mean that $\pi$ is a $\mathbf{Q}$-rational morphism and the highest weight line $l_\pi$ in $\overline{\mathbf{Q}}^N$ of $\pi$ is defined over $\mathbf{Q}$. Assume $\pi(R)$ is the stabilizer of $l_\pi$ in $\pi(G)$. Then the map $g \mapsto \pi(g^{-1})l_\pi$ gives rise to a $\mathbf{Q}$-rational embedding of $R\backslash G$ into the projective space $\mathbf{P}^{N-1}$. If $R_1$ denotes the maximal parabolic subgroup of $GL_N$ defined in a similar fashion as in Example 4.1, then $\mathbf{P}^{N-1}$ is identified with $R_1\backslash GL_N$, and the height $H_{R_1}$ is defined on $R_1(\mathbf{A})^1\backslash GL_N(\mathbf{A})^1$. The composition of $\pi$ and $H_{R_1}$ gives a height function $H_{R,\pi}$ on $R(\mathbf{A})^1\backslash G(\mathbf{A})^1$. Then the generalized Hermite constant $\gamma_{R,\pi}$ is defined to be

$$\gamma_{R,\pi} = \max_{[g]\in R(\mathbf{A})^1\backslash G(\mathbf{A})^1} F_{R,\pi}(g)\,, \quad \text{where } F_{R,\pi}(g) = \min_{[v]\in R(\mathbf{Q})\backslash G(\mathbf{Q})} H_{R,\pi}(vg)\,.$$

Second remark is a base change from $\mathbf{Q}$ to a number field $\mathsf{k}$. The adelic definition of the generalized Hermite constants is immediately extended to reductive groups defined over $\mathsf{k}$. Thus one can define $\gamma_{R,\pi}(\mathsf{k})$ for a connected reductive group $G$ defined over $\mathsf{k}$, a parabolic $\mathsf{k}$-subgroup $R \subset G$ and a strongly $\mathsf{k}$-rational representation $\pi$. In this notation, the constant $\gamma_{R,\pi}(\mathbf{Q})$ means $\gamma_{R,\pi}$. We write $\gamma_n(\mathsf{k})$ (resp. $\gamma_n(\mathsf{k})_1$ and $\gamma_{n,k}(\mathsf{k})$) for the generalized Hermite constant of $R_1\backslash GL_n$ (resp. $(R_1\cap SL_n)\backslash SL_n$ and $R_k\backslash GL_n$) defined over $\mathsf{k}$. When $\mathsf{k}$ is a totally real number field, $\gamma_2(\mathsf{k})_1$ was implicitly occurred in Cohn's paper [19, §5, §9]. Newman [44, Chapter XI] defined $\gamma_n(\mathsf{k})_1$ for imaginary quadratic fields $\mathsf{k}$. For a general number filed $\mathsf{k}$, $\gamma_n(\mathsf{k})_1$ was studied by Icaza [32] in terms of Humbert forms, i.e., $\gamma_n(\mathsf{k})_1$ is relating with the function $F_*$ as follows:

$$(\gamma_n(\mathsf{k})_1)^{2n} = \max_{[a]\in \mathsf{k}_{\mathbf{R}}^+\backslash P_n(\mathsf{k}_{\mathbf{R}})/GL_n(\mathsf{o}_{\mathsf{k}})} F_*(a)\,,$$

(cf. [65, §3]). As a generalization of Rankin's constant, Thunder [61] defined $\gamma_{n,k}(\mathsf{k})$ by using twisted heights on Grassmann varieties. The above definition of $\gamma_{R,\pi}(\mathsf{k})$ was given in [63]. Meyer [41] investigated $\gamma_{R,\pi}(\mathsf{k})$ when $G = GL_n$ and $\pi$ is a Schur module realized as a polynomial representation, and established Voronoï's theorem for $F_{R,\pi}$ by using Bavard's theory. Among other results in [41], we present here only the algebraicity of $\gamma_{R,\pi}(\mathsf{k})$:

**Theorem 4.1 (Meyer).** *Let $\mathsf{k}$ be an arbitrary algebraic number field. When $G = GL_n$, all $\gamma_{R,\pi}(\mathsf{k})$ are algebraic numbers.*

In the case that $\mathsf{k}$ is a function field of one variable over a finite field, $\gamma_Q(\mathsf{k})$ was studied in [64].

## 4.2. *Generalized Hermite constants of $Sp_n$*

Let $n = 2m$ be an even integer. We consider a symplectic group

$$G(\mathbf{Q}) = Sp_n(\mathbf{Q}) = \left\{ g \in GL_{2m}(\mathbf{Q}) \;:\; {}^t g \begin{pmatrix} 0 & -\mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix} g = \begin{pmatrix} 0 & -\mathrm{I}_m \\ \mathrm{I}_m & 0 \end{pmatrix} \right\}.$$

For $1 \le k \le m$, $Q_k$ denotes the maximal parabolic subgroup of $G$ given as follows:

$$\begin{aligned} Q_k(\mathbf{Q}) &= U_k(\mathbf{Q})L_k(\mathbf{Q})\,, \\ L_k(\mathbf{Q}) &= \left\{ \delta(a,b) = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b_{11} & 0 & b_{12} \\ 0 & 0 & {}^t a^{-1} & 0 \\ 0 & b_{21} & 0 & b_{22} \end{pmatrix} \;:\; \begin{array}{l} a \in GL_k(\mathbf{Q}) \\ b = (b_{ij}) \in Sp_{2(m-k)}(\mathbf{Q}) \end{array} \right\}, \\ U_k(\mathbf{Q}) &= \left\{ \begin{pmatrix} \mathrm{I}_k & * & * & * \\ 0 & \mathrm{I}_{m-k} & * & 0 \\ 0 & 0 & \mathrm{I}_k & 0 \\ 0 & 0 & * & \mathrm{I}_{m-k} \end{pmatrix} \in G(\mathbf{Q}) \right\}. \end{aligned}$$

The module of $\mathbf{Q}$-rational characters $\mathbf{X}^*_{\mathbf{Q}}(L_k)$ of $L_k$ is a free $\mathbf{Z}$-module of rank 1 and its base is given by

$$\widehat{\alpha}_{Q_k}(\delta(a,b)) = \det a\,.$$

We fix a good maximal compact subgroup $K$ of $G(\mathbf{A})$ so that $G(\mathbf{A})$ has the Iwasawa decomposition $G(\mathbf{A}) = Q_k(\mathbf{A})K$. Since $G$ and $Q_k$ satisfy both conditions ($\mathrm{C}_1$) and ($\mathrm{C}_2$), one has

$$\gamma_{Q_k} = \max_{[g]\in G(\mathbf{Z})\backslash G(\mathbf{R})/K_\infty} \min_{\gamma\in G(\mathbf{Z})} H^\infty_{Q_k}(\gamma g)\,. \tag{4}$$

We restrict ourselves to the case $k = m$. An element of $L_m(\mathbf{A})$ is denoted by

$$\delta(a) = \begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, \qquad (a \in GL_m(\mathbf{A})).$$

By definition, we have

$$H^\infty_{Q_m}(u\delta(a)h) = |\det a|^{-1}, \qquad (u \in U_m(\mathbf{R}),\ \delta(a) \in L_m(\mathbf{R}),\ h \in K_\infty).$$

Let

$$\mathsf{H}_m = \{Z \in M_m(\mathbf{C}) \ :\ \operatorname{Re} Z \in V_m,\ \operatorname{Im} Z \in P_m\}$$

be the Siegel upper half space. The group $G(\mathbf{R})$ acts on $\mathsf{H}_m$ by

$$g\langle Z\rangle = (aZ+b)(cZ+d)^{-1}, \qquad (g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{R}),\ Z \in \mathsf{H}_m).$$

Since it is possible to choose the maximal compact subgroup $K_\infty$ as the stabilizer of $Z_0 = \sqrt{-1}\mathrm{I}_m \in \mathsf{H}_m$ in $G(\mathbf{R})$, we have $\operatorname{Im}\{(u\delta(a)h)\langle Z_0\rangle\} = a^t a$, and hence

$$H^\infty_{Q_m}(g) = (\det \operatorname{Im}\{g\langle Z_0\rangle\})^{-1/2}$$

for any $g \in G(\mathbf{R})$. Combining this with (4), we get

$$\gamma_{Q_m} = \max_{[g]\in G(\mathbf{Z})\backslash G(\mathbf{R})/K_\infty} \min_{\gamma\in G(\mathbf{Z})} (\det \operatorname{Im}\{\gamma g\langle Z_0\rangle\})^{-1/2}\,.$$

Since $g\langle Z_0\rangle$ runs over a fundamental domain of $G(\mathbf{Z})\backslash\mathsf{H}_m$ , we have

$$\gamma_{Q_m} = \frac{1}{\displaystyle\min_{[Z]\in G(\mathbf{Z})\backslash \mathsf{H}_m} \max_{\gamma\in G(\mathbf{Z})} (\det \operatorname{Im}\{\gamma\langle Z\rangle\})^{1/2}}\,.$$

Note that

$$\det \operatorname{Im}\{\gamma\langle Z\rangle\} = |\det(cZ+d)|^{-2} \det \operatorname{Im} Z \ \text{ for } \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathbf{Z}).$$

Siegel's fundamental domain $\mathsf{S}_m$ of $G(\mathbf{Z})\backslash\mathsf{H}_m$ is given as follows:

$$\mathsf{S}_m = \left\{ Z \in \mathsf{H}_m \ :\ \begin{array}{l} \bullet\ |\det(cZ+d)| \geq 1 \text{ for all } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in G(\mathbf{Z}) \\ \bullet\ \operatorname{Im} Z \in \mathsf{M}_m, \quad |\operatorname{Re} Z_{ij}| \leq 1/2 \text{ for all } i, j \end{array} \right\},$$

where $\mathsf{M}_m$ denotes Minkowski's domain:

$$\left\{ Y \in P_m \ : \ \begin{array}{l} \bullet\ {}^t x Y x \geq Y_{ii} \text{ for all } x \in \mathbf{Z}^m \text{ with } \gcd(x_i, \cdots, x_m) = 1 \\ \bullet\ Y_{j,j+1} > 0, \quad i = 1, \cdots, m, \ j = 1, \cdots, m-1 \end{array} \right\}.$$

From

$$Z \in \mathsf{S}_m \implies \max_{\gamma \in G(\mathbf{Z})} \det \operatorname{Im} \{\gamma\langle Z \rangle\} = \det \operatorname{Im} Z,$$

it follows that

$$\gamma_{Q_m} = \frac{1}{\min\limits_{Z \in \mathsf{S}_m} (\det \operatorname{Im} Z)^{1/2}},$$

namely

$$\min_{Z \in \mathsf{S}_m} \det \operatorname{Im} Z = \frac{1}{(\gamma_{Q_m})^2}.$$

If $m = 1$, we have $\min_{Z \in \mathsf{S}_1} \det \operatorname{Im} Z = \sqrt{3}/2$. Recently, Kawamura [33] determined the actual value of $\min_{Z \in \mathsf{S}_2} \det \operatorname{Im} Z$ by using Gottschling's description of $\mathsf{S}_2$.

**Theorem 4.2 (Kawamura).** *One has* $\min_{Z \in \mathsf{S}_2} \det \operatorname{Im} Z = 2/3$, *and hence* $\gamma_{Q_2} = \sqrt{3/2}$. *This minimum is attained only when* $Z = Z_8$ *or* $-\overline{Z_8}$, *where*

$$Z_8 = \frac{1}{3}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \sqrt{-1}\frac{\sqrt{2}}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.$$

The boundary $\partial\mathsf{S}_2$ of $\mathsf{S}_2$ is described by 28 polynomials in 6 real variables. Hayata [30] computed 0-dimensional cells of $\partial\mathsf{S}_2$. There are at least 170 0-dimensional cells of $\partial\mathsf{S}_2$. The points $Z_8$ and $-\overline{Z_8}$ are contained in Hayata's list.

For general $n = 2m$, we have the following bound by [63, Example 3]:

$$\min_{Z \in \mathsf{S}_m} \det \operatorname{Im} Z \leq \left\{ \frac{1}{m+1} \cdot \frac{\prod\limits_{j=1}^{[\frac{m-1}{2}]} \xi(2j+1)}{\prod\limits_{j=1}^{m} \xi(m+j)} \right\}^{\frac{2}{m+1}},$$

where $\xi(s)$ denotes the zeta function $\pi^{-s/2}\Gamma(s/2)\zeta(s)$.

**Problem 4.1.** *Give a good lower bound of* $\min_{Z \in \mathsf{S}_m} \det \operatorname{Im} Z$.

**Problem 4.2.** *Formulate a Voronoï type theorem for the function* $Z \mapsto \det \operatorname{Im} Z$ *on* $\mathsf{S}_m$*.*

We finish this example by some observation. Let

$$R_m(\mathbf{Q}) = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \ : \ a, d \in GL_m(\mathbf{Q}),\ b \in M_m(\mathbf{Q}) \right\} .$$

be a maximal parabolic subgroup of $GL_n$. We have $\gamma_{R_m} = \gamma_{n,m}$ as seen in Example 4.1. It is obvious that $Sp_n(\mathbf{A}) \subset GL_n(\mathbf{A})$, $Sp_n(\mathbf{A}) \cap R_m(\mathbf{A}) = Q_m(\mathbf{A})$ and $H_{R_m}(g) = H_{Q_m}(g)^2$ for $g \in Sp_n(\mathbf{R})$. Since $Q_m(\mathbf{Q})\backslash Sp_n(\mathbf{Q}) \subset R_m(\mathbf{Q})\backslash GL_n(\mathbf{Q})$, the inequality

$$F_{R_m}(g) \leq F_{Q_m}(g)^2 \leq (\gamma_{Q_m})^2$$

holds for any $g \in Sp_n(\mathbf{A})$. If the maximum of $F_{R_m}$ is attained on a point in $Sp_n(\mathbf{A})$, $\gamma_{n,m}$ would be bounded by $(\gamma_{Q_m})^2$. This happens when $m = 1$ or 2 and in fact we have $\gamma_{2,1} = (\gamma_{Q_1})^2$ and $\gamma_{4,2} = (\gamma_{Q_2})^2$.

### 4.3. *Variation of the set of minimal points*

We return to the general setting of **4.1**. We write $Y_Q$ and $X_Q$ for $Q(\mathbf{A})^1\backslash G(\mathbf{A})^1$ and $Q(\mathbf{Q})\backslash G(\mathbf{Q})$, respectively. For $g \in G(\mathbf{A})^1$, the set $S_Q(g)$ of minimal points in $X_Q$ is defined as

$$S_Q(g) = \{[x] \in X_Q \ : \ H_Q(xg) = F_Q(g)\} .$$

By Northcott's theorem, $S_Q(g)$ is a finite set. We prove the following:

**Proposition 4.1.** *For* $g \in G(\mathbf{A})^1$*, there is a neighbourhood* $\mathcal{U}$ *of* $g$ *in* $G(\mathbf{A})^1$ *such that* $S_Q(g') \subset S_Q(g)$ *for all* $g' \in \mathcal{U}$*.*

**Proof.** We define the operator norm $\|g\|_Q$ of $g \in G(\mathbf{A})^1$ by

$$\|g\|_Q = \sup_{[y] \in Y_Q} \frac{H_Q(yg)}{H_Q(y)} = \max_{h \in K} H_Q(hg) .$$

It is easy to see the following properties:

- $\|g_1 g_2\|_Q \leq \|g_1\|_Q \|g_2\|_Q$ for all $g_1, g_2 \in G(\mathbf{A})^1$.
- $\|h_1 g h_2\|_Q = \|g\|_Q$ for all $g \in G(\mathbf{A})^1$ and $h_1, h_2 \in K$.
- $\|h\|_Q = 1$ for all $h \in K$.
- $g \mapsto \|g\|_Q$ is continuous on $G(\mathbf{A})^1$.

We fix a $g \in G(\mathbf{A})^1$ and put $C = \min_{[x] \in X_Q \setminus S_Q(g)} H_Q(xg)$. Then $F_Q(g) < C$ and we can take a constant $\delta$ so that $1 < \delta < C/F_Q(g)$. Since $g \mapsto F_Q(g)$ is continuous on $G(\mathbf{A})^1$, the set

$$\mathcal{U} = \left\{ u \in G(\mathbf{A})^1 \ : \ \|u^{-1}\|_Q < \frac{C}{\delta F_Q(g)}, \ \frac{F_Q(gu)}{F_Q(g)} < \delta \right\}$$

is a neighbourhood of the identity in $G(\mathbf{A})^1$. Let $u \in \mathcal{U}$ and $[x] \in S_Q(gu)$. We have

$$\|u^{-1}\|_Q^{-1} H_Q(xg) \le H_Q(xgu) = F_Q(gu)\,.$$

If $[x] \notin S_Q(g)$, then $C \le H_Q(xg)$ and

$$\delta F_Q(g) \le \delta F_Q(g) H_Q(xg)/C < \|u^{-1}\|_Q^{-1} H_Q(xg) \le F_Q(gu) < \delta F_Q(g)\,.$$

This is a contradiction. Therefore, we have $[x] \in S_Q(g)$, and hence $S_Q(gu) \subset S_Q(g)$ for any $u \in \mathcal{U}$. □

This proposition is a generalization of Proposition 2.2. As a consequence, one can define the local maximality of $S_Q(g)$. This leads us to similar problems as in **2.3**, e.g.,

**Problem 4.3.** *If $g \in G(\mathbf{A})^1$ is an extreme point of $F_Q$, is $S_Q(g)$ locally maximal?*

**Problem 4.4.** *Bound $\sharp S_Q(g)$ by a constant independent of $g$.*

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