Commutator relations between q-root vectors of the positive part of $U_q(\hat{\mathrm{sl}}_2)$

Let $\mathbf{Z}_+ = \{n \in \mathbf{Z} | n \geq 0\}$, so $\mathbf{N} = \mathbf{Z}_+ \setminus \{0\}$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}. \tag{0.1}$$

Let $\mathbf{K} = \mathbf{C}(q)$. Let $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$. Define a $\mathbf{K}$-algebra $U = U_q = U_q\widehat{\mathrm{sl}}_2$ by generators

$$K_i^{\pm 1},\ E_i,\ F_i \quad (i \in \{0,1\}) \tag{0.2}$$

and relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1,\ K_i K_j = K_j K_i, \tag{0.3}$$

$$K_i E_j K_i^{-1} = q^{a_{ij}} E_j,\ K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \tag{0.4}$$

$$[E_i, F_j] = \delta_{ij} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \tag{0.5}$$

$$E_i^3 E_j - [3] E_i^2 E_j E_i + [3] E_i E_j E_i^2 - E_j E_i^3 \ (i \neq j), \tag{0.6}$$

$$F_i^3 F_j - [3] F_i^2 F_j F_i + [3] F_i F_j F_i^2 - F_j F_i^3 \ (i \neq j). \tag{0.7}$$

Let U^+ (resp. U^0, resp. U^-) be the subalgebra (with 1) of U generated by E_i (resp. $K_i^{\pm 1}$, resp. F_i) with $i \in \{0,1\}$. Then, as a $\mathbf{K}$-linear space, $U \simeq U^+ \otimes U^0 \otimes U^-$ ($XZY \leftarrow X \otimes Z \otimes Y$). As a $\mathbf{K}$-algebra (with 1), U^+ (resp. U^0, resp. U^-) can also be defined with the above generators and the relations (0.6) (resp. (0.3), resp. (0.7)). Define a $\mathbf{K}$-algebra automorphism Ω of U by $\Omega(E_i) = F_i$, $\Omega(K_i) = K_i^{-1}$ and $\Omega(F_i) = E_i$. Then $\Omega(U^+) = U^-$ and $\Omega^2 = \mathrm{id}$. Let $U^+_{0,0} = \mathbf{K} (\subset U^+)$ and for m, $n \in \mathbf{N}$, let $U^+_{m,0} = \mathbf{K}E_0^m$, $U^+_{0,n} = \mathbf{K}E_1^n$ and $U^+_{m,n} = E_0 U^+_{m-1,n} + E_1 U^+_{m,n-1}$. If $m < 0$ or $n < 0$, let $U^+_{m,n} = \{0\}$. Let $U^-_{m,n} = \Omega(U^+_{-m,-n})$. For m, $n \in \mathbf{Z}$, let $U_{m,n} = \oplus_{x,y=-\infty}^{+\infty} \mathrm{Span}_{\mathbf{K}}(U^+_{x,y} U^0 U^-_{m-x,n-y})$. Let Q be a rank-two free $\mathbf{Z}$-module with a basis $\{\alpha_0, \alpha_1\}$, so $Q = \mathbf{Z}\alpha_0 \oplus \mathbf{Z}\alpha_1$. Define a bi-additive map $(\ ,\ ) : Q \times Q \to \mathbf{Z}$ by $(\alpha_i, \alpha_j) = a_{ij}$. For $\mu =$

$m\alpha_0 + n\alpha_1 \in Q$, let $U_\mu = U_{m,n}$. For $X_\mu \in U_\mu$ and $X_\nu \in U_\nu$ with μ and $\nu \in Q$, let $[\![X_\mu, X_\nu]\!] = X_\mu X_\nu - q^{(\mu,\nu)} X_\nu X_\mu$. For $X_\lambda \in U_\lambda$, $X_\mu \in U_\mu$ and $X_\nu \in U_\nu$ with λ, μ and $\nu \in Q$ and for i, $j \in \{0,1\}$, we have

$$[\![[\![X_\lambda, X_\mu]\!], X_\nu]\!] = [\![X_\lambda, [\![X_\mu, X_\nu]\!]]\!] + q^{-(\mu,\nu)}[[\![X_\lambda, X_\nu]\!], X_\mu]_{q^{(\mu,\nu-\lambda)}} \tag{0.8}$$

$$[\![X_\lambda, [\![X_\mu, X_\nu]\!]]\!] = [\![[\![X_\lambda, X_\mu]\!], X_\nu]\!] + q^{-(\lambda,\mu)}[X_\mu, [\![X_\lambda, X_\nu]\!]]_{q^{(\mu,-\nu+\lambda)}}, \tag{0.9}$$

$$[\![E_i, F_jK_j^{-1}]\!] = \delta_{ij}\frac{1-K_i^{-2}}{q-q^{-1}},\ [\![F_jK_j, E_i]\!] = \delta_{ij}\frac{K_i^2-1}{q-q^{-1}}, \tag{0.10}$$

$$[\![[\![E_i, X_\mu]\!], F_jK_j^{-1}]\!] = [\![E_i, [\![X_\mu, F_jK_j^{-1}]\!]]\!] + \delta_{ij}[(\mu,\alpha_i)]X_\mu \tag{0.11}$$

and

$$[\![K_jF_j, [\![X_\mu, E_i]\!]]\!] = [\![[\![K_jF_j, X_\mu]\!], E_i]\!] + \delta_{ij}[(\mu,\alpha_i)]X_\mu. \tag{0.12}$$

For $i \in \{0,1\}$, define a **K**-algebra automorphism $T_i : U \to U$ as follows; in the following, $j \in \{0,1\} \setminus \{i\}$.

$$\begin{cases} T_i(K_i) = K_i^{-1},\ T_i(K_j) = K_i^2K_j, \\ T_i(E_i) = -F_iK_i,\ T_i(E_j) = \frac{q^{-2}}{[2]}[\![[\![E_j, E_i]\!], E_i]\!], \\ T_i(F_i) = -K_i^{-1}E_i,\ T_i(F_j) = \frac{1}{[2]}[\![[\![F_j, F_i]\!], F_i]\!]. \end{cases} \tag{0.13}$$

Then we have

$$\begin{cases} T_i^{-1}(K_i) = K_i^{-1},\ T_i^{-1}(K_j) = K_i^2K_j, \\ T_i^{-1}(E_i) = -K_i^{-1}F_i,\ T_i^{-1}(E_j) = \frac{q^{-2}}{[2]}[\![E_i, [\![E_i, E_j]\!]]\!], \\ T_i^{-1}(F_i) = -E_iK_i,\ T_i^{-1}(F_j) = \frac{1}{[2]}[\![F_i, [\![F_i, F_j]\!]]\!]. \end{cases} \tag{0.14}$$

Note that if $i \neq j$,

$$\begin{aligned} (0.15)\quad & T_i([\![E_i, E_j]\!]) \\ &= [\![-F_iK_i, \frac{q^{-2}}{[2]}[\![[\![E_j, E_i]\!], E_i]\!]]\!] \\ &= -\frac{1}{[2]}[\![F_iK_i, [\![[\![E_j, E_i]\!], E_i]\!]]\!] \\ &= -\frac{1}{[2]}([(\alpha_i,\alpha_j)] + [(\alpha_i, \alpha_i+\alpha_j)])[\![E_j, E_i]\!] \quad \text{(by (0.12))} \\ &= [\![E_j, E_i]\!]. \end{aligned}$$

Define a **K**-algebra automorphism τ of U by $\tau(K_i) = K_j$, $\tau(E_i) = E_j$ and $\tau(F_i) = F_j$ $(i \neq j)$. Then by (0.15), $T_0\tau([\![E_1, E_0]\!]) = [\![E_1, E_0]\!]$. Set $E_{\alpha_1} := E_1$ and $E_{\alpha_0} = E_{\delta-\alpha_1} := E_0$. For $n \in \mathbf{N}$, set

(0.16)
$$\begin{cases} E_{n\delta+\alpha_1} & := \frac{1}{[2]}[E_\delta, E_{(n-1)\delta+\alpha_1}] = -(-1)^{\delta_{n0}}(T_0\tau)^{-n}(E_1), \\ E_{n\delta} & := q^{-2}[\![E_{(n-1)\delta+\alpha_1}, E_{\alpha_0}]\!], \\ E_{(n-1)\delta+\alpha_0} = E_{n\delta-\alpha_1} & := -\frac{1}{[2]}[E_\delta, E_{(n-1)\delta-\alpha_1}] = -(-1)^{\delta_{n0}}(T_0\tau)^{n}(E_0). \end{cases}$$

We have

(0.17)
$$[E_{m\delta}, E_{n\delta}] = 0,$$

(0.18)
$$E_{n\delta} = q^{-2}[\![E_{(n-r-1)\delta+\alpha_1}, E_{r\delta+\alpha_0}]\!] \Longleftrightarrow -(-1)^{\delta_{r0}}(T_0\tau)^r(E_{n\delta}) = (-1)^{\delta_{rn-1}}E_{n\delta},$$

(0.19)
$$[E_{r\delta}, E_{n\delta+\alpha_1}] = (q^2 - q^{-2})\sum_{k=1}^{r-1} q^{2(1-k)}E_{(r-k)\delta}E_{(n+k)\delta+\alpha_1} + [2]q^{2(1-r)}E_{(n+r)\delta+\alpha_1},$$

(0.20)
$$[E_{r\delta}, E_{n\delta-\alpha_1}] = -(q^2 - q^{-2})\sum_{k=1}^{r-1} q^{2(1-k)}E_{(n+k)\delta-\alpha_1}E_{(r-k)\delta} - [2]q^{2(1-r)}E_{(n+r)\delta-\alpha_1},$$

(0.21)
$$[\![E_{n\delta+\alpha_1}, E_{(n+r)\delta+\alpha_1}]\!] = -[\![E_{(n+r-1)\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!]$$

and

(0.22)
$$[\![E_{(n+r)\delta-\alpha_1}, E_{n\delta-\alpha_1}]\!] = -[\![E_{(n+1)\delta-\alpha_1}, E_{(n+r-1)\delta-\alpha_1}]\!].$$

If $X \in U_{m,n}$, we write $\eta(X) = m + n$; we agree that $\eta(0) = n$ for any $n \in \mathbf{Z}_+$. We prove (0.17)-(0.21) by induction on $\eta(\cdot)$.

(1) We first note

(0.23)
$$\begin{aligned} & [E_{2\delta}, E_{\alpha_1}] \\ = & -[E_{\alpha_1}, E_{2\delta}] \\ = & -q^{-2}[E_{\alpha_1}, [\![E_{\delta+\alpha_1}, E_0]\!]] \\ = & -q^{-2}[\![[\![E_{\alpha_1}, E_{\delta+\alpha_1}]\!], E_0]\!] - q^{-2}[E_{\delta+\alpha_1}, E_\delta]_{q^4} \\ & \text{(by (0.9) since } -(\alpha_1, \delta+\alpha_1) = -2 \\ & \text{and } (\delta+\alpha_1, -\alpha_0+\alpha_1) = 4) \\ = & -0 - q^{-2}E_{\delta+\alpha_1}E_\delta + q^2E_\delta E_{\delta+\alpha_1} \text{(by (0.6))} \\ = & q^{-2}[2]E_{2\delta+\alpha_1} + (q^2 - q^{-2})E_\delta E_{\delta+\alpha_1} \end{aligned}$$

and

$$[E_\delta^2, E_1] \tag{0.24}$$
$$= [2](E_\delta E_{\delta+\alpha_1} + E_{\delta+\alpha_1} E_\delta)$$
$$= -[2]^2 E_{2\delta+\alpha_1} + 2[2] E_\delta E_{\delta+\alpha_1}.$$

Set $\widetilde{E}_{2\delta} := -\frac{q-q^{-1}}{2} E_\delta^2 + E_{2\delta}$. Then, by (0.23)-(0.24), we have

$$[\widetilde{E}_{2\delta}, E_1] = \frac{[4]}{2} E_{2\delta+\alpha_1}. \tag{0.25}$$

By (0.17) for $(m,n) = (1,2)$ and (0.25), we have

$$[\widetilde{E}_{2\delta}, E_{n\delta+\alpha_1}] = \frac{[4]}{2} E_{(n+2)\delta+\alpha_1}. \tag{0.26}$$

(2) We prove (0.21) for $r \in \{1,2,3\}$. Note that if $r = 1$, it is equivalent to

$$[\![E_{n\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!] = 0. \tag{0.27}$$

If $n = 0$, (0.27) follows from (0.6). By (0.26)-(0.27), we have

$$\begin{aligned} 0 &= (\frac{1}{2[2]^2}(\mathrm{ad}E_\delta)^2 - \frac{2}{[4]}(\mathrm{ad}\widetilde{E}_{2\delta}))([\![E_{n\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!]) \\ &= [\![E_{(n+1)\delta+\alpha_1}, E_{(n+2)\delta+\alpha_1}]\!]. \end{aligned}$$

Applying $\mathrm{ad}E_\delta$ to (0.27), we have (0.21) for $r = 2$.
We calculate

$$\begin{aligned} 0 &= \frac{1}{[2]}[E_\delta, [\![E_{n\delta+\alpha_1}, E_{(n+k)\delta+\alpha_1}]\!] + [\![E_{(n+k-1)\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!]] \\ &= [\![E_{(n+2)\delta+\alpha_1}, E_{(n+k)\delta+\alpha_1}]\!] + [\![E_{(n+1)\delta+\alpha_1}, E_{(n+k+1)\delta+\alpha_1}]\!] \\ &\quad + [\![E_{(n+k)\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!] + [\![E_{(n+k-1)\delta+\alpha_1}, E_{(n+2)\delta+\alpha_1}]\!] \\ &= [\![E_{(n+1)\delta+\alpha_1}, E_{(n+k+1)\delta+\alpha_1}]\!] + [\![E_{(n+k)\delta+\alpha_1}, E_{(n+1)\delta+\alpha_1}]\!] \\ &\qquad \text{(by induction on } k \text{ and (0.21) for } r = 1 \text{ or } 2) \end{aligned}$$

Then we have proved (0.21) by induction on $\eta(\cdot)$. Similarly we have (0.22).

We calculate

$$
\begin{aligned}
&[E_{\alpha_1}, E_{(m+1)\delta}] \\
&= q^{-2}[E_{\alpha_1}, [\![E_{m\delta+\alpha_1}, E_{\alpha_0}]\!]] \\
&= q^{-2}([\![[\![E_{\alpha_1}, E_{m\delta+\alpha_1}]\!], E_{\alpha_0}]\!] + q^{-2}[E_{m\delta+\alpha_1}, [\![E_{\alpha_1}, E_{\alpha_0}]\!]]_{q^4}) \quad \text{(by (0.9))} \\
&= q^{-2}(-[\![[\![E_{(m-1)\delta+\alpha_1}, E_{\delta+\alpha_1}]\!], E_{\alpha_0}]\!] + q^{-2}[E_{m\delta+\alpha_1}, [\![E_{\alpha_1}, E_{\alpha_0}]\!]]_{q^4}) \\
&= -[E_{(m-1)\delta+\alpha_1}, E_{2\delta}] - q^2[E_{(m+1)\delta}, E_{\delta+\alpha_1}]_{q^{-4}} + q^{-2}[E_{m\delta+\alpha_1}, E_\delta]_{q^4} \\
&= \frac{1}{[2]}[E_\delta, \Big(-[E_{(m-2)\delta+\alpha_1}, E_{2\delta}] - q^2[E_{m\delta}, E_{\alpha_1}]_{q^{-4}} + q^{-2}[E_{(m-1)\delta+\alpha_1}, E_\delta]_{q^4}\Big)] \\
&= \frac{1}{[2]}[E_\delta, \Big(-[E_{(m-2)\delta+\alpha_1}, E_{2\delta}] \\
&\quad -q^2((1-q^{-4})E_{m\delta}E_{\alpha_1} + q^{-4}[E_{m\delta}, E_{\alpha_1}]) \\
&\quad +q^{-2}((1-q^4)E_\delta E_{(m-1)\delta+\alpha_1} + [E_{(m-1)\delta+\alpha_1}, E_\delta])\Big)] \\
&= \Big((q^2-q^{-2})E_\delta E_{m\delta+\alpha_1} + [2]q^{-2}E_{(m+1)\delta+\alpha_1}\Big) - (q^2-q^{-2})E_{m\delta}E_{\delta+\alpha_1} \\
&\quad -q^{-2}\Big((q^2-q^{-2})\sum_{k=1}^{m-1} q^{2(1-k)}E_{(m-k)\delta}E_{(1+k)\delta+\alpha_1} + q^{2(1-m)}E_{(m+1)\delta+\alpha_1}\Big) \\
&\quad -(q^2-q^{-2})E_\delta E_{m\delta+\alpha_1} - q^{-2}[2]E_{(m+1)\delta+\alpha_1} \\
&= -(q^2-q^{-2})\sum_{k=1}^{m} q^{2(1-k)}E_{(m+1-k)\delta}E_{k\delta+\alpha_1} - q^{-2m}E_{(m+1)\delta+\alpha_1}.
\end{aligned}
$$

Then, by induction on $\eta(\cdot)$, we have (0.19).

Similarly we have (0.20).

$$
\begin{aligned}
&[\![E_{x\delta+\alpha_1}, E_{y\delta-\alpha_1}]\!] \\
&= [\![E_{x\delta+\alpha_1}, -\frac{1}{[2]}[E_\delta, E_{(y-1)\delta-\alpha_1}]]\!] \\
&= -\frac{1}{[2]}\Big([\![[E_{x\delta+\alpha_1}, E_\delta], E_{(y-1)\delta-\alpha_1}]\!] + [E_\delta, [\![E_{x\delta+\alpha_1}, E_{(y-1)\delta-\alpha_1}]\!]\Big) \\
&= [\![E_{(x+1)\delta+\alpha_1}, E_{(y-1)\delta-\alpha_1}]\!] - \frac{q^2}{[2]}[E_\delta, E_{(x+y-1)\delta}] \\
&= q^2E_{(x+y)\delta} - (y-1)\frac{q^2}{[2]}[E_\delta, E_{(x+y-1)\delta}]
\end{aligned}
$$

$$
\begin{aligned}
[E_{x\delta}, E_{y\delta}] &= [q^{-2}[\![E_{(x-1)\delta+\alpha_1}, E_{\alpha_0}]\!], E_{y\delta}] \\
&= q^{-2}\Big([\![[E_{(x-1)\delta+\alpha_1}, E_{y\delta}], E_{\alpha_0}]\!] + [\![E_{(x-1)\delta+\alpha_1}, [E_{\alpha_0}, E_{y\delta}]]\!]\Big) \\
&= q^{-2}\Big([\![-(q^2-q^{-2})\sum_{k=1}^{y-1} q^{2(1-k)} E_{(y-k)\delta} E_{(x-1+k)\delta+\alpha_1} \\
&\quad -[2]q^{2(1-y)} E_{(x-1+y)\delta+\alpha_1}, E_{\alpha_0}]\!] \\
&\quad +[\![E_{(x-1)\delta+\alpha_1}, (q^2-q^{-2})\sum_{k=1}^{y-1} q^{2(1-k)} E_{k\delta+\alpha_0} E_{(y-k)\delta} + [2]q^{2(1-y)} E_{k\delta+\alpha_0}]\!]\Big) \\
&= q^{-2}\Big(-(q^2-q^{-2})\sum_{k=1}^{y-1} q^{2(1-k)}(E_{(y-k)\delta} q^2 E_{(x+k)\delta} \\
&\quad +q^2(-(q^2-q^{-2})\sum_{e=1}^{y-k-1} q^{2(1-e)} E_{e\delta+\alpha_0} E_{(y-k-e)\delta} \\
&\quad -[2]q^{2(1-(y-k))} E_{(y-k)\delta+\alpha_0}) E_{(x-1+k)\delta+\alpha_1}) - [2]q^{2(1-y)} q^2 E_{(x+y)\delta}\Big) \\
&\quad +q^{-2}\Big((q^2-q^{-2})\sum_{k=1}^{y-1} q^{2(1-k)}(q^2 E_{(x+k)\delta} E_{(y-k)\delta} \\
&\quad +q^2 E_{k\delta+\alpha_0}(-(q^2-q^{-2})\sum_{e=1}^{y-k-1} q^{2(1-e)} E_{(y-k-e)\delta} E_{e\delta+\alpha_1} \\
&\quad -[2]q^{2(1-y+k)} E_{(y-k+x-1)\delta+\alpha_1})) \\
&\quad +[2]q^{2(1-y)}(q^2 E_{(x+y)\delta} - y\frac{q^2}{[2]}[E_\delta, E_{(x+y-1)\delta}])\Big) \\
&= (q^2-q^{-2})\sum_{k=1}^{y-1} q^{2(1-k)}[E_{(x+k)\delta}, E_{(y-k)\delta}] + yq^{2(1-y)}[E_{(x+y-1)\delta}, E_\delta] \\
&= (q^2-q^{-2})[E_{(x+1)\delta}, E_{(y-1)\delta}] + q^{-2}[E_{(x+1)\delta}, E_{(y-1)\delta}] + q^{2(1-y)}[E_{(x+y-1)\delta}, E_\delta] \\
&= q^2[E_{(x+1)\delta}, E_{(y-1)\delta}] + q^{2(1-y)}[E_{(x+y-1)\delta}, E_\delta] \\
&= \frac{q^{2y}-q^{-2y}}{q^2-q^{-2}}[E_{(x+y-1)\delta}, E_\delta] \\
&= \frac{q^{2y}-q^{-2y}}{q^{2x}-q^{-2x}}[E_{y\delta}, E_{x\delta}]
\end{aligned}
$$

Hence we have $[E_{x\delta}, E_{y\delta}] = 0$, i.e., (0.17).