A Serre-type theorem of the elliptic Lie algebras and superalgebras with rank more than or equal to two

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Abstract

In this paper, we give a finite number of defining relations for the elliptic Lie algebras and superalgebras with rank≥ 2.

Introduction

In 1985, K. Saito [S] introduced the notion of the *extended affine root systems*; we also call them the SEARS's (see also [A]). Let R be an SEARS. Let $\mathcal{V} := \mathbb{R}R$ and $\mathcal{V}^0 := \{v \in \mathcal{V} | s_\alpha(v) = v \, \text{for all} \; \alpha \in R\}$, where s_α denotes the reflection with respect to an α. Let $m := \mathcal{V}^0$ and $l := \dim \mathcal{V} - m$. We say that the m is the *nullity* of the R and say that the l is the *rank* of the R. The R is *reduced* if $\mathbb{R}\alpha \cap R = \{\alpha, -\alpha\}$ for all $\alpha \in R$. We notice that if $m = 0$, the R is a finite root system, and that if $m = 1$, the R is an affine root system. If $m = 2$, the R is called an *elliptic root system* (or ERS for short); see also Subsec. 1.3. (The notation $\mathcal{V}$ is used only in Introduction.)

In 1997, B. Allison, S. Azam, S. Berman, Y. Gao and A. Pianzola [AABGP] introduced and studied root systems defined by different axioms from those of the SEARS's. In 2002, S. Azam [A] showed that there exists a natural one-to-one correspondence between their root systems and the SEARS's.

Let R be an ERS. Let $\widetilde{\pi} : \mathcal{V} \to \mathcal{V}/\mathcal{V}^0$ be a natural projective map. The R is called *simply-laced* if $l \geq 2$ and $\widetilde{\pi}(R)$ is a simply-laced finite root system. In 2000, K. Saito and D. Yoshii [SY] studied the Lie algebra $\mathfrak{g}_R$ with a simply-laced ERS R, and gave a Serre-type theorem for $\mathfrak{g}_R$; in other words, they gave a finite number of defining relations of $\mathfrak{g}_R$ satisfied by Chevalley generators.

In this paper, for all the (not only reduced but also non-reduced) ERS's R with $l \geq 2$, we give a Serre-type theorem for a Lie algebra or superalgebra $\mathfrak{g}_R$ having the property that the system formed by its real roots is isomorphic to the R (see Theorems 2.1 and 6.1). In the process of giving the Serre-type theorem, we also give a proof of the classification theorem for the above R's (see Theorems 1.2 and 2.2). The proof of the classification theorem seems to be easier than that in [S] for the reduced marked ERS's (see below and see Theorem 1.1); one of the reason is that we use the $\mathfrak{g}_R$ in the proof. The $\mathfrak{g}_R$ is not a Lie algebra but a Lie superalgebra if and only if the R is not reduced. (In the text, $\mathfrak{g}_R$ shall be denoted as $\mathfrak{g}_{\mathcal{D}}$.)

Let R be an ERS. If there exists a one dimensional subspace $G_{\mathbb{R}}$ of $\mathcal{V}^0$ such that $G_{\mathbb{R}} \cap \mathbb{Z}R \neq \{0\}$ and $\pi(R)$ is a reduced affine root system, where $\pi : \mathcal{V} \to \mathcal{V}/G_{\mathbb{R}}$ is the natural projection, then we call the R the *reduced marked ERS*. (More precisely, in the text, we call the pair $(R \otimes_{\mathbb{R}} \mathbb{C}, G_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C})$ the reduced marked ERS (see Subsec. 1.3).) If R is an reduced marked ERS with $l \geq 2$, then the Serre-type theorem for the $\mathfrak{g}_R$ has already been given by the author [Ya2] in 2004.

We also give universality theorems of the $\mathfrak{g}_R$ with $l \geq 2$. (see Theorems 5.1 and 5.2). By one of them, we see that if the R satisfies the condition that there exists a basis $\{\delta,\, a\}$ of $\mathcal{V}^0$ such that $(\alpha + \mathcal{V}^0) \cap R = \alpha + \mathbb{Z}\delta + \mathbb{Z}a$ for all $\alpha \in R$, then the $\mathfrak{g}_R$ is isomorphic to the universal central extension of the Lie (super)algebra $\mathfrak{g}^f_R \otimes \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}]$, where $\mathfrak{g}^f_R$ is a finite dimensional simple Lie (super)algebra whose root system is isomorphic to the $\widetilde{\pi}(R)$, i,e., the $\mathfrak{g}_R$ is the toroidal Lie (super)algebra (see [MEY]). We also treat quantum tori elliptic Lie algebras (see Subsec. 5.2).

1 Preliminary

1.1 Pre-elliptic base system

Let l be a fixed positive integer. Through out this paper, we assume $l \geq 2$. Let $\mathcal{E}$ be an $l+4$-dimensional $\mathbb{C}$-vector space. Let $J : \mathcal{E} \times \mathcal{E} \to \mathbb{C}$ be a non-degenerate symmetric bilinear form. Let $\mathcal{E}^\times := \{x \in \mathcal{E} | J(x,x) \neq 0\}$. If $x \in \mathcal{E}^\times$, we call x a *non-isotropic element*, let $x^\vee := \frac{2x}{J(x,x)}$ and define $s_x \in \mathrm{GL}(\mathcal{E})$ by $s_x(y) = y - J(x^\vee, y)x$. Let $\{\alpha_0, \ldots, \alpha_l,\, \Lambda_\delta,\, a,\, \Lambda_a\}$ be a basis of $\mathcal{E}$ such that the $(l+1) \times (l+1)$-matrix $A := (J(\alpha_i^\vee, \alpha_j))_{0 \leq i,j \leq l}$ is an affine type generalized Cartan matrix [K1], [K2], and such that $J(\Lambda_\delta, \alpha_i) = \delta_{i0}$, $J(\Lambda_\delta, \Lambda_\delta) = 0$, $J(a, \alpha_i) = 0$, $J(a, \Lambda_\delta) = 0$, $J(a,a) = 0$, $J(\Lambda_a, \alpha_i) = 0$, $J(\Lambda_a, a) = 1$ and $J(\Lambda_a, \Lambda_a) = 0$. The A is called $\mathbf{A}_l^{(1)}$ ($l \geq 2$), $\mathbf{B}_l^{(1)}$ ($l \geq 3$), $\mathbf{C}_l^{(1)}$ ($l \geq 2$), $\mathbf{D}_l^{(1)}$ ($l \geq 4$), $\mathbf{E}_l^{(1)}$ ($l = 6,\ 7.8$), $\mathbf{F}_4^{(1)}$ ($l = 4$), $\mathbf{G}_2^{(1)}$ ($l = 2$), $\mathbf{A}_{2l}^{(2)}$ ($l \geq 2$), $\mathbf{A}_{2l-1}^{(2)}$ ($l \geq 3$), $\mathbf{D}_{l+1}^{(2)}$ ($l \geq 2$), $\mathbf{E}_6^{(2)}$ ($l = 4$) or $\mathbf{D}_4^{(3)}$ ($l = 2$). (See [K2, Tables 1-4] and [K1, §4.8 TABLE Aff 1,2,3].) Since we have assumed $l \geq 2$, A is neither $\mathbf{A}_1^{(1)}$ nor $\mathbf{A}_2^{(2)}$. Let $\Pi := \{\alpha_0, \ldots, \alpha_l\}$. Let W be the subgroup of $\mathrm{GL}(\mathcal{E})$ generated by s_α ($\alpha \in \Pi$), i.e., W is the affine Weyl group.

For a subset S of Π, let W_S be the subgroup of W generated by s_α ($\alpha \in S$). Let $l(w)$ be the length of $w \in W$ with respect to s_α ($\alpha \in \Pi$).

Lemma 1.1. *Let α, $\beta \in \Pi$ and $w \in W$ be such that $w(\alpha) = \beta$ and $l(w) > 0$. Then there exists a $\gamma \in \Pi \setminus \{\alpha\}$ and a $w' \in W_{\{\alpha,\gamma\}}$ such that $w'(\alpha) \in \{\alpha, \gamma\}$ and $l(w') + l(w(w')^{-1}) = l(w)$.*

This can be proved by a well-known argument (see [J, Proof of Proposition 8.20]).

A function $f : \Pi \to \mathbb{C}$ is called *W-invariant* if $f(\alpha) = f(\beta)$ for every $(\alpha, \beta) \in \Pi \times \Pi$ with $\beta = w(\alpha)$ for some $w \in W$. By Lemma 1.1, we see the following.

Lemma 1.2. *Keep the notation as above. A function $f : \Pi \to \mathbb{C}$ is W-invariant if and only if $f(\alpha) = f(\beta)$ for every $(\alpha, \beta) \in \Pi \times \Pi$ with $J(\alpha^\vee, \beta) = J(\alpha, \beta^\vee) = -1$.*

Let $k : \Pi \to \mathbb{N}$ be a W-invariant function such that G.C.D.$\{k(\alpha)|\alpha \in \Pi\} = 1$. Let $g : \Pi \to 2^{\mathbb{Z}}$ be a W-invariant function, where $2^{\mathbb{Z}}$ is the power set of $\mathbb{Z}$, i.e., the set of the subsets of $\mathbb{Z}$. If $g(\alpha) = \emptyset$ for every α, g is also denoted by 0. We call a quintuple $\mathcal{D} = \mathcal{D}(\mathcal{E}, \Pi, a, k, g)$ of such $\mathcal{E}$, Π, a, k and g a *pre-elliptic base system* (PEBS for short). For $x \in \mathcal{E}$ and a subset B of $\mathbb{C}$, let $Bx := \{bx \in \mathcal{E}|b \in B\}$; moreover, for a subset X of $\mathcal{E}$, let $BX := \sum_{x\in X} Bx$. (If B is an empty set $\emptyset$, then $BX = \emptyset$.) For subsets S and T of $\mathcal{E}$, let $S + T := \{x + y \in \mathcal{E}|x \in S,\ y \in T\}$; if $T = \{x\}$, let $x + S := T + S$. (If $S = \emptyset$, $S + T = \emptyset$.)

Let $\mathcal{D}$ be a PEBS. Let

$$R(k, g) := \bigcup_{w\in W} w\Big(\bigcup_{\alpha\in\Pi} \big((\alpha + \mathbb{Z}k(\alpha)a) \cup (2\alpha + g(\alpha)k(\alpha)a)\big)\Big). \tag{1.1}$$

Then

$$R(k, g) \subset \Big((\mathbb{Z}_+\Pi + \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi + \mathbb{Z}a)\Big) \setminus \mathbb{Z}a, \tag{1.2}$$

where $\mathbb{Z}_\pm = \{n \in \mathbb{Z}| \pm n \geq 0\}$.

Lemma 1.3. *Keep the notation as above. Let S be a subset of Π. Let $\lambda \in (\mathbb{Z}S) \cap \mathcal{E}^\times$. Then there exists a $w \in W_S$ such that*

$$w(\lambda) \in \Big(\bigcup_{\alpha\in S} \mathbb{Z}\alpha\Big) \cup \Big(\mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}_-S)\Big).$$

Proof. By the definition of Π, there exists a $\kappa \in \mathbb{C} \setminus \{0\}$ such that $\kappa J(\alpha_i, \alpha_j) \in \mathbb{R}$ $(0 \leq i, j \leq l)$, and $\kappa J(\alpha_i, \alpha_i) > 0$ $(0 \leq i \leq l)$. Then the symmetric bilinear form $(\kappa J)_{|\mathbb{R}\Pi\times\mathbb{R}\Pi} : \mathbb{R}\Pi \times \mathbb{R}\Pi \to \mathbb{R}$ is positive definite. For $\mu = \sum_{\alpha\in\Pi} b_\alpha \alpha \in \mathbb{R}\Pi$ with $b_\alpha \in \mathbb{R}$, let $ht(\mu) := \sum_{\alpha\in\Pi} b_\alpha \in \mathbb{R}$.

Let λ be as in the statement. We may assume $\lambda \in (\mathbb{Z}_+S \cup \mathbb{Z}_-S) \setminus \{0\}$. Moreover we may assume $\lambda \in \mathbb{Z}_+S$. We use an induction on $ht(\lambda) \in \mathbb{N}$. If $ht(\lambda) = 1$, then $\lambda \in S$. We assume $ht(\lambda) > 1$. We may assume $\lambda \notin \mathbb{N}\alpha$ for any $\alpha \in S$. Since $\lambda \in \mathcal{E}^\times$, $\kappa J(\lambda, \lambda) > 0$. Hence there there exists an $\alpha \in S$ such that $\kappa J(\lambda, \alpha) > 0$. Notice that $s_\alpha(\lambda) = \lambda - \frac{2\kappa J(\lambda,\alpha)}{\kappa J(\alpha,\alpha)}\alpha$. If $s_\alpha(\lambda) \notin \mathbb{Z}S \setminus (\mathbb{Z}_+S \cup \mathbb{Z}_-S)$, then $s_\alpha(\lambda) \in \mathbb{Z}_+S$ and $ht(s_\alpha(\lambda)) < ht(\lambda)$. This completes the proof. ☐

For a subset S of Π, let

$$R(k,g)_S := R(k,g) \cap ((\oplus_{\alpha\in S}\mathbb{C}\alpha) \oplus \mathbb{C}a).$$

Lemma 1.4. *Keep the notation as above. Then*

$$R(k,g)_S = \bigcup_{w\in W_S} w\Big(\bigcup_{\alpha\in S}\big((\alpha+\mathbb{Z}k(\alpha)a) \cup (2\alpha+g(\alpha)k(\alpha)a)\big)\Big).$$

In particular, for $\alpha \in \Pi$, *we have*

$$R(k,g)_{\{\alpha\}} = \bigcup_{\epsilon\in\{\pm 1\}} \big((\epsilon\alpha+\mathbb{Z}k(\alpha)a) \cup (2\epsilon\alpha+g(\alpha)k(\alpha)a).$$

Proof. If $|S| = 1$, then the lemma follows immediately from the definition. From this, together with (1.2) and Lemma 1.3, the lemma for a general S follows; notice that $w(a) = a$ for all $w \in W_S$. ☐

1.2 Elliptic base systems and quasi-elliptic base systems

Here we introduce the notions of an elliptic base system (EBS for short) and an quasi-elliptic base system (QEBS for short). In Theorem 2.2, we shall show that these notions are equivalent. In Theorem 1.2, we shall show that an EBS can be regarded as a 'base system' of the elliptic root system in the sense of [S].

Let $\mathcal{D}$ be an PEBS with $l \geq 2$. For $\alpha \in \Pi$, Let

$$\Pi_c(\alpha) := \{\beta\in\Pi | \beta\neq\alpha,\ J(\beta,\alpha)\neq 0\}.$$

Define a subset Π^B of Π by

$$\Pi^B := \{\alpha\in\Pi | \forall\beta\in\Pi_c(\alpha),\ J(\alpha^\vee,\beta) = -2\}.$$

We call $\mathcal{D}$ an *an quasi elliptic base system* (QEBS for short) if the following hold.

(1) If $\alpha \in \Pi$, $\beta \in \Pi_c(\alpha)$ and $J(\beta^\vee,\alpha) = -1$, then $\frac{k(\beta)}{k(\alpha)} \in \mathbb{Z}$ and $J(\alpha^\vee,\beta)\frac{k(\alpha)}{k(\beta)} \in \mathbb{Z}$.
(2) $g(\alpha) = \emptyset$ if $\alpha \notin \Pi^B$.
(3) If $\alpha \in \Pi^B$ and $\beta \in \Pi_c(\alpha)$, then $g(\alpha)\frac{k(\alpha)}{k(\beta)} = \emptyset$, $\mathbb{Z}$, $2\mathbb{Z}$ or $2\mathbb{Z}+1$.

We call a PEBS $\mathcal{D}$ an *elliptic base system* (EBS for short) if the following holds.

$$\forall\alpha\in R(k,g), \quad s_\alpha(R(k,g)) = R(k,g).$$

Lemma 1.5. *Let $\mathcal{D}$ a PEBS with $l \geq 2$. If $\mathcal{D}$ is an EBS, then it is also a QEBS.*

Proof. The axiom (1) follows from Lemma 1.4 and the following (*c.f.* [S, Proof of (6.1) Assertion]).

$$\begin{cases} s_\alpha s_{\alpha+mk(\alpha)a}(\beta) = \beta + mJ(\alpha^\vee, \beta)k(\alpha)a, \\ s_\beta s_{\beta+mk(\beta)a}(\alpha) = \alpha + mk(\beta)a \end{cases}$$

for $m \in \mathbb{Z}$.

If $\beta \in \Pi_c(\alpha)$, then $\mathbb{Z} \ni J((2\alpha)^\vee, \beta) = \frac{J(\alpha^\vee,\beta)}{2}$. Hence, if $g(\alpha) \neq \emptyset$, then $\alpha \in \Pi^B$, which implies the axiom (2).

Let $\alpha \in \Pi^B$. Assume $g(\alpha) \neq \emptyset$. Let $n \in \mathbb{Z}$ be such that $2\alpha + nk(\alpha)a \in R(k, g)$. Then

$$s_\alpha s_{\alpha \pm k(\alpha)a}(2\alpha + nk(\alpha)a) = 2\alpha + (n \mp 4)k(\alpha)a$$

and

$$s_\alpha s_{2\alpha+nk(\alpha)a}(2\alpha + nk(\alpha)a) = 2\alpha - nk(\alpha)a\,.$$

Hence $g(\alpha) = \mathbb{Z}$, $2\mathbb{Z}$, $2\mathbb{Z}+1$, $4\mathbb{Z}$ or $4\mathbb{Z}+2$. Let $\beta \in \Pi_c(\alpha)$. Then

$$s_\beta s_{\beta+mk(\beta)a}(2\alpha + nk(\alpha)a) = 2\alpha + (nk(\alpha) - 2mk(\beta))a$$

Hence $g(\alpha) = \mathbb{Z}$, $2\mathbb{Z}$ or $2\mathbb{Z}+1$ if $k(\beta) = k(\alpha)$. Moreover

$$s_\alpha s_{2\alpha+nk(\alpha)a}(\beta + mk(\beta)a) = \beta + (mk(\beta) - nk(\alpha))a\,.$$

Hence $g(\alpha) = 2\mathbb{Z}$, $4\mathbb{Z}$ or $4\mathbb{Z}+2$ if $k(\beta) = 2k(\alpha)$. This implies the axiom (3) and completes the proof. □

Converse of Lemma 1.5 shall be given in Theorem 2.2.

If $\mathcal{D}(\mathcal{E}, \Pi, a, k, 0)$ is a QEBS, i.e., $g = 0$, then it is called an *special* QEBS (SQEBS for short).

1.3 Elliptic root systems

Keep the notation in §1. Notice that $l \geq 2$. For a subset S of $\mathcal{E}$, let $S^\perp := \{x \in \mathcal{E} | \forall y \in S,\ J(x, y) = 0\}$. Following [S], we say that a subset R of $\mathcal{E}$ is an *elliptic root system* (ERS for short) of rank l if it satisfies the following.

(1.3) $$\forall x,\, \forall y \in \mathbb{R}R,\ J(x, x)J(y, y) \in \mathbb{R}_+,$$

$$\dim_{\mathbb{C}}(\mathbb{C}R \cap R^{\perp}) = 2, \tag{1.4}$$

$$\dim_{\mathbb{C}} \mathbb{C}R = l + 2 = \mathrm{rank}_{\mathbb{Z}}\mathbb{Z}R, \tag{1.5}$$

$$\forall\alpha \in R,\ s_{\alpha}(R) = R, \tag{1.6}$$

$$\forall\alpha,\, \forall\beta \in R,\ J(\alpha^{\vee}, \beta) \in \mathbb{Z}, \tag{1.7}$$

$$\text{If } R = R_1 \cup R_2,\ R_2 \subset (R_1)^{\perp} \text{ for some } R_1,\ R_2 \subset R, \text{ then } R_1 \neq \emptyset \text{ or } R_2 \neq \emptyset, \tag{1.8}$$

where $\mathbb{R}_+ = \{x \in \mathbb{R} | x \geq 0\}$.

Let R be an ERS. We call the $\mathcal{E}$ for the R the *base space*. A one dimensional subspace G of $\mathbb{C}R \cap R^{\perp}$ is called a *marking line* if $G \cap \mathbb{Z}R \neq \{0\}$. The pair (R, G) is called a *marked elliptic root system* (MERS for short). An MERS (R, G) is called a *reduced marked elliptic root system* (RMERS for short) if

$$\forall\alpha,\, \forall\beta \in R,\ 2\alpha - \beta \notin G\,. \tag{1.9}$$

By [S], we have the following.

Theorem 1.1 ((6.4) of [S]). *If (R, G) be an RMERS, then there exists an SQEBS $\mathcal{D}(\mathcal{E}, \Pi, a, k, 0)$ such that $R = R(k, 0)$, $G = \mathbb{C}a$ and $\mathcal{E}$ is the base space of R. If $\mathcal{D}(\mathcal{E}, \Pi, a, k, 0)$ is an SQEBS, then $(R(k, 0), \mathbb{C}a)$ is an RMERS; in particular, $\mathcal{D}(\mathcal{E}, \Pi, a, k, 0)$ is an EBS.*

In [S], we do not need the assumption $l \geq 2$ for Theorem 1.1.

Theorem 1.2. *(1) If $\mathcal{D}$ is an EBS, then $(R(k, g), \mathbb{C}a)$ is an MERS.*

(2) Let (R, G) be an MERS. Then there exists an EBS $\mathcal{D}$ such that $R = R(k, g)$, $G = \mathbb{C}a$ and $\mathcal{E}$ is the base space of R.

Proof. The statement (1) is clear. We prove the statement (2). Let

$$R' := \{\alpha \in R \,|\, \frac{\alpha}{2} \notin R + G\}.$$

Let $R'' := R \setminus R'$. We have

$$\frac{1}{2}R'' \subset R' + G \tag{1.10}$$

because, if $\alpha \in R''$ is such that $\frac{\alpha}{2} \notin (R'+G)$, then $\frac{\alpha}{4} \in R+G$ and $J(\alpha^\vee, \frac{\alpha}{4}) = \frac{1}{2} \notin \mathbb{Z}$, contradiction.

We show

$$\mathbb{Z}R' = \mathbb{Z}R. \tag{1.11}$$

Clearly $\mathbb{Z}R' \subset \mathbb{Z}R$ holds. Let $\beta \in R''$. By (1.10), $\beta = 2\alpha + x$ for some $\alpha \in R'$ and some $x \in G$. Notice that

$$\forall \gamma \in R,\ \sigma_\gamma(R') = R' \quad \text{and} \quad \sigma_\gamma(R'') = R''\,. \tag{1.12}$$

It follows that $R' \ni \sigma_\alpha\sigma_\beta(\alpha) = \sigma_\alpha(\alpha - \beta) = -\alpha - (\beta - 4\alpha) = 3\alpha - \beta = \alpha - x$. Hence $x \in \mathbb{Z}R'$. Hence $\mathbb{Z}R \subset \mathbb{Z}R'$, as desired.

By (1.11), we see that (R', G) is an RMERS whose base space is $\mathcal{E}$. By Theorem 1.1, there exists an SQEBS $\mathcal{D}(\mathcal{E}, \Pi, a, k, 0)$ such that $R(k,0) = R'$ and $G = \mathbb{C}a$. It follows from Lemma 1.4 that

$$(\alpha + G) \cap R = (\alpha + G) \cap R' = (\alpha + G) \cap R(k,0) = \alpha + \mathbb{Z}k(\alpha)a \tag{1.13}$$

for $\alpha \in \Pi$. To complete the proof, it suffices to show that

$$\forall \alpha \in \Pi,\ R'' \cap (2\alpha + G) \subset 2\alpha + \mathbb{Z}k(\alpha)a\,.$$

Let $\beta \in R'' \cap (2\alpha + G)$. Let $x := \beta - 2\alpha$. Then $\sigma_\alpha\sigma_\beta(\alpha) = 3\alpha - \beta = \alpha - x$. By (1.13), we have $x \in \mathbb{Z}k(\alpha)a$, as desired. □

2 Elliptic algebras and superalgebras

2.1 Definition with generators and relations

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. For $\alpha \in \Pi$, let

$$c(\alpha) := \begin{cases} 2 & \text{if } g(\alpha) = \mathbb{Z} \text{ or } 2\mathbb{Z}+1, \\ 1 & \text{otherwise,} \end{cases}$$

and let

$$\alpha^* := c(\alpha)\alpha + k(\alpha)a \in \mathcal{E}.$$

For $\alpha \in R(k,g)$, let

$$p(\alpha) := \begin{cases} 1 & \text{if } 2\alpha \in R(k,g), \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\mathcal{A} := \{(\alpha,\beta,y) \in \Pi \times \Pi \times \mathbb{N} \,|\, \alpha \neq \beta,\ J(\alpha,\beta) \neq 0,\ k(\alpha)y = k(\beta)\}.$$

Let

$$\mathcal{B} := \bigcup_{\alpha\in\Pi} \{\alpha,\ -\alpha,\ \alpha^*,\ -\alpha^*\}.$$

For μ, $\nu \in R(k,g)$ with $\nu \neq \mu$, let

$$x_{\mu,\nu} := \begin{cases} 1 - J(\mu^\vee,\nu) & \text{if } \mu,\ \nu \in \mathcal{B},\ \mu \neq \nu \text{ and } J(\mu^\vee,\nu) < 0, \\ 0 & \text{if } \mu,\ \nu \in \mathcal{B},\ \mu \neq \nu \text{ and } J(\mu^\vee,\nu) \geq 0. \end{cases}$$

Keep the notation as above. Let $\mathfrak{g}_{\mathcal{D}} = \mathfrak{g}_{\mathcal{D}(\mathcal{E},\Pi,a,k,g)}$ be the Lie superalgebra defined by generators

$$h_\sigma\,(\sigma \in \mathcal{E}), E_\mu\,(\mu \in \mathcal{B}) \tag{2.1}$$

with parities

$$p(h_\sigma) = 0,\ p(E_\mu) = p(\mu) \tag{2.2}$$

and defining relations

$$\begin{cases} xh_\sigma + yh_\tau = h_{x\sigma+y\tau} & \text{if } x,\ y \in \mathbb{C} \text{ and } \sigma,\ \tau \in \mathcal{E}, \\ [h_\sigma, h_\tau] = 0 & \text{if } \sigma,\ \tau \in \mathcal{E}, \\ [h_\sigma, E_\mu] = J(\sigma,\mu)E_\mu & \text{if } \sigma \in \mathcal{E} \text{ and } \mu \in \mathcal{B}, \\ [E_\mu, E_{-\mu}] = h_{\mu^\vee} & \text{if } \mu \in \mathcal{B}, \\ (\mathrm{ad}E_\mu)^{x_{\mu,\nu}} E_\nu = 0 & \text{if } \mu,\ \nu \in \mathcal{B} \text{ and } \mu \neq \nu, \\ c(\alpha)(\mathrm{ad}E_{\alpha^*})^y E_\beta = (\mathrm{ad}E_\alpha)^{c(\alpha)y} E_{\beta^*} & \text{if } (\alpha,\beta,y) \in \mathcal{A}, \\ c'(\alpha)(\mathrm{ad}E_{-\alpha^*})^y E_{-\beta} = (\mathrm{ad}E_{-\alpha})^{c(\alpha)y} E_{-\beta^*} & \text{if } (\alpha,\beta,y) \in \mathcal{A}, \\ (\mathrm{ad}E_\alpha)^i (\mathrm{ad}E_{\alpha^*})^{y-i} E_\beta = 0 & \text{if } (\alpha,\beta,y) \in \mathcal{A} \text{ and } 1 \leq i \leq y-1, \\ (\mathrm{ad}E_{-\alpha})^i (\mathrm{ad}E_{-\alpha^*})^{y-i} E_{-\beta} = 0 & \text{if } (\alpha,\beta,y) \in \mathcal{A} \text{ and } 1 \leq i \leq y-1, \end{cases} \tag{2.3}$$

where $c'(\alpha) = (-1)^{c(\alpha)+1}c(\alpha)$.

For $\mu \in \mathcal{E}$, let $\mathfrak{g}_{\mathcal{D},\mu} := \{X \in \mathfrak{g}_{\mathcal{D}} | \forall h_\sigma,\ [h_\sigma, X] = J(\sigma,\mu)X\}$. Define the sub Lie superalgebra $\mathfrak{h}_{\mathcal{D}}$ of $\mathfrak{g}_{\mathcal{D}}$ by $\mathfrak{h}_{\mathcal{D}} := \{h_\sigma \in \mathfrak{g}_{\mathcal{D},0} | \sigma \in \mathcal{E}\}$.

Lemma 2.1. *Keep the notation as above. Then* $h_\sigma \neq 0$ *for* $\sigma \in \mathcal{E} \setminus \{0\}$*. In particular,* $\dim \mathfrak{h}_{\mathcal{D}} = l + 4$.

Proof of the lemma shall be given in Subsec. 4.2.

2.2 Main theorem

We see that there exists a unique $\delta \in \mathbb{Z}_+\Pi$ such that

$$\mathbb{Z}\delta = \{\lambda \in \mathbb{Z}\Pi | J(\lambda, \lambda) = 0\}.$$

Let $\mathbb{Z}^{2,\prime} := \mathbb{Z}^2 \setminus \{(0,0)\}$.

Theorem 2.1. *Let $\mathcal{D} = \mathcal{D}(\mathcal{E}, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. Then we have*

$$\mathfrak{g}_{\mathcal{D}} = \mathfrak{h}_{\mathcal{D}} \bigoplus \Big(\bigoplus_{\nu \in R(k,g)} \mathfrak{g}_{\mathcal{D},\nu} \Big) \bigoplus \Big(\bigoplus_{(m,n) \in \mathbb{Z}^{2,\prime}} \mathfrak{g}_{\mathcal{D},m\delta+na} \Big).$$

Moreover $\mathfrak{h}_{\mathcal{D}} = \mathfrak{g}_{\mathcal{D},0}$*, and* $\dim \mathfrak{g}_{\mathcal{D},\alpha} = 1$ *for all* $\alpha \in R(k,g)$.

Proof of the theorem shall be given in Subsec. 3.2.

From now until the end of Subsec. 2.2, we suppose that we have proved Theorem 2.1. Let $\nu \in R(k,g)$. Let $E'_{\pm\nu} \in \mathfrak{g}_{\mathcal{D},\pm\nu}$ be such that $[E'_\nu, E'_{-\nu}] = h_{\nu^\vee}$. By Theorem 2.1, $E'_{\pm\nu}$ are locally nilpotent. Hence we can define $n_\nu \in \mathrm{Aut}(\mathfrak{g}_{\mathcal{D}})$ by

$$n_\nu = \begin{cases} \exp \mathrm{ad} E'_\nu \exp \mathrm{ad}(-E'_{-\nu}) \exp \mathrm{ad} E'_\nu & \text{if } p(\nu) = 0, \\ \exp(\frac{1}{4}\mathrm{ad}[E'_\nu, E'_\nu]) \exp(\frac{1}{4}\mathrm{ad}[E'_{-\nu}, E'_{-\nu}]) \exp(\frac{1}{4}\mathrm{ad}[E'_\nu, E'_\nu]) & \text{if } p(\nu) = 1. \end{cases} \tag{2.4}$$

Theorem 2.2. *Let $\mathcal{D}$ be a PEBS with $l \geq 2$. Then $\mathcal{D}$ is an EBS if and only if it is a QEBS.*

Proof. The 'only-if'-part follows from Lemma 1.5. Here we only prove the 'if'-part. Recall n_ν ($\nu \in R(k,g)$) from (2.4). Notice that $n_\nu(\mathfrak{g}_{\mathcal{D},\lambda}) = \mathfrak{g}_{\mathcal{D},s_\nu(\lambda)}$. Then, by Theorem 2.1, we see that $\mathcal{D}$ is an EBS. ☐

3 Proof of Theorem 2.1

In this section, we suppose that we have proved Lemma 2.1.

3.1 Rank one and two subsystems

Let S be a finite subset of $R(k,g)$. Assume that the elements of S are linearly independent and that the square matrix $A_S := (J(\alpha^\vee, \beta))_{\alpha,\beta \in S}$ is an affine type generalized Cartan matrix in the sense of [K1, §4.8]. Then we call the S the *affine type subset* of $R(k,g)$. Let

EW_S be the subgroup of $\mathrm{GL}(\mathcal{E})$ generated by s_μ ($\mu \in S$). Let $S^{\rm odd} := \{\alpha \in S | p(\alpha) = 1\}$ and let

$$R(k,g)^S := \bigcup_{w \in EW_S} w\Big(\bigcup_{\alpha \in S \setminus S^{\rm odd}} \{\alpha\} \bigcup \bigcup_{\alpha \in S^{\rm odd}} \{\alpha, 2\alpha\} \Big).$$

Then we see that $R(k,g)^S$ is the affine type (real) root system with the base S; $R(k,g)^S$ is reduced if and only if $S^{\rm odd} = \emptyset$. For the pair $(A_S, S^{\rm odd})$ of the above A_S and $S^{\rm odd}$, we define the the Dynkin diagram $\Gamma(A_S, S^{\rm odd})$ in the same manner as in [K2]. If $\Gamma(A_S, S^{\rm odd})$ is called $\mathbf{X}$ in the tables [K2, Tables 1-4], we say that the name of S is $\mathbf{X}$. The following two lemmas follow from Lemma 1.4 and the well-known fact [M, Appendixes 1-2].

Lemma 3.1. *Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Let $\alpha \in \Pi$. Then $\{\alpha, -\alpha^*\}$ is an affine type subset of $R(k,g)$, and we have $R(k,g)_{\{\alpha\}} = R(k,g)^{\{\alpha,-\alpha^*\}}$.*

The name $\mathbf{X}$ of the $\{\alpha, -\alpha^*\}$ is given in the Table below.

$g(\alpha)$	$\emptyset$	$2\mathbb{Z}+1$	$\mathbb{Z}$	$2\mathbb{Z}$	$4\mathbb{Z}+2$	$4\mathbb{Z}$
$\mathbf{X}$	$\mathbf{A}_1^{(1)}$	$\mathbf{A}_2^{(2)}$	$\mathbf{B}^{(1)}(0,1)$	$\mathbf{C}^{(2)}(2)$	$\mathbf{A}^{(4)}(0,2)$	$\mathbf{A}^{(4)}(0,2)$

Table 1

Lemma 3.2. *Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Let α, $\beta \in \Pi$ be such that $J(\alpha, \beta^\vee) = -1$. Let*

$$\gamma := \begin{cases} s_\alpha(-\beta^*) & \textit{if } k(\alpha) = k(\beta),\ 1 \leq -J(\alpha^\vee, \beta) \leq 2 \textit{ and } g(\alpha) = \emptyset \textit{ or } 2\mathbb{Z}, \\ s_\beta s_\alpha(-\beta^*) & \textit{if } J(\alpha^\vee, \beta) = -3 \textit{ and } k(\alpha) = k(\beta), \\ s_\alpha s_\beta(-\alpha^*) & \textit{if } 3k(\alpha) = k(\beta), \\ s_\beta(-\alpha^*) & \textit{otherwise.} \end{cases} \tag{3.1}$$

Then $\{\alpha, \beta, \gamma\}$ is an affine subset of $R(k,g)$ and we have $R(k,g)_{\{\alpha,\beta\}} = R(k,g)^{\{\alpha,\beta,\gamma\}}$.

The name $\mathbf{Y}$ of the $\{\alpha, \beta, \gamma\}$ is given in the following table.

$\mathbf{Y}$	γ	$J(\alpha^\vee,\beta)$	$k(\beta)/k(\alpha)$	$g(\alpha)$
$\mathbf{A}_2^{(1)}$	$s_\alpha(-\beta^*)$	-1	1	$\emptyset$
$\mathbf{C}_2^{(1)}$	$s_\alpha(-\beta^*)$	-2	1	$\emptyset$
$\mathbf{G}_2^{(1)}$	$s_\beta s_\alpha(-\beta^*)$	-3	1	$\emptyset$
$\mathbf{D}_3^{(2)}$	$s_\beta(-\alpha^*)$	-2	2	$\emptyset$
$\mathbf{D}_4^{(3)}$	$s_\alpha s_\beta(-\alpha^*)$	-3	3	$\emptyset$
$\mathbf{A}_4^{(2)}$	$s_\beta(-\alpha^*)$	-2	1	$2\mathbb{Z}+1$
$\mathbf{B}^{(1)}(0,2)$	$s_\beta(-\alpha^*)$	-2	1	$\mathbb{Z}$
$\mathbf{A}^{(2)}(0,3)$	$s_\alpha(-\beta^*)$	-2	1	$2\mathbb{Z}$
$\mathbf{C}^{(2)}(3)$	$s_\beta(-\alpha^*)$	-2	2	$2\mathbb{Z}$
$\mathbf{A}^{(4)}(0,4)$	$s_\beta(-\alpha^*)$	-2	2	$4\mathbb{Z}+2$ or $4\mathbb{Z}$

TABLE 2

3.2 Embedded rank two affine (super)algebras

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. For $\mu \in \mathcal{B}$, we define $n_\mu \in \mathrm{Aut}(\mathfrak{g}_{\mathcal{D}})$ in the same way as in (2.4) with $E_{\pm\mu}$ in place of $E'_{\pm\mu}$.

Lemma 3.3. *Keep the notation as in Lemma 3.2. Let $S := \{\alpha, \beta, \gamma\}$. If γ is defined as $s_{\nu_1}\cdots s_{\nu_r}(-\mu^*)$ in (3.1), we let*

$$E_{\pm\gamma} := n_{\nu_1}\cdots n_{\nu_r}(E_{\mp\mu^*}). \tag{3.2}$$

Then we have

$$\begin{cases} (\mathrm{ad}E_{\pm\mu})^{1-J(\mu^\vee,\nu)}E_{\pm\nu} = 0 & \textit{if } \mu,\ \nu \in S \textit{ with } \mu \neq \nu, \\ [E_\mu, E_{-\nu}] = \delta_{\mu\nu}h_{\mu^\vee} & \textit{if } \mu,\ \nu \in S. \end{cases} \tag{3.3}$$

Proof. If the name of S is neither $\mathbf{A}_4^{(2)}$ nor $\mathbf{B}(0,2)^{(1)}$, the equalities (3.3) is proved in a similar way to [Ya2, §2.3]. We assume that the name of S is $\mathbf{A}_4^{(2)}$ or $\mathbf{B}(0,2)^{(1)}$. It is clear that $[E_\gamma, E_{-\gamma}] = h_{s_\beta(-(\alpha^*)^\vee)} = h_{\gamma^\vee}$. We have

$$E_\gamma = n_\beta(E_{-\alpha^*}) = 2^{-1}[E_{-\beta},[E_{-\beta},E_{-\alpha^*}]] = 4^{-1}[E_{-\beta},[E_{-\alpha},[E_{-\alpha},E_{-\beta^*}]]].$$

It is clear that $[E_\gamma, E_{-\beta}] = 0$ and $(\mathrm{ad}E_\beta)^3E_\gamma = 0$. We have

$$\begin{aligned} [E_\gamma, E_{-\alpha}] &= 4^{-1}[[E_{-\beta},E_{-\alpha}],[E_{-\alpha},[E_{-\alpha},E_{-\beta^*}]]] \\ &= 2^{-1}[[E_{-\beta},E_{-\alpha}],[E_{-\beta},E_{-\alpha^*}]] \\ &= -2^{-1}n_\beta([[E_{-\beta},E_{-\alpha}],E_{-\alpha^*}]) \\ &= -2^{-1}n_\beta([[E_{-\beta},E_{-\alpha^*}],E_{-\alpha}]) \\ &= 4^{-1}n_\beta([[E_{-\alpha},[E_{-\alpha},E_{-\beta^*}]],E_{-\alpha}]) = 0 \end{aligned}$$

and

$$\begin{aligned}
[E_\gamma, E_\alpha] &= 4^{-1}[E_{-\beta}, \{(-1)^{p(\alpha)}(2+J(\alpha^\vee,\beta))+J(\alpha^\vee,\beta)\}[E_{-\alpha},E_{-\beta^*}]] \\
&= -2^{-1}[E_{-\beta},[E_{-\alpha},E_{-\beta^*}]] \\
&= 2^{-1}[[E_{-\alpha},E_{-\beta}],E_{-\beta^*}] \\
&= -4^{-1}n_\alpha([E_{-\beta},[E_{-\alpha},[E_{-\alpha},E_{-\beta^*}]]]) \\
&= -2^{-1}n_\alpha([E_{-\beta},[E_{-\beta},E_{-\alpha^*}]]) = 0.
\end{aligned}$$

Thus we get the equalities (3.3); the other equalities than the above ones can be proved similarly. □

For a subset S of Π, let $\mathfrak{g}_{\mathcal{D}}^S$ be the subalgebra of $\mathfrak{g}_{\mathcal{D}}$ generated by h_σ ($\sigma\in\mathcal{E}$) and E_μ, $E_{-\mu}$, E_{μ^*}, $E_{-\mu^*}$ ($\mu\in S$). For $\nu\in\mathcal{E}$, let $\mathfrak{g}_{\mathcal{D},\nu}^S := \mathfrak{g}_{\mathcal{D}}^S\cap\mathfrak{g}_{\mathcal{D},\nu}$. Let $m_S := \min\{k(\alpha)|\alpha\in S\}$.

Lemma 3.4. *Let $\mathcal{D}$ be a QEBS with $l\geq 2$. Let S be a subset of Π such that $|S|=1$ or 2. Then we have*

$$\mathfrak{g}_{\mathcal{D}}^S = \mathfrak{h}_{\mathcal{D}}\bigoplus\Big(\bigoplus_{\lambda\in R(k,g)_S}\mathfrak{g}_{\mathcal{D},\lambda}^S\Big)\bigoplus\Big(\bigoplus_{n\in\mathbb{Z}\setminus\{0\}}\mathfrak{g}_{\mathcal{D},nm_Sa}^S\Big). \tag{3.4}$$

Moreover we have $\dim\mathfrak{g}_{\mathcal{D},\lambda}^S = 1$ *for* $\lambda\in R(k,g)_S$. *Furthermore*

$$\mathfrak{g}_{\mathcal{D},nm_Sa}^S = \mathfrak{g}_{\mathcal{D},nm_Sa}^{\{\alpha\}}\oplus\mathfrak{g}_{\mathcal{D},nm_Sa}^{\{\beta\}} \tag{3.5}$$

if $S=\{\alpha,\beta\}$.

Proof. We first assume $|S|=1$ and $S=\{\alpha\}$. Recall the affine subset $T:=\{\alpha,-\alpha^*\}$ from Lemma 3.1. It follows from Lemma 2.1 that $E_{\pm\mu}\neq 0$ for $\mu\in T$, since $[E_\mu,E_{-\mu}] = h_{\mu^\vee}\neq 0$. Since $E_{\pm\mu}$ ($\mu\in T$) satisfy the Serre relations (see (2.3)), the lemma follows from the well-known argument [K1, Corollary 5.12] (see also [K2, Proposition 1.6]).

Assume $|S|=2$, and $S=\{\alpha,\beta\}$. If $J(\beta^\vee,\alpha)=0$, the lemma follows from the same argument as above. Assume $J(\beta^\vee,\alpha)=-1$. Recall γ and $E_{\pm\gamma}$ from Lemma 3.3. Let $U:=\{\alpha,\beta,\gamma\}$. We have $E_{\pm\gamma}\in\mathfrak{g}_{\mathcal{D}}^S$ since $E_{\pm\gamma} = n_{\nu_1}\cdots n_{\nu_r}E_{\mp\mu^*}$. Let $\mathfrak{g}_{\mathcal{D}}^{(U)}$ be the sub Lie superalgebra of $\mathfrak{g}_{\mathcal{D}}^S$ generated by $\mathfrak{h}_{\mathcal{D}}$ and $E_{\pm\omega}$ ($\omega\in U$). Then $E_{\pm\mu^*} = n_{\nu_r}^{-1}\cdots n_{\nu_1}^{-1}E_{\mp\gamma}\in\mathfrak{g}_{\mathcal{D}}^{(U)}$. Let $\rho\in\{\alpha,\beta\}$ be such that $\rho\neq\mu$ (see Lemma 3.2 for μ). By the sixth and seventh equalities of (2.3), we have $n_\mu E_{\pm\rho^*} = n_{\mu^*}E_{\pm\rho}$. Hence $E_{\pm\rho^*} = n_\mu^{-1}n_{\mu^*}E_{\pm\rho}\in\mathfrak{g}_{\mathcal{D}}^{(U)}$. Hence $\mathfrak{g}_{\mathcal{D}}^{(U)} = \mathfrak{g}_{\mathcal{D}}^S$. By Lemma 3.3, $E_{\pm\omega}$ ($\omega\in U$) satisfy the Serre relations (3.3). Using the well-known argument [K1, Corollary 5.12] again, we have (3.4) for the S. The equality (3.5) follows from the fact that

$$(\mathbb{Z}_{\pm}\alpha+\mathbb{Z}_{\mp}\beta+\mathbb{Z}a)\cap R(k,g)_S=\emptyset, \tag{3.6}$$

and that $\mathfrak{g}_{\mathcal{D}}^{S}$ is generated by $\mathfrak{g}_{\mathcal{D},\lambda}^{\{\alpha\}}$ ($\lambda\in R(k,g)_{\{\alpha\}}$) and $\mathfrak{g}_{\mathcal{D},\nu}^{\{\beta\}}$ ($\nu\in R(k,g)_{\{\beta\}}$) (See also Lemma 1.4). This completes the proof. $\square$

Keep the notation as above. Let S be a subset of Π. Let $ED_S:=\mathbb{Z}S+\mathbb{Z}m_Sa$. Define the subsets of $ED_{S,+}$ and $ED_{S,-}$ of ED_S by $ED_{S,\pm}:=(\mathbb{Z}_{\pm}S+\mathbb{Z}m_Sa)\setminus\mathbb{Z}m_Sa$. Define the sub Lie superalgebras $\mathfrak{n}_{\mathcal{D}}^{S,+}$, $\mathfrak{n}_{\mathcal{D}}^{S,-}$ $\mathfrak{l}_{\mathcal{D}}^{S,+}$ and $\mathfrak{l}_{\mathcal{D}}^{S,-}$ of $\mathfrak{g}_{\mathcal{D}}^{S}$ by

$$\mathfrak{n}_{\mathcal{D}}^{S,\pm}:=\bigoplus_{\lambda\in EQ_{S,\pm}}\mathfrak{g}_{\mathcal{D},\lambda}^{S}$$

and

$$\mathfrak{l}_{\mathcal{D}}^{S,\pm}:=\bigoplus_{n\in\mathbb{Z}_{\pm}\setminus\{0\}}\mathfrak{g}_{\mathcal{D},nm_sa}^{S}.$$

Lemma 3.5. *Keep the notation as above. Then the following hold.*

(1) $\mathfrak{g}_{\mathcal{D}}^{S}=\mathfrak{n}_{\mathcal{D}}^{S,+}\oplus\mathfrak{l}_{\mathcal{D}}^{S,+}\oplus\mathfrak{h}_{\mathcal{D}}\oplus\mathfrak{l}_{\mathcal{D}}^{S,-}\oplus\mathfrak{n}_{\mathcal{D}}^{S,-}$.

(2) $\mathfrak{n}_{\mathcal{D}}^{S,+}$ *(respectively,* $\mathfrak{l}_{\mathcal{D}}^{S,+}$, $\mathfrak{l}_{\mathcal{D}}^{S,-}$ *or* $\mathfrak{n}_{\mathcal{D}}^{S,-}$*) is generated by* $\mathfrak{n}_{\mathcal{D}}^{\{\alpha\},+}$ *(respectively,* $\mathfrak{l}_{\mathcal{D}}^{\{\alpha\},+}$, $\mathfrak{l}_{\mathcal{D}}^{\{\alpha\},-}$ *or* $\mathfrak{n}_{\mathcal{D}}^{\{\alpha\},-}$*) with* $\alpha\in S$.

(3) $\dim\mathfrak{g}_{\mathcal{D},\lambda}^{S}=1$ *for* $\lambda\in R(k,g)_{\{\alpha\}}$ *if* $\alpha\in\Pi$.

Proof. If $|S|=1$ or 2, the lemma follows from Lemma 3.4. Assume $|S|\geq 3$. From Lemma 3.4 and (3.6) when $J(\alpha^\vee,\beta)\neq 0$, or from the defining relations (2.3) when $J(\alpha^\vee,\beta)=0$, it follows that

$$[\mathfrak{n}_{\mathcal{D}}^{\{\alpha\},\pm},\mathfrak{n}_{\mathcal{D}}^{\{\beta\},\mp}]=\{0\}$$

and

$$[\mathfrak{l}_{\mathcal{D}}^{\{\alpha\},+}\oplus\mathfrak{l}_{\mathcal{D}}^{\{\alpha\},-},\mathfrak{n}_{\mathcal{D}}^{\{\beta\},\pm}]\subset\mathfrak{n}_{\mathcal{D}}^{\{\beta\},\pm}.$$

for α, $\beta\in\Pi$ with $\alpha\neq\beta$. Then the lemma follows from this fact and Lemma 2.1. $\square$

Proof of Theorem 2.1. The theorem follows from Lemmas 1.3, 1.4 and 3.5. $\square$

4 Proof of Lemma 2.1

4.1 An affine Lie superalgebra

Here we first recall the definition of the contragredient Lie superalgebras. Let $\bar{I}$ be a finite set. Let $\bar{I}^{\mathrm{odd}}$ be a subset of $\bar{I}$. Define a map $\bar{p}:\bar{I}\to\{0,1\}$ by $\bar{p}(i)=1$ $(i\in\bar{I}^{\mathrm{odd}})$ and $\bar{p}(j)=0$ $(j\in\bar{I}\setminus\bar{I}^{\mathrm{odd}})$. Let $\bar{A}:=(\bar{a}_{ij})_{i,j\in\bar{I}}$ be an $\bar{I}\times\bar{I}$ matrix. Let $\bar{\mathfrak{H}}$ be the $2|\bar{I}|$-dimensional $\mathbb{C}$-vector space. Let $\bar{\mathfrak{H}}^*$ be the dual space of $\bar{\mathfrak{H}}$. Let $\{\bar{\alpha}_i,\,\bar{\gamma}_i\,(i\in\bar{I})\}$ be a basis of $\bar{\mathfrak{H}}$. Let $\{\bar{h}_i,\,\bar{t}_i\,(i\in\bar{I})\}$ be a basis of $\bar{\mathfrak{H}}^*$. We assume that $\bar{\alpha}_i(\bar{h}_j)=\bar{a}_{ji}$, $\bar{\alpha}_i(\bar{t}_j)=\bar{\gamma}_i(\bar{h}_j)=\delta_{ji}$ and $\bar{\gamma}_i(\bar{t}_j)=0$. Let $\bar{\Pi}:=\{\bar{\alpha}_i\,(i\in\bar{I})\}$. Let $\bar{\Pi}^\vee:=\{\bar{h}_i\,(i\in\bar{I})\}$. For the datum $\bar{D}:=(\bar{A},\bar{I},\bar{I}^{\mathrm{odd}},\bar{\Pi},\bar{\Pi}^\vee)$, we define a Lie superalgebra $\bar{\mathfrak{G}}'_{\bar{D}}:=\bar{\mathfrak{G}}'(\bar{A},\bar{I}^{\mathrm{odd}})$ by generators

$$\bar{h}'_i,\,\bar{t}'_i,\;\bar{E}'_i,\,\bar{F}'_i\,(i\in\bar{I})$$

with parities

$$p(\bar{h}'_i)=p(\bar{t}'_i)=0,\;p(\bar{E}'_i)=p(\bar{F}'_i)=\bar{p}(i)$$

and defining relations

$$\begin{cases}[\bar{h}'_i,\bar{h}'_j]=[\bar{h}'_i,\bar{t}'_j]=[\bar{t}'_i,\bar{t}'_j]=0,\\ [\bar{h}'_i,\bar{E}'_j]=\bar{a}_{ij}\bar{E}'_j,[\bar{t}'_i,\bar{E}'_j]=\delta_{ij}\bar{E}'_j\ ,[\bar{h}'_i,\bar{F}'_j]=-\bar{a}_{ij}\bar{F}'_j\ ,[\bar{t}'_i,\bar{F}'_j]=-\delta_{ij}\bar{F}'_j,\\ [\bar{E}'_i,\bar{F}'_j]=\delta_{ij}\bar{h}'_i\,.\end{cases}$$

Let $\bar{\mathfrak{H}}'$ be the subalgebra of $\bar{\mathfrak{G}}'_{\bar{D}}$ generated by $\bar{h}'_i$, $\bar{t}'_i$. Let $\mathfrak{r}$ be the ideal of $\bar{\mathfrak{G}}'_{\bar{D}}$ which is maximal among the ones $\mathfrak{r}'$ such that $\mathfrak{r}'\cap\bar{\mathfrak{H}}'=\{0\}$. We denote by $\bar{\mathfrak{G}}_{\bar{D}}=\bar{\mathfrak{G}}(\bar{A},\bar{I}^{\mathrm{odd}})$ the quotient Lie superalgebra $\bar{\mathfrak{G}}'_{\bar{D}}/\mathfrak{r}$. In this paper, we call the $\bar{\mathfrak{G}}_{\bar{D}}$ the *contragredient Lie superalgebra*. Let $\bar{\pi}:\bar{\mathfrak{G}}'_{\bar{D}}\to\bar{\mathfrak{G}}_{\bar{D}}$ be a natural projective map. Notice that $\dim\bar{\pi}(\bar{\mathfrak{H}}')=2|\bar{I}|$. By abuse of notation, we shall also denote $\bar{\pi}(\bar{\mathfrak{H}}')$, $\bar{\pi}(\bar{h}'_i)$ and $\bar{\pi}(\bar{t}'_i)$ by $\bar{\mathfrak{H}}$, $\bar{h}_i$ and $\bar{t}_i$, respectively. We shall also denote $\bar{\pi}(\bar{E}'_i)$ and $\bar{\pi}(\bar{F}'_i)$ by $\bar{E}_i$ and $\bar{F}_i$, respectively.

Keep the notation as above. Let $\bar{I}^{\mathrm{pos}}:=\{i\in\bar{I}|\bar{a}_{ii}\neq 0\}$. Let $\bar{I}^{\mathrm{null}}:=\bar{I}\setminus\bar{I}^{\mathrm{pos}}$ Define the square matrix $\bar{A}^{\mathrm{pos}}$ by $\bar{A}^{\mathrm{pos}}:=(\bar{a}_{ij})_{i,j\in\bar{I}^{\mathrm{pos}}}$. We say that $\bar{D}$ is an *handy datum* if the following hold.

(1) If $|\bar{I}^{\mathrm{pos}}|<|\bar{I}|$, then $\bar{A}^{\mathrm{pos}}$ is a finite-type generalized Cartan matrix; $\bar{A}^{\mathrm{pos}}$ may be not necessarily irreducible.

(2) If $|\bar{I}^{\mathrm{pos}}|=|\bar{I}|$, $\bar{A}$ is an affine type generalized Cartan matrix.

(3) If i, $j\in\bar{I}^{\mathrm{null}}$ with $i\neq j$, then $(\bar{a}_{ij},\bar{a}_{ji})=(0,0)$, $(2,2)$.

(4) If $i\in\bar{I}^{\mathrm{pos}}$ and $j\in\bar{I}^{\mathrm{null}}$, then $(\bar{a}_{ij},\bar{a}_{ji})=(0,0)$, $(-1,-1)$ or $(-1,-2)$.

(5) $\bar{I}^{\mathrm{null}}\subset\bar{I}^{\mathrm{odd}}$.

(6) If $i\in\bar{I}^{\mathrm{pos}}\cap\bar{I}^{\mathrm{odd}}$ and $j\in\bar{I}^{\mathrm{null}}$, then $\bar{a}_{ij}=\bar{a}_{ji}=0$.

(7) If $i \in \bar{I}^{\mathrm{pos}} \cap \bar{I}^{\mathrm{odd}}$ and $j \in \bar{I}^{\mathrm{pos}} \setminus \{i\}$, then $|\bar{a}_{ij}| \geq |\bar{a}_{ji}|$ and $\bar{a}_{ij} = -2$ or -4.

(8) If $i \in \bar{I}^{\mathrm{null}}$, then there exists a unique $j \in \bar{I}^{\mathrm{null}}$ such that $i \neq j$ and $\bar{a}_{ij} \neq 0$.

(9) There exists a nondegenerate symmetric form $\bar{J} : \bar{\mathfrak{H}}^* \times \bar{\mathfrak{H}}^* \to \mathbb{C}$ such that $\bar{J}(\bar{\alpha}_i, \bar{\alpha}_i) = 0$ if and only if $\bar{a}_{ii} = 0$, and such that $\bar{J}(\bar{\alpha}_i, \bar{\alpha}_j) = b_i^{-1}\bar{a}_{ij}$, $\bar{J}(\bar{\gamma}_i, \bar{\alpha}_j) = b_i^{-1}\delta_{ij}$, $\bar{J}(\bar{\gamma}_i, \bar{\gamma}_j) = 0$, where $b_i := 2/\bar{J}(\bar{\alpha}_i, \bar{\alpha}_i)$ if $\bar{a}_{ii} \neq 0$; and, otherwise, $b_i := 1$.

For $\sigma \in \bar{\mathfrak{H}}^*$, let $\bar{h}_\sigma \in \bar{\mathfrak{H}}$ be such that $\tau(\bar{h}_\sigma) = \bar{J}(\tau, \sigma)$ for all $\tau \in \bar{\mathfrak{H}}^*$. Then $\bar{h}_i = b_i \bar{h}_{\bar{\alpha}_i}$ and $\bar{t}_i = b_i \bar{h}_{\bar{\gamma}_i}$.

Lemma 4.1. *Let $\bar{D}$ be a handy datum. Then the following hold for $\bar{\mathfrak{G}}(\bar{A}, \bar{I}^{\mathrm{odd}})$.*

(1) $(\mathrm{ad}\bar{E}_i)^{1-\bar{a}_{ij}}\bar{E}_j = 0$ for $i \in \bar{I}^{\mathrm{pos}}$ and $j \in \bar{I} \setminus \{i\}$.

(2) $[\bar{E}_i, \bar{E}_j] = 0$ if $\bar{a}_{ij} = 0$. In particular it follows that if $\bar{a}_{ii} = 0$, then $[\bar{E}_i, \bar{E}_i] = 0$ and $(\mathrm{ad}\bar{E}_i)^2 X = 0$ for any homogeneous element X of $\bar{\mathfrak{G}}$.

(3) $[\bar{E}_j, [[\bar{E}_i, \bar{E}_j], \bar{E}_m]] = 0$ if $\bar{a}_{jj} = 0$ and $-\bar{a}_{ji} = \bar{a}_{jm} \neq 0$.

(4) $[[\bar{E}_i, \bar{E}_j], \bar{E}_m] = [[\bar{E}_i, \bar{E}_m], \bar{E}_j]$ if $\bar{a}_{ii} = 2$, $\bar{a}_{jj} = \bar{a}_{mm} = 0$, $\bar{a}_{ji} = \bar{a}_{mi} = -1$ and $\bar{a}_{jm} = 2$.

(5) The same formulas as (1)-(4) with $\bar{F}_i$'s in place of $\bar{E}_i$'s hold.

(6) There exists an invariant form $\bar{J} : \bar{\mathfrak{G}} \times \bar{\mathfrak{G}} \to \mathbb{C}$ such that $\bar{J}(\bar{h}_\sigma, \bar{h}_\tau) = \bar{J}(\sigma, \tau)$. (By abuse of notation, we use the same symbol $\bar{J}$ for the bilinear forms on $\bar{\mathfrak{G}}$ and $\bar{\mathfrak{H}}^*$.)

See [Ya1, Proposition 6.7.1] for the proof.

Let D be an handy datum. Let $\mathbb{C}[t, t^{-1}]$ be the Laurent polynomial algebra. Let

$$\mathfrak{L}(\bar{D}) := \bar{\mathfrak{G}}(\bar{A}, \bar{I}^{\mathrm{odd}}) \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}v \oplus \mathbb{C}w.$$

We view $\mathfrak{L}(\bar{D})$ as a Lie superalgebra in the following way. The parity of $X \otimes t^n$ is the same as the one of X for a homogeneous element X of $\bar{\mathfrak{G}}(\bar{A}, \bar{I}^{\mathrm{odd}})$; the parities of v and w are 0. The Lie super bracket of $\mathfrak{L}(\bar{D})$ is given by

$$\begin{aligned}&[X \otimes t^m + a_1 v + b_1 w, Y \otimes t^n + a_2 v + b_2 w] \\ &\quad = [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}\bar{J}(X, Y)v + b_1 n Y \otimes t^n - b_2 m X \otimes t^m.\end{aligned}$$

for homogeneous elements X, Y of $\bar{\mathfrak{G}}(\bar{A}, \bar{I}^{\mathrm{odd}})$. We shall also denote by $\bar{J}$ the invariant form on $\mathfrak{L}(\bar{D})$ defined by

$$\bar{J}(X \otimes t^m + a_1 v + b_1 w, Y \otimes t^n + a_2 v + b_2 w) = \delta_{m+n,0}\bar{J}(X, Y) + a_1 b_2 + b_1 a_2.$$

4.2 Unfolding

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. We define a map $k^\vee : \Pi \to \{1,2,3,4\}$ by the following.

(1) If $J(\beta^\vee, \alpha) = -1$ and $g(\alpha) = \emptyset$ or $\mathbb{Z}$, then $k^\vee(\alpha) = \frac{k(\beta)}{k(\alpha)} k^\vee(\beta)$.
(2) If $J(\alpha^\vee, \beta) = -2$, $k(\alpha) = 2k(\beta)$ and $g(\alpha) = 2\mathbb{Z}$, then $k^\vee(\alpha) = 2k^\vee(\beta)$.
(3) If $J(\alpha^\vee, \beta) = -2$, $k(\alpha) = k(\beta)$ and $g(\alpha) = 2\mathbb{Z}+1$ or $2\mathbb{Z}$, then $k^\vee(\alpha) = k^\vee(\beta) = 2$.
(4) If $J(\alpha^\vee, \beta) = -2$ and $g(\alpha) = 4\mathbb{Z}$ or $4\mathbb{Z}+2$, then $k^\vee(\alpha) = 3$ and $k^\vee(\beta) = 2$.
(5) $k^\vee(\alpha) = 4k^\vee(\beta)$ if and only if $4k(\alpha) = k(\beta)$.

For the above $\mathcal{D}$, we define a handy datum $\bar{D}_{\mathcal{D}} = (\bar{A}_{\mathcal{D}}, \bar{I}_{\mathcal{D}}, \bar{I}^{\mathrm{odd}}_{\mathcal{D}}, \bar{\Pi}_{\mathcal{D}}, \bar{\Pi}^\vee_{\mathcal{D}})$ in the following way. Let

$$\bar{I}_{\mathcal{D}} := \{(\alpha, x) \in \Pi \times \{1,2,3,4\} | 1 \leq x \leq k^\vee(\alpha)\}.$$

We define a square matrix $\bar{A}_{\mathcal{D}} = (\bar{a}_{(\alpha,x),(\beta,y)})_{(\alpha,x),(\beta,y)\in \bar{I}_{\mathcal{D}}}$ in the following way.

(1) If $J(\alpha^\vee, \beta) = 0$, then $\bar{a}_{(\alpha,x),(\beta,y)} = 0$.

(2) Let $\alpha \in \Pi$. If $g(\alpha) = \emptyset$ or $\mathbb{Z}$ then $\bar{a}_{(\alpha,x),(\alpha,y)} = 2\delta_{xy}$. If $g(\alpha) = 2\mathbb{Z}+1$, then $\bar{a}_{(\alpha,x),(\alpha,y)} = 3\delta_{xy} - 1$. If $g(\alpha) = 2\mathbb{Z}$, then $\bar{a}_{(\alpha,x),(\alpha,y)} = 2 - 2\delta_{xy}$. If $g(\alpha) = 4\mathbb{Z}$ or $4\mathbb{Z}+2$, then $\bar{a}_{(\alpha,1),(\alpha,1)} = \bar{a}_{(\alpha,3),(\alpha,3)} = \bar{a}_{(\alpha,1),(\alpha,2)} = \bar{a}_{(\alpha,2),(\alpha,1)} = 0$, $\bar{a}_{(\alpha,2),(\alpha,2)} = \bar{a}_{(\alpha,1),(\alpha,3)} = \bar{a}_{(\alpha,3),(\alpha,1)} = 2$, $\bar{a}_{(\alpha,2),(\alpha,3)} = -1$ and $\bar{a}_{(\alpha,3),(\alpha,2)} = -2$

(3) Assume $J(\beta^\vee, \alpha) = -1$. Then

$$\begin{aligned}
&(\bar{a}_{(\alpha,x),(\beta,y)}, \bar{a}_{(\beta,y),(\alpha,x)}) \\
&\quad = \begin{cases}
(0,0) & \text{if } k^\vee(\alpha) = 4,\ k^\vee(\beta) = 2 \text{ and } x - y \notin 2\mathbb{Z}, \\
(0,0) & \text{if } k^\vee(\beta) \leq k^\vee(\alpha) \leq \frac{3}{2} k^\vee(\beta) \text{ and } x \neq y, \\
(-2,-1) & \text{if } 2 \leq k^\vee(\alpha) \leq 3,\ k^\vee(\beta) = 2,\ \bar{a}_{(\alpha,x),(\alpha,x)} = 0 \text{ and } x = y, \\
(-1,-1) & \text{if } g(\alpha) = 2\mathbb{Z}+1 \text{ and } x = y, \\
(\frac{k(\alpha)}{k(\beta)} J(\alpha^\vee, \beta), -1) & \text{otherwise.}
\end{cases}
\end{aligned}$$

Lemma 4.2. *Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Then there exists a unique homomorphism $\pi_{\mathcal{D}} : \mathfrak{g}_{\mathcal{D}} \to \mathfrak{L}(\bar{D}_{\mathcal{D}})$ satisfying the following properties:*

(i)

$$\pi_{\mathcal{D}}(E_{\pm\alpha}) = \begin{cases}
\sum_{x=1}^{k^\vee(\alpha)} \bar{E}_{\pm(\alpha,x)} & \text{if } g(\alpha) = \emptyset,\ \mathbb{Z} \text{ or } 2\mathbb{Z}, \\
\sqrt{2}(\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) & \text{if } g(\alpha) = 2\mathbb{Z}+1, \\
\sqrt{2}\bar{E}_{\pm(\alpha,2)} \pm \frac{1}{\sqrt{2}}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,3)}] & \text{if } g(\alpha) = 4\mathbb{Z}+2, \\
\bar{E}_{\pm(\alpha,1)} \pm [\bar{E}_{\pm(\alpha,3)}, \bar{E}_{\pm(\alpha,2)}] & \text{if } g(\alpha) = 4\mathbb{Z},
\end{cases}$$

$$\pi_{\mathcal{D}}(E_{\pm\alpha^*}) = \begin{cases} \sum_{x=1}^{k^\vee(\alpha)} \exp(\pm\frac{\pi\sqrt{-1}(2j-1-k^\vee(\alpha))}{k^\vee(\alpha)})\bar{E}_{\pm(\alpha,x)} \otimes t^{\pm k(\alpha)} & \text{if } g(\alpha)=\emptyset \text{ or } 2\mathbb{Z}, \\ \pm\frac{1}{4}\sum_{x=1}^{k^\vee(\alpha)} \exp(\pm\frac{\pi\sqrt{-1}(2j-1-k^\vee(\alpha))}{k^\vee(\alpha)})[\bar{E}_{\pm(\alpha,x)}, \bar{E}_{\pm(\alpha,x)}] \otimes t^{\pm k(\alpha)} & \text{if } g(\alpha)=\mathbb{Z}, \\ -\sqrt{-1}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,2)}] \otimes t^{\pm 1} & \text{if } g(\alpha)=2\mathbb{Z}+1, \\ (\bar{E}_{\pm(\alpha,1)} + \sqrt{-1}[\bar{E}_{\pm(\alpha,3)}, \bar{E}_{\pm(\alpha,2)}]) \otimes t^{\pm 1} & \text{if } g(\alpha)=4\mathbb{Z}+2, \\ \sqrt{2}(\bar{E}_{\pm(\alpha,2)} + \frac{\sqrt{-1}}{2}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,3)}]) \otimes t^{\pm 1} & \text{if } g(\alpha)=4\mathbb{Z}, \end{cases}$$

(ii) There exists an $\kappa \in \mathbb{C} \setminus \{0\}$ such that $\bar{J}(\pi_{\mathcal{D}}(h_\mu), \pi_{\mathcal{D}}(h_\nu)) = \kappa J(\mu,\nu)$ for μ, $\nu \in \mathcal{B}$.

(iii) $\pi_{\mathcal{D}}(h_a) = \kappa v$, $\pi_{\mathcal{D}}(h_{\Lambda_a}) = w$ and $\pi_{\mathcal{D}}(h_{\Lambda_\delta}) = \sum_{i=1}^{k^\vee(\alpha_0)} \bar{t}_{(\alpha_0,i)}$.
In particular, $\pi_{\mathcal{D}}(h_\sigma) \neq 0$ for all $\sigma \in \mathcal{E}$.

This can be proved directly by using Lemma 4.1.

Proof of Lemma 2.1. The lemma follows immediately from Lemma 4.2. □

Keep the notation as in Lemma 4.2. We shall also denote by J the invariant form on $\mathfrak{g}_{\mathcal{D}}$ defined by $\frac{1}{\kappa}\bar{J}$.

5 Invariant form and central extension

5.1 Invariant form and a universal property

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Following the notation in [S], we say that $\mathcal{D}$ is $A_l^{(1,1)}$ if the $(l+1)\times(l+1)$-matrix $A = (J(\alpha^\vee,\beta))_{\alpha,\beta\in\Pi}$ is $\mathbf{A}_l^{(1)}$ (see also Subsec. 1.1 for the name $\mathbf{A}_l^{(1)}$). Notice that if $\mathcal{D}$ is $A_l^{(1,1)}$, then $p(\alpha)=0$ and $g(\alpha)=\emptyset$ for all $\alpha \in \Pi$.

Theorem 5.1. *Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Assume that $\mathcal{D}$ is not $A_l^{(1,1)}$. Let $\mathfrak{g}'_{\mathcal{D}}$ be a Lie superalgebra satisfying the following conditions.*

(1) $\mathfrak{g}'_{\mathcal{D}}$ includes $\mathfrak{h}_{\mathcal{D}}$ as a subalgebra.
(2)

$$\mathfrak{g}'_{\mathcal{D}} = \mathfrak{h}_{\mathcal{D}} \bigoplus \Big(\bigoplus_{\nu\in R(k,g)} \mathfrak{g}'_{\mathcal{D},\nu} \Big) \bigoplus \Big(\bigoplus_{(m,n)\in\mathbb{Z}^{2,\prime}} \mathfrak{g}'_{\mathcal{D},m\delta+na} \Big),$$

and $\dim \mathfrak{g}'_{\mathcal{D},\nu} = 1$ *for $\nu \in R(k,g)$, where $\mathfrak{g}'_{\mathcal{D},\sigma} := [X \in \mathfrak{g}'_{\mathcal{D}} | [h,X] = \sigma(h)X \ (h \in \mathfrak{h}_{\mathcal{D}})]$.*

(3) There exists an invariant form J' on $\mathfrak{g}'_{\mathcal{D}}$ such that $J'(h_\sigma, h_\tau) = J'(\sigma,\tau)$ $(\sigma,\, \tau \in \mathcal{E})$ and such that $\operatorname{Ker} J' \subset \bigoplus_{(m,n)\in\mathbb{Z}^{2,\prime}} \mathfrak{g}'_{\mathcal{D},m\delta+na}$.

Then there exists an epimorphism $\eta : \mathfrak{g}_{\mathcal{D},\sigma} \to \mathfrak{g}'_{\mathcal{D},\sigma}$ such that $\eta(h_\sigma) = h_\sigma$ $(\sigma \in \mathcal{E})$ and $J' \circ (\eta \times \eta) = J$.

Proof. Using the same argument as in [K1, Theorem 2.2], we can choose non-zero elements E'_μ of $\mathfrak{g}'_{\mathcal{D},\nu}$ $(\nu \in R(k,g))$ so that they, together with h_σ's, satisfy the relations (2.3). Then the theorem follows from Theorem 2.1. □

Corollary 5.1. *Let $\mathcal{D} = \mathcal{D}(\mathcal{E},\Pi,a,k,g)$ be a QEBS with $l \geq 2$. Assume that $g(\alpha_0) = 4\mathbb{Z}$. Let $\mathcal{D}' = \mathcal{D}(\mathcal{E},\Pi,a,k,g')$ be the QEBS obtained from the $\mathcal{D}$ by replacing g by g' such that $g'(\alpha_0) = 4\mathbb{Z}+2$ and $g'(\alpha_i) = g(\alpha_i)$ $(1 \leq i \leq l)$. Then there exists an isomorphism $\xi : \mathfrak{g}_{\mathcal{D}} \to \mathfrak{g}_{\mathcal{D}'}$ such that $\xi(h_{\alpha_0}) = h_{\alpha_0^*}$, $\xi(h_{\alpha_i}) = h_{\alpha_i}$ $(1 \leq i \leq l)$, $\xi(h_a) = h_a$, $\xi(h_{\Lambda_\delta}) = h_{\Lambda_\delta}$ and $\xi(h_{\Lambda_a}) = h_{\Lambda_a-\Lambda_\delta}$.*

This can be proved easily by using Theorem 5.1.

5.2 A Lie algebra with the quantum tori

Here we recall a Lie algebra studied in [BGK]. Let $q \in \mathbb{C}\setminus\{0\}$. Let $\mathbb{C}_q = \mathbb{C}_q[s^{\pm1}, t^{\pm1}]$ be the $\mathbb{C}$-algebra defined by generators $s^{\pm1}$, $t^{\pm1}$ and defining relations $ts = qst$. Let $\mathrm{M}_{l+1}(\mathbb{C}_q)$ be the $\mathbb{C}_q$-algebra of the $(l+1)\times(l+1)$-matrices over $\mathbb{C}_q$. Let $\widehat{\mathrm{M}}_{l+1}(\mathbb{C}_q) := \mathrm{M}_{l+1}(\mathbb{C}_q) \oplus \mathbb{C}c_1 \oplus \mathbb{C}c_2 \oplus \mathbb{C}d_1 \oplus \mathbb{C}d_2$. We regard $\widehat{\mathrm{M}}_{l+1}(\mathbb{C}_q)$ as a $\mathbb{C}$-Lie algebra by

$$\begin{aligned}&[s^{x_1}t^{x_2}E_{ij}, s^{y_1}t^{y_2}E_{mn}] \\ &\quad = s^{x_1+y_1}t^{x_2+y_2}(\delta_{jm}q^{x_2y_1}E_{in} - \delta_{in}q^{x_1y_2}E_{mj}) + \delta_{x_1+y_1,0}\delta_{x_2+y_2,0}q^{x_2y_1}(x_1c_1+x_2c_2)\end{aligned}$$

and

$$[c_i, s^{x_1}t^{x_2}E_{mn}] = 0,\ [d_i, s^{x_1}t^{x_2}E_{mn}] = x_i s^{x_1}t^{x_2}E_{mn},\ [c_i,c_j] = [c_i,d_j] = [d_i,d_j] = 0.$$

We define an symmetric invariant form $\bar{J}_q$ on $\widehat{\mathrm{M}}_{l+1}(\mathbb{C}_q)$ by $\bar{J}_q(s^{x_1}t^{x_2}E_{ij}, s^{y_1}t^{y_2}E_{mn}) = \delta_{x_1+y_1,0}\delta_{x_2+y_2,0}q^{x_2y_1}$, $\bar{J}_q(s^{x_1}t^{x_2}E_{ij}, c_i) = \bar{J}_q(s^{x_1}t^{x_2}E_{ij}, d_i) = 0$, $\bar{J}_q(c_i,c_j) = \bar{J}_q(d_i,d_j) = 0$ and $\bar{J}_q(c_i,d_j) = \delta_{ij}$.

Let $\mathcal{D}$ be $A_l^{(1,1)}$. Let $\mathfrak{g}^q_{\mathcal{D}}$ be the Lie algebra defined be the generators (2.1) and the defining relations obtained from the ones (2.3) by replacing $[E_{\pm\alpha_0^*}, E_{\pm\alpha_l}] = [E_{\pm\alpha_0}, E_{\pm\alpha_l^*}]$ with

$$q^{\pm1}[E_{\pm\alpha_0^*}, E_{\pm\alpha_l}] = [E_{\pm\alpha_0}, E_{\pm\alpha_l^*}].$$

Then there exists a unique homomorphism $\pi^q_{\mathcal{D}} : \mathfrak{g}^q_{\mathcal{D}} \to \widehat{\mathrm{M}}_{l+1}(\mathbb{C}_q)$ such that $\pi^q_{\mathcal{D}}(E_{\alpha_i}) = E_{ii+1}$, $\pi^q_{\mathcal{D}}(E_{\alpha^*_i}) = tE_{ii+1}$, $\pi^q_{\mathcal{D}}(E_{-\alpha_i}) = E_{i+1i}$, $\pi^q_{\mathcal{D}}(E_{-\alpha^*_i}) = t^{-1}E_{i+1i}$ $(1 \leq i \leq l)$, $\pi^q_{\mathcal{D}}(E_{\alpha_0}) = sE_{l+11}$, $\pi^q_{\mathcal{D}}(E_{\alpha^*_0}) = stE_{l+11}$, $\pi^q_{\mathcal{D}}(E_{-\alpha_0}) = s^{-1}E_{1l+1}$, $\pi^q_{\mathcal{D}}(E_{-\alpha^*_0}) = qs^{-1}t^{-1}E_{1l+1}$, $\pi^q_{\mathcal{D}}(h_{\Lambda_\delta}) = d_1$ and $\pi^q_{\mathcal{D}}(h_{\Lambda_a}) = d_2$. We see that similar results to Theorems 2.1 and 5.1 also hold for $\mathfrak{g}^q_{\mathcal{D}}$ with $\pi^q_{\mathcal{D}}$.

5.3 Central extension

We first recall the definition of the universal central extension of a Lie superalgebra. See [IK] for more detail. Let $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ be a Lie superalgebra. Let $\mathrm{Der}(\mathfrak{a}) = [\mathfrak{a}, \mathfrak{a}]$. We say that $\mathfrak{a}$ is perfect if $\mathrm{Der}(\mathfrak{a}) = \mathfrak{a}$. We say that a Lie superalgebra epimorphism $P : \mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1 \to \mathfrak{a}$ is a central extension if $[\ker P, \mathfrak{u}] = \{0\}$ and $\ker P = (\mathfrak{u}_0 \cap \ker P) \oplus (\mathfrak{u}_1 \cap \ker P)$ (we do not assume $\ker P \subset \mathfrak{u}_0$). We say that a central extension $V : \mathfrak{u} \to \mathfrak{a}$ is universal if $\mathfrak{u} = \mathrm{Der}(\mathfrak{u})$ and if for any central extension $W : \mathfrak{b} \to \mathfrak{a}$, there exists a homomorphism $M : \mathfrak{u} \to \mathfrak{b}$ such that $W \circ M = V$. Notice that if $P : \mathfrak{u} \to \mathfrak{a}$ is a central extension and if x, $y \in \mathfrak{a}$ are homogeneous elements, then there exists a unique $z \in \mathfrak{u}$ such that $z \in [P^{-1}(\{x\}), P^{-1}(\{y\})]$; we denote the z by $N(P, x, y)$. Notice that if $x \in \mathfrak{a}_i$ and $y \in \mathfrak{a}_i$, then $N(P, x, y) \in \mathfrak{u}_{i+j}$.

In this subsection, the following easy lemma is also necessary.

Lemma 5.1. *Let* $\mathfrak{a} = \mathfrak{a}_0 \oplus \mathfrak{a}_1$ *be a Lie superalgebra such that* $\mathfrak{a} = \mathrm{Der}(\mathfrak{a}) \oplus \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n$, $k_i \in \mathfrak{a}_0$, $[k_i.k_j] = 0$, *and* $\mathrm{Der}(\mathfrak{a}) = \oplus_{x \in \mathbb{C}^n} \mathfrak{a}'_x$, *where* $\mathfrak{a}'_x = \{X \in \mathrm{Der}(\mathfrak{a}) | [k_i, X] = x_i X\}$. *Assume that* $\mathfrak{a}$ *is presented by generators* k_i *(*$1 \leq i \leq n$*)* $a_p \in \mathrm{Der}(\mathfrak{a})$ *(*$p \in P$*) with* $a_p \in \mathfrak{a}'_{x(p)}$ *for some* $x(p) = (x(p)_1, \ldots, x(p)_n) \in \mathbb{C}^n$ *and defining relations* $f_t = 0$ *(*$t \in T$*) and* $[k_i, a_p] = x(p)_i a_p$, $[k_i.k_j] = 0$, *where* f_t*'s are assumed to be expressed only by the elements* a_p *(*$p \in P$*). Then* $\mathrm{Der}(\mathfrak{a})$ *is also presented by the generators* a_p *(*$p \in P$*) and the defining relations* $f_t = 0$ *(*$t \in T$*).*

Proof. Let $\mathfrak{c} := \mathbb{C}k_1 \oplus \cdots \oplus \mathbb{C}k_n$. Let $\mathfrak{b}$ be the Lie superalgebra generated by the generators b_p ($p \in P$) and the defining relations $g_t = 0$ ($t \in T$), where g_t is expressed by replacing a_p of f_t with b_p. Let $\mathfrak{b}_x = \{Y \in \mathfrak{b} | [k_i, Y] = x_i Y\}$ for $x \in \mathbb{C}^n$. Then $\mathfrak{b} = \oplus_{x \in \mathbb{C}^n} \mathfrak{b}_x$. We can define a Lie superalgebra $\mathfrak{d} = \mathfrak{b} \oplus \mathfrak{c}$ by $[b + \sum y_i k_i, b' + \sum y'_i k_i] = [b, b'] + (\sum y_i x'_i) b' - (\sum y'_i x_i) b$. We see that there exists an isomorphism $\Phi : \mathfrak{d} \to \mathfrak{a}$ such that $\Phi(b_p) = a_p$ and $\Phi(k_i) = k_i$. ☐

Lemma 5.2. *Keep the notation as in Lemma 3.2. Let* $S := \{\alpha, \beta\}$ *and* $T := S \cup \{\gamma\}$. *Then* $\mathrm{Der}(\mathfrak{g}^S_{\mathcal{D}})$ *is isomorphic to the contragredient Lie superalgebra* $\mathrm{Der}(\bar{\mathfrak{G}}(A_T, T^{\mathrm{odd}}))$.

Proof. Recall from the proof of Lemma 3.4 that the generators E_ω ($\omega \in T$) of $\mathrm{Der}(\mathfrak{g}^S_{\mathcal{D}})$ satisfy the Serre relations (see also (3.3)). It is well-known that $\mathrm{Der}(\bar{\mathfrak{G}}(A_T, T^{\mathrm{odd}}))$ is presented by the Chevalley generators and the Serre relations (see also Lemma 5.1), which is shown by the same argument as that for the Gabber-Kac theorem. Hence there exists an epimorphism $\theta : \mathrm{Der}(\bar{\mathfrak{G}}(A_T, T^{\mathrm{odd}})) \to \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S)$. By Lemma 3.4, we see that $\dim(\mathfrak{h}_{\mathcal{D}} \cap \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S)) = 3 = \dim(\bar{\mathfrak{H}} \cap \mathrm{Der}(\bar{\mathfrak{G}}(A_T, T^{\mathrm{odd}})))$. By this fact and by the definition of $\bar{\mathfrak{G}}(A_T, T^{\mathrm{odd}})$ (see Subsec. 4.1), we see that the θ is an isomorphism. $\square$

Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Recall the homomorphism $\pi_{\mathcal{D}} : \mathfrak{g}_{\mathcal{D}} \to \mathfrak{L}(\bar{D}_{\mathcal{D}})$ from Lemma 4.2. Let $\varpi : \mathrm{Im}\pi_{\mathcal{D}} \to \mathrm{Im}\pi_{\mathcal{D}}/\pi_{\mathcal{D}}(\mathbb{C}h_\delta \oplus \mathbb{C}h_a)$ be the natural projective map. For a subset S of Π, let $\mathfrak{p}_{\mathcal{D}}^S := (\varpi \circ \pi_{\mathcal{D}})(\mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S))$, and define the epimorphism $\varrho_S : \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S) \to \mathfrak{p}_{\mathcal{D}}^S$ by $\varrho_S = (\varpi \circ \pi_{\mathcal{D}})_{|\mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S)}$.

Lemma 5.3. *Keep the notation as above. Assume that* $S = \{\alpha, \beta\}$ *and* $J(\alpha, \beta^\vee) = -1$. *Then* ϱ_S *is a universal central extension.*

Proof. Let $f : \mathfrak{b} \to \mathfrak{p}_{\mathcal{D}}^S$ be a central extension. Recall $T = S \cup \{\gamma\}$ from Lemma 5.2. For $\mu \in T$, let $h'_{\mu^\vee} := N(f, \varrho_S(E_\mu), \varrho_S(E_{-\mu}))$. and $E'_{\pm\mu} := N(f, \varrho_S(\pm\frac{1}{2}h_{\mu^\vee}), \varrho_S(E_{\pm\mu}))$. Let μ, $\nu \in T$. We see that $[h'_{\mu^\vee}, E'_{\pm\nu}] = [h'_{\mu^\vee}, \pm 2^{-1}[h'_{\nu^\vee}, E'_{\pm\nu}]] = \pm 2^{-1}[h'_{\nu^\vee}, [h'_{\mu^\vee}, E'_{\pm\nu}]] = \pm 2^{-1}J(\mu^\vee, \pm\nu)[h'_{\nu^\vee}, E'_{\pm\nu}] = J(\mu^\vee, \pm\nu)E'_{\pm\nu}$. Similarly we have $[h'_{\mu^\vee}, h'_{\nu^\vee}] = 0$. Assume $\mu \neq \nu$. Then we have $0 = [xh'_{\mu^\vee} + yh'_{\nu^\vee}, [E'_\mu, E'_{-\nu}]] = J(x\mu^\vee + y\nu^\vee, \mu - \nu)[E'_\mu, E'_{-\nu}]$ for x, $y \in \mathbb{C}$. Hence $[E'_\mu, E'_{-\nu}] = 0$. Similarly we have $(\mathrm{ad}E'_{\pm\mu})^{1-J(\mu^\vee,\nu)}E'_{\pm\nu} = 0$. Then the lemma follows from Lemma 5.2. $\square$

Theorem 5.2. *Let* $\mathcal{D}$ *be a QEBS with* $l \geq 2$. *Then* $\varrho_\Pi : \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}) \to \mathfrak{p}_{\mathcal{D}}^\Pi$ *is the universal central extension. (In particular, if* $g(\alpha) = \emptyset$ *for all* $\alpha \in \Pi$ *and if the Cartan matrix* A *is* $\mathbf{X}_l^{(1)}$ *for some* $\mathbf{X} = \mathbf{A}, \ldots, \mathbf{G}$, *then* $\mathrm{Der}(\mathfrak{g}_{\mathcal{D}})$ *is the toroidal Lie algebra in the sense of [MEY].)*

Proof. Let $f : \mathfrak{b} \to \mathfrak{p}_{\mathcal{D}}^\Pi$ be a central extension. For $\alpha \in \Pi$, let $h'_{\alpha^\vee} := N(f, \varrho_\Pi(E_\alpha), \varrho_\Pi(E_{-\alpha})]$, $h'_{(\alpha^*)^\vee} := N(f, \varrho_\Pi(E_{\alpha^*}), \varrho_\Pi(E_{-\alpha^*})]$, $E'_{\pm\alpha} := N(f, \varrho_\Pi(\pm\frac{1}{2}h_{\alpha^\vee}), \varrho_\Pi(E_\alpha)]$
and $E'_{\pm\alpha^*} := N(f, \varrho_\Pi(\pm\frac{1}{2c(\alpha)}h_{\alpha^\vee}), \varrho_\Pi(E_{\pm\alpha^*})]$.

Let α, $\beta \in \Pi$ be such that $J(\alpha, \beta^\vee) = -1$. Let $S = \{\alpha, \beta\}$. By Lemma 5.3, there exists a homomorphism $g_S : \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}^S) \to f^{-1}(\mathfrak{p}_{\mathcal{D}}^S)$ such that $\varrho_S = f \circ g_S$. Then we have

$$h'_{\mu^\vee} = g_S(h_{\mu^\vee}),\ E'_{\pm\mu} = g_S(E_{\pm\mu}) \tag{5.1}$$

for $\mu \in \{\alpha, \beta, \alpha^*, \beta^*\}$. Then the elements of (5.1) satisfy the equalities (2.3).

Let α, $\beta \in \Pi$ be such that $J(\alpha, \beta) = 0$. Let $\mu \in \{\pm\alpha, \pm\alpha^*\}$ and $\nu \in \{\pm\beta, \pm\beta^*\}$. Let $\varepsilon \in \mathbb{C}$ be such that $J(\alpha^\vee, \mu) + \varepsilon J(\beta^\vee, \nu) \neq 0$. Then we have

$$\begin{aligned} 0 &= [h'_{\alpha^\vee} + \varepsilon h'_{\beta^\vee}, [E'_\mu, E'_\nu]] \\ &= [[h'_{\alpha^\vee} + \varepsilon h'_{\beta^\vee}, E'_\mu], E'_\nu] + [E'_\mu, [h'_{\alpha^\vee} + \varepsilon h'_{\beta^\vee}, E'_\nu]] \\ &= [[h'_{\alpha^\vee}, E'_\mu], E'_\nu] + [E'_\mu, [\varepsilon h'_{\beta^\vee}, E'_\nu]] \\ &= (J(\alpha^\vee, \mu) + \varepsilon J(\beta^\vee, \nu))[E'_\mu, E'_\nu]. \end{aligned}$$

Hence we have

$$[E'_\mu, E'_\nu] = 0. \tag{5.2}$$

Using (5.2), we have $[h'_{\alpha^\vee}, E'_\nu] = 0$, $[h'_{\beta^\vee}, E'_\mu] = 0$ and $[h'_{\alpha^\vee}, h'_{\beta^\vee}] = 0$.

By the above argument and by Lemma 5.1, we see that there exists a homomorphism $\vartheta : \mathrm{Der}(\mathfrak{g}_{\mathcal{D}}) \to \mathfrak{b}$ such that $f \circ \vartheta = \varrho_\Pi$. □

Notice that if $\mathcal{D}$ is $A_l^{(1,1)}$, a similar result to Theorem 5.2 also holds for $\mathfrak{g}^q_{\mathcal{D}}$ with $\pi^q_{\mathcal{D}}$ (see also Subsec. 4.3).

6 Elliptic root base

Let $\mathcal{D} = \mathcal{D}(\mathcal{E}, \Pi, a, k, g)$ be a QEBS with $l \geq 2$. Here by extending the notion given in [S], we introduce an *elliptic root basis* of $R(k,g)$ for the $\mathcal{D}$. Recall $\delta \in \mathbb{Z}_+\Pi$ from Subsec. 2.2. For $\alpha \in \Pi$, let $x_\alpha \in \mathbb{Z}_+$ $(\alpha \in \Pi)$ be the coefficient of δ, i.e., $\delta = \sum_{\alpha\in\Pi} x_\alpha \alpha$. Let $m_\alpha := \frac{c(\alpha)I(\alpha,\alpha)x_\alpha}{k(\alpha)}$. Let $m_{\max} := \max\{m_\alpha | \alpha \in \Pi\}$ and $\Pi_{\max} := \{\alpha \in \Pi | m_\alpha = m_{\max}\}$. For a subset S of Π, let $S^* := \{\alpha^* | \alpha \in S\}$. Let $\Gamma(R,G) := \Pi \cup \Pi^*_{\max}$ (where R and G denote $R(k,g)$ and $\mathbb{C}a$ respectively). We call the $\Gamma(R,G)$ the *elliptic root basis* of $R(k,g)$. For a subset S of Π, let $\Gamma(R,G;S) := \Gamma(R,G) \cap (S \cup S^*)$. Recall the Lie superalgebra $\mathfrak{g}_{\mathcal{D}}$ from Subsec. 2.1.

Theorem 6.1. *Let $\mathcal{D}$ be a QEBS with $l \geq 2$. Then the Lie superalgebra $\mathfrak{g}_{\mathcal{D}}$ can also presented by the generators*

$$h_\sigma\,(\sigma \in \mathcal{E}), E_\mu,\ E_{-\mu}\,(\mu \in \Gamma(R,G)) \tag{6.1}$$

with the same parities as (2.2) and the following relations.

$$\left\{\begin{array}{l}\textit{The same relations as the ones among those in (2.3) expressed}\\ \textit{only by the same symbols as in (6.1),}\end{array}\right. \tag{6.2}$$

$$[E_{\pm\alpha^*}, [E_{\pm\alpha}, E_{\pm\beta}]] = 0,\ \textit{if}\ \Gamma(R,G;\{\alpha,\ \beta\}) = \{\alpha, \alpha^*, \beta\}\ \textit{and}\ \tfrac{J(\alpha^\vee,\beta)}{J(\beta^\vee,\alpha)} = c(\alpha), \tag{6.3}$$

$$\left\{\begin{array}{l}[E_{\pm\alpha^*}, [E_{\pm\alpha}, E_{\pm\beta}]] = [[E_{\pm\alpha^*}, E_{\pm\beta}], [E_{\pm\alpha}, E_{\pm\beta}]] = 0,\\ \textit{if}\ \Gamma(R,G;\{\alpha,\ \beta\}) = \{\alpha, \alpha^*, \beta\},\ g(\alpha) = \emptyset\ \textit{and}\ \frac{J(\beta^\vee,\alpha)}{J(\alpha^\vee,\beta)} = 2\ \textit{or}\ 3,\end{array}\right. \tag{6.4}$$

$$\left\{\begin{array}{l}[(\mathrm{ad}E_{\pm\beta})^{-J(\beta^\vee,\alpha)}E_{\pm\alpha}, (\mathrm{ad}E_{\pm\beta^*})^{-J((\beta^*)^\vee,\gamma)}E_{\pm\gamma}] = 0,\\ \textit{if}\ \Gamma(R,G;\{\alpha,\ \beta,\ \gamma\}) = \{\alpha, \beta, \beta^*, \gamma\}.\end{array}\right. \tag{6.5}$$

This can be proved by the same argument as that for [Ya2, Theorem 4.1].

Appendix

As mentioned in the text, especially in Theorem 1.1, K. Saito [S] (see also [SY]) introduced the notion of the ERS, and showed that every RMERS is realized as the $R(k,0)$ for some SEBS $\mathcal{D}(\mathcal{E},\Pi,a,k,0)$. However, there exists a reduced ERS which is not realized as RMERS. As mentioned in Introduction, the authors of [AABGP] introduced the notion of extended affine root systems (EARS for short), which is different from the SEARS's introduced in [S]. It is known (see [A]) that there exists a natural one-to-one correspondence between the reduced SEARS's and the EARS's. Here we also use the terminology and notation in [AABGP, Construction 2.32 and Theorem 2.37]. By Theorems 1.2 and 2.2, we see that if an EARS has the nullity equal to two and $4 \leq \dim \mathcal{V}$, the corresponding reduced ERS is realized as $R(k,g)$ for some QEBS $\mathcal{D} = \mathcal{D}(\mathcal{E},\Pi,a,k,g)$ with $g(\alpha) = \emptyset$ or $2\mathbb{Z}+1$ ($\alpha \in \Pi$). Let $\mathcal{D}$ be a QEBS with $l \geq 2$ such that the name of $A(= A_\Pi)$ is $\mathbf{D}^{(2)}_{l+1}$. We may assume that there exists $\varepsilon_i \in \mathcal{E}$ ($1 \leq i \leq l$) such that $J(\varepsilon_i,\varepsilon_j) = \delta_{ij}$, $\alpha_0 = \delta - \varepsilon_1$, $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ ($2 \leq i \leq l-1$) and $\alpha_l = \varepsilon_l$. Then the corresponding EARS is $R(X,S,L,E)$ is such that

$$X = \begin{cases} B_l & \text{if } g(\alpha_0) = g(\alpha_l) = \emptyset \\ BC_l & \text{otherwise,} \end{cases}$$

and $S = ((2\mathbb{Z}+1)\delta + \mathbb{Z}k(\alpha_0)a) \cup (2\mathbb{Z}\delta + \mathbb{Z}k(\alpha_l)a))$, $L = 2\mathbb{Z}\delta + \mathbb{Z}k(\alpha_i)a$ ($2 \leq i \leq l-1$) and $E = ((4\mathbb{Z}+2)\delta + g(\alpha_0)a) \cup (4\mathbb{Z}\delta + g(\alpha_l)a))$. Here if $g(\alpha_0) = \emptyset$, then $(4\mathbb{Z}+2)\delta + g(\alpha_0)a) = \emptyset$; if $g(\alpha_1) = \emptyset$, then $(4\mathbb{Z}+2)\delta + g(\alpha_1)a) = \emptyset$. Strictly saying, if $E = \emptyset$, then $R(X,S,L,E)$ is denoted as $R(X,S,L)$.

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