

## Elliptic Lie superalgebras of rank $\geq 2$

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### Abstract

In this paper, we introduce a class of Lie superalgebras  $\mathfrak{g}$ , defined with generators and relations. We show that non-isotropic roots of our Lie superalgebra  $\mathfrak{g}$  form a non-reduced elliptic root system  $R$  of rank  $\geq 2$  in the sense of K. Saito [10]. We also show that  $\mathfrak{g}$  is maximal among Lie superalgebras associated with  $R$ .

*Keywords:* elliptic Lie algebras; toroidal Lie algebras; extended affine Lie algebras; elliptic Lie superalgebras

## Introduction

In 1985, K. Saito [10] introduced the notion of *n-extended affine root systems*. If  $n = 0$  (respectively,  $n = 1$ ), it is an irreducible finite root system (respectively, an affine root system). In [10], he also intensively studied 2-extended affine root systems, which are now called *elliptic root systems* (see [11]). Since then, various attempts have been made to construct Lie algebras whose non-isotropic roots form those root systems. Among them are *toroidal Lie algebras* [9] and *extended affine Lie algebras* [1]. See [12, Introduction] for the history.

In 2000, K. Saito and D. Yoshii [12] constructed Lie algebras by using the Borcherds lattice vertex algebras, called them *simply-laced elliptic Lie algebras* and showed that they are isomorphic to the *ADE*-type (2-variable) toroidal Lie algebras of rank  $\geq 2$ . They also gave two other definitions for their Lie algebras. One uses generators and relations. The other uses (affine-type) Heisenberg Lie algebras; this was generalized by D. Yoshii [16] in order to define Lie algebras associated with the reduced elliptic root systems, and he called them *elliptic Lie algebras*. In 2004, the author [15] gave defining relations of the elliptic Lie algebras of rank  $\geq 2$  (see also Remark 1.1).

In this paper, we introduce a class of Lie superalgebras  $\mathfrak{g}$ . We also show that non-isotropic roots of  $\mathfrak{g}$  form a non-reduced elliptic root system  $R$  (see Theorem 1.1 (1)). Moreover, we show *maximality* of  $\mathfrak{g}$ , which means that there exists a natural epimorphism  $f : \mathfrak{g} \rightarrow \mathfrak{g}'$  if the non-isotropic roots of a Lie superalgebra  $\mathfrak{g}'$  form  $R$  (see Theorem 1.1 (3)).

We point out that our defining relations are closely related to defining relations, called *Drinfeld realization*, of the quantum affine algebras due to V.G. Drinfeld [5, Theorems 3 and 4].

The paper is organized as follows. In Section 1, we state our main theorem. In Section 2, we use maps  $k$  and  $g$  to express elliptic root systems, which is a key to this paper. (The idea of  $k$  has been given by K. Saito [10].) In Section 3, we prove our main theorem. In Section 4, we show that the Cartan subalgebra  $\mathfrak{h}$  of  $\mathfrak{g}$  has a desirable dimension; this fact is used in Section 3.

## 1 Preliminary

Let  $\mathbb{Z}_+ = \{x \in \mathbb{Z} | x \geq 0\}$  and  $\mathbb{Z}_- = \{x \in \mathbb{Z} | x \leq 0\}$ . Then  $\mathbb{N} \cup \{0\} = \mathbb{Z}_+$ .

### 1.1 Extended affine root systems

**Definition 1.1.** Let  $l \in \mathbb{N}$  and  $n \in \mathbb{Z}_+$ . Let  $V$  be an  $(l + n)$ -dimensional  $\mathbb{R}$ -linear space having a positive semi-definite symmetric bilinear form  $(, ) : V \times V \rightarrow \mathbb{R}$  such that  $\dim V^0 = n$ , where we set  $V^0 = \{v \in V | (v, v) = 0\}$ . Let  $V^\times = V \setminus V^0$ . For  $v \in V^\times$ , let  $v^\vee = \frac{2v}{(v, v)}$  and define  $s_v \in \text{GL}(V)$  by  $s_v(w) = w - (v^\vee, w)v$  ( $w \in V$ ). Let  $R$  be a subset of  $V$ . Then  $R$  (or more precisely,  $(R, V)$ ) is an  $(n)$ -*extended affine root system* if  $R$  satisfies the following axioms:

- (AX1)  $R \subset V^\times$ ,  $V = \mathbb{R}R$ .
- (AX2)  $\text{rank}_{\mathbb{Z}} \mathbb{Z}R = n + l (= \dim_{\mathbb{R}} V)$ .
- (AX3)  $(\alpha^\vee, \beta) \in \mathbb{Z}$  for  $\alpha, \beta \in R$ .
- (AX4)  $s_\alpha(R) = R$  for all  $\alpha \in R$ .
- (AX5) If  $R = R_1 \cup R_2$  and  $(\alpha, \beta) = 0$  for  $\alpha \in R_1$  and  $\beta \in R_2$ , then  $R_1 = \emptyset$  or  $R_2 = \emptyset$ .

(see [10, (1.2) Definition 1 and (1.3) Note 2 iii].)

Let  $R$ ,  $l$  and  $n$  be as in Definition 1.1. We call  $l$  (respectively,  $n$ ) the *rank* (respectively, *nullity*) of  $R$ . If  $n = 0$  (respectively,  $n = 1$ ), then  $R$  is an *irreducible finite root system* (respectively, an *affine root system*). If  $n = 2$ , then  $R$  is an *elliptic root system* (see [10], [11] and [12]).

We say that two extended affine root systems  $(R, V)$  and  $(R', V')$  are *isomorphic* if there exist an  $\mathbb{R}$ -linear bijective map  $f : V \rightarrow V'$  and  $c \in \mathbb{R}_{>0}$  such that  $f(R) = R'$  and  $(f(v), f(w)) = c(v, w)$  for  $v, w \in V$ .

Let  $R$  be an extended affine root system. As is well-known, we have the following (see Definition 1.1 (AX3,4)).

$$(1.1) \quad R \cap \mathbb{R}\alpha = \{\alpha, -\alpha\}, \{\alpha, 2\alpha, -\alpha, -2\alpha\} \text{ or } \{\alpha, \frac{1}{2}\alpha, -\alpha, -\frac{1}{2}\alpha\}$$

( $\alpha \in R$ ). We call  $R$  *reduced* (respectively, *non-reduced*) if  $R \cap 2R = \emptyset$  (respectively,  $R \cap 2R \neq \emptyset$ ). We always let  $\pi : V \rightarrow V/V^0$  denote the canonical map.

If  $R$  is an irreducible finite root system or an affine root system, then a subset  $\Pi$  of  $R$  formed by  $(l + n)$ -linearly independent elements is a *base* if

$$(1.2) \quad R = (R \cap \mathbb{Z}_+\Pi) \cup (R \cap \mathbb{Z}_-\Pi).$$

If  $R$  is an affine root system, we let  $\delta = \delta(\Pi) \in \mathbb{N}\Pi$  be such that  $\mathbb{Z}\delta = \mathbb{Z}R \cap V^0$  (see Theorem 1.2 (1) and Lemma 1.1). If  $R$  is an affine root system and  $\alpha_0, \dots, \alpha_l$  are the elements of a base  $\Pi$  of  $R$ , we let  $A = A(\Pi) = ((\alpha_i^\vee, \alpha_j))_{0 \leq i, j \leq l}$ . Then  $A$  is a generalized Cartan matrix of affine-type (see [6]).

## 1.2 Main theorem

Assume  $R$  to be an elliptic root system. Let  $\{a, b\}$  be a basis of  $\mathbb{Z}R \cap V^0$ . Let  $\pi_a : V \rightarrow V/\mathbb{R}a$  be the canonical map. Let  $\Pi$  be a subset of  $R$  satisfying the following conditions.

(FS1)  $|\pi_a(\Pi)| = |\Pi|$ . (Here  $|\mathcal{S}|$  denotes the cardinal number of a set  $\mathcal{S}$ .)

(FS2)  $\pi_a(\Pi)$  is a base of the affine root system  $\pi_a(R)$ .

(FS3) For  $\alpha \in \Pi$ , if there exists  $\alpha' \in R$  such that  $\pi_a(\alpha) = \pi_a(\alpha')$  and  $2\alpha' \in R$ , then  $2\alpha \in R$ .

We call such  $\Pi \cup \{a\}$  a *fundamental-set* of  $R$ . We see that  $\Pi \cup \{a\}$  is a

basis of  $\mathbb{Z}R$ . We also denote by  $\delta = \delta(\Pi)$  the element of  $\mathbb{N}\Pi$  such that  $\pi_a(\delta) = \delta(\pi_a(\Pi))$ . We let  $A = A(\Pi) = A(\pi_a(\Pi))$ . Notice that  $\{a, \delta\}$  is a basis of  $\mathbb{Z}R \cap V^0$ .

Let  $\Pi'$  be a subset of  $R$  such that  $|\pi_a(\Pi')| = |\Pi'|$  and  $\pi_a(\Pi')$  is a base of  $\pi_a(R)$ . For  $\alpha' \in \Pi'$ , replace  $\alpha'$  with  $\alpha$  if there exists  $\alpha \in R$  such that  $\pi_a(\alpha) = \pi_a(\alpha')$  and  $2\alpha \in R$ . Then  $\Pi' \cup \{a\}$  is a fundamental-set of  $R$ .

Let  $R$  be an elliptic root system. Let  $l \geq 2$ . Let  $\Pi \cup \{a\}$  be a fundamental-set of  $R$ . Let  $\mathcal{P}$  be the subset of the power set  $2^{\mathbb{Z}}$  of  $\mathbb{Z}$  formed by  $\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1, 4\mathbb{Z}$ , i.e.,  $\mathcal{P} = \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1, 4\mathbb{Z}\}$ . We can define maps  $k : \Pi \rightarrow \{1, 2, 3, 4\}$  and  $g : \Pi \rightarrow \mathcal{P}$  by

$$(1.3) \quad (\alpha + \mathbb{R}a) \cap R = \alpha + \mathbb{Z}k(\alpha)a, \quad (2\alpha + \mathbb{R}a) \cap R = 2\alpha + g(\alpha)k(\alpha)a$$

for  $\alpha \in \Pi$  (see (2.2)). (If  $g(\alpha) = \emptyset$ , then  $2\alpha + g(\alpha)k(\alpha)a = \emptyset$ .) We shall see that  $R$  is determined by the datum  $(\Pi, k, g)$  of the above  $\Pi$ ,  $k$  and  $g$  (See Theorems 2.1 and 2.2), so we shall also denote  $R$  by  $R(\Pi, k, g)$ .

Let a datum  $(\Pi, k, g)$  be as above. For  $\alpha \in \Pi$ , define  $c(\alpha)$ ,  $\alpha^*$ ,  $p(\pm\alpha)$ ,  $p(\pm\alpha^*)$  by

$$(1.4) \quad c(\alpha) = \begin{cases} 2 & \text{if } g(\alpha) = \mathbb{Z} \text{ or } 2\mathbb{Z} + 1, \\ 1 & \text{if } g(\alpha) = \emptyset, 2\mathbb{Z} \text{ or } 4\mathbb{Z}, \end{cases}$$

$$\alpha^* = c(\alpha)\alpha + k(\alpha)a,$$

$$p(\alpha) = p(-\alpha) = \begin{cases} 1 & \text{if } g(\alpha) = \mathbb{Z}, 2\mathbb{Z} \text{ or } 4\mathbb{Z}, \\ 0 & \text{if } g(\alpha) = \emptyset \text{ or } 2\mathbb{Z} + 1, \end{cases}$$

and

$$p(\alpha^*) = p(-\alpha^*) = \begin{cases} 1 & \text{if } g(\alpha) = 2\mathbb{Z}, \\ 0 & \text{if } g(\alpha) = \emptyset, \mathbb{Z}, 2\mathbb{Z} + 1 \text{ or } 4\mathbb{Z}. \end{cases}$$

(We shall see that  $\{\alpha, -\alpha^*\}$  is a base of the rank-one affine root system  $R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$  (see Lemma 3.2 (2)). We shall also see that for  $\gamma \in \{\alpha, -\alpha, \alpha^*, -\alpha^*\}$ , it follows that  $p(\gamma) = 1$  if and only if  $2\gamma \in R$  (see Lemma 3.2 (1)).)

Let

$$\mathcal{A} = \{(\alpha, \beta, y) \in \Pi \times \Pi \times \mathbb{N} \mid \alpha \neq \beta, (\alpha, \beta) \neq 0, k(\alpha)y = k(\beta)\}.$$

Let  $\mathcal{B}_+ = \{\alpha, \alpha^* | \alpha \in \Pi\}$  and  $\mathcal{B} = \mathcal{B}_+ \cup (-\mathcal{B}_+)$ . For  $\mu, \nu \in \mathcal{B}$  with  $\mu \neq \nu$  and  $\mu + \nu \neq 0$ , we define  $x_{\mu, \nu} \in \mathbb{N}$  by

$$x_{\mu, \nu} = \begin{cases} 1 - (\mu^\vee, \nu) & \text{if } (\mu^\vee, \nu) < 0, \\ 1 & \text{if } (\mu^\vee, \nu) \geq 0. \end{cases}$$

Let  $\alpha_0, \dots, \alpha_l$  be the elements of  $\Pi$ . We assume that  $\{\pi(\alpha_1), \dots, \pi(\alpha_l)\}$  is a base of the irreducible finite root system  $\pi(R(\Pi, k, g))$ . Let  $\mathcal{E} = \mathcal{E}(\Pi, k, g)$  be the  $(l+4)$ -dimensional  $\mathbb{C}$ -linear space with a basis  $\{\alpha_0, \dots, \alpha_l, a, \Lambda_\delta, \Lambda_a\}$ . We identify  $V = \mathbb{R}R(\Pi, k, g)$  with the  $\mathbb{R}$ -linear subspace of  $\mathcal{E}$  spanned by  $\alpha_0, \dots, \alpha_l, a$ . We extend  $(, )$  on  $V$  to the non-degenerate symmetric bilinear form on  $\mathcal{E}$  by  $(\delta, \Lambda_\delta) = (a, \Lambda_a) = 1$ ,  $(\Lambda_\delta, \Lambda_\delta) = (\Lambda_a, \Lambda_a) = (\Lambda_\delta, \Lambda_a) = 0$ ,  $(\Lambda_a, \delta) = (\Lambda_\delta, a) = 0$ , and  $(\Lambda_\delta, \alpha) = (\Lambda_a, \alpha) = 0$  ( $\alpha \in \Pi \setminus \{\alpha_0\}$ ).

**Definition 1.2.** Let  $l \geq 2$  and let  $(\Pi, k, g)$  be as above. Let  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  be the  $\mathbb{C}$ -Lie superalgebra defined by generators

$$h_\sigma \ (\sigma \in \mathcal{E}), \quad E_\mu \ (\mu \in \mathcal{B})$$

with parities

$$p(h_\sigma) = 0, \quad p(E_\mu) = p(\mu)$$

and defining relations

- (SR1)  $xh_\sigma + yh_\tau = h_{x\sigma+y\tau}$  if  $x, y \in \mathbb{C}$  and  $\sigma, \tau \in \mathcal{E}$ ,
- (SR2)  $[h_\sigma, h_\tau] = 0$  if  $\sigma, \tau \in \mathcal{E}$ ,
- (SR3)  $[h_\sigma, E_\mu] = (\sigma, \mu)E_\mu$  if  $\sigma \in \mathcal{E}$  and  $\mu \in \mathcal{B}$ ,
- (SR4)  $[E_\mu, E_{-\mu}] = h_{\mu^\vee}$  if  $\mu \in \mathcal{B}_+$ ,
- (SR5)  $(\text{ad}E_\mu)^{x_{\mu, \nu}} E_\nu = 0$  if  $\mu, \nu \in \mathcal{B}$  are such that  $\mu \neq \nu$  and  $\mu + \nu \neq 0$ ,
- (SR6)  $c(\alpha)(\text{ad}E_{\alpha^*})^y E_\beta = (\text{ad}E_\alpha)^{c(\alpha)y} E_{\beta^*}$  if  $(\alpha, \beta, y) \in \mathcal{A}$ ,
- (SR7)  $(-1)^{c(\alpha)+1} c(\alpha)(\text{ad}E_{-\alpha^*})^y E_{-\beta} = (\text{ad}E_{-\alpha})^{c(\alpha)y} E_{-\beta^*}$  if  $(\alpha, \beta, y) \in \mathcal{A}$ ,
- (SR8)  $(\text{ad}E_\alpha)^i (\text{ad}E_{\alpha^*})^{y-i} E_\beta = 0$  if  $(\alpha, \beta, y) \in \mathcal{A}$  and  $1 \leq i \leq y-1$ ,
- (SR9)  $(\text{ad}E_{-\alpha})^i (\text{ad}E_{-\alpha^*})^{y-i} E_{-\beta} = 0$  if  $(\alpha, \beta, y) \in \mathcal{A}$  and  $1 \leq i \leq y-1$ .

Let  $\mathfrak{h} = \mathfrak{h}(\Pi, k, g)$  be the  $\mathbb{C}$ -subspace of  $\mathfrak{g}$  formed by the elements  $h_\sigma$  ( $\sigma \in \mathcal{E}$ ). For  $\mu \in \mathcal{E}$ , let  $\mathfrak{g}_\mu = \{X \in \mathfrak{g} \mid [h_\sigma, X] = (\sigma, \mu)X \ (\sigma \in \mathcal{E})\}$ .

Then our main theorem can be stated as follows.

**Theorem 1.1.** *Let  $(R, V)$  be an elliptic root system of  $l \geq 2$ , where  $l = \dim V/V^0$ . Let  $\Pi \cup \{a\}$  be a fundamental-set of  $R$ . Define maps  $k : \Pi \rightarrow \{1, 2, 3, 4\}$  and  $g : \Pi \rightarrow \mathcal{P}$  by (1.3) (so  $R = R(\Pi, k, g)$ ). Assume that  $g(\alpha) \neq \emptyset$  for some  $\alpha \in \Pi$ . Let  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  be as in Definition 1.2. Then we have the following.*

(1) *We have*

$$(1.5) \quad \mathfrak{g} = \mathfrak{h} \bigoplus \left( \bigoplus_{\alpha \in R(\Pi, k, g)} \mathfrak{g}_\alpha \right) \bigoplus \left( \bigoplus_{(m, n) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \mathfrak{g}_{m\delta + na} \right).$$

*Moreover, it follows that  $\dim \mathfrak{h} = l + 4$ , i.e.,  $h_\sigma \neq 0$  for  $\sigma \in \mathcal{E} \setminus \{0\}$  and that  $\dim \mathfrak{g}_\alpha = 1$  for  $\alpha \in R = R(\Pi, k, g)$ .*

(2) *There exists a super-symmetric invariant form  $(, ) : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{C}$  such that*

$$(1.6) \quad (h_\sigma, h_\tau) = (\sigma, \tau) \quad (\sigma, \tau \in \mathcal{E}).$$

(3) *Let  $\mathfrak{g}'$  be a  $\mathbb{C}$ -Lie superalgebra satisfying the following conditions.*

(3-i)  *$\mathfrak{g}'$  includes  $\mathfrak{h}$  as a sub-Lie superalgebra.*

(3-ii)  *$\mathfrak{g}'$  has the same properties as those in (1) with  $\mathfrak{g}'$  in place of  $\mathfrak{g}$ .*

(3-iii)  *$\mathfrak{g}'$  has a super-symmetric invariant form satisfying the same properties as that in (1.6).*

(3-iv)  *$\mathfrak{g}'$  is generated by  $\mathfrak{h}$  and  $\mathfrak{g}'_\mu$  ( $\mu \in R(\Pi, k, g)$ ).*

*Then there exists an epimorphism  $f : \mathfrak{g} \rightarrow \mathfrak{g}'$  such that  $f(h_\sigma) = h_\sigma$  ( $\sigma \in \mathcal{E}$ ).*

*Proof of the theorem shall be given in Subsections 3.3 and 4.8.*

*Remark 1.1.* (1) A result similar to Theorem 1.1 for  $g$  such that  $g(\alpha) = \emptyset$  for all  $\alpha \in \Pi$  has been given in [15].

(2) By Theorem 2.1 and Lemma 2.1, we shall see that  $R(\Pi, k, g)$  is reduced if and only if  $g(\Pi) \subset \{\emptyset, 2\mathbb{Z} + 1\}$ . There exist two isomorphism classes of the reduced elliptic root systems  $R(\Pi, k, g)$  of  $l \geq 2$  which are not isomorphic to  $R(\Pi', k', g')$  for any  $(\Pi', k', g')$  such that  $g'(\Pi) = \{\emptyset\}$  (see [1] and [2]).

(3) If  $(\Pi, k, g)$  is as in the statement of Theorem 1.1, by (2.5), we shall see  $3 \notin k(\Pi)$ . For a general  $(\Pi, k, g)$ , by (2.5), we shall see that if  $3 \in k(\Pi)$ , then  $\pi(R(\Pi, k, g))$  is  $G_2$ .

(4) See [3] for the classification of the non-reduced elliptic root systems.

### 1.3 Notation $R_{\text{sh}}$ , $R_{\text{lg}}$ and $R_{\text{ex}}$

Here we recall the names of the irreducible finite root systems and the generalized Cartan matrices of affine-type. If  $R$  is an irreducible finite root system, then  $R$  is one of the following:  $A_l$  ( $l \geq 1$ ),  $B_l$  ( $l \geq 3$ ),  $C_l$  ( $l \geq 2$ ),  $D_l$  ( $l \geq 4$ ),  $E_l$  ( $l = 6, 7, 8$ ),  $G_2$  ( $l = 2$ ) and  $BC_l$  ( $l \geq 1$ ) (see [4, Chapter VI §1 Definition 1 ( $SR_1$ )-( $SR_3$ )] and [1]). If  $l = 2$ , we use  $A_2$ ,  $C_2$  or  $G_2$ ; we don't use  $B_2$ . If  $R$  is an affine root system and  $\Pi$  is a base of  $R$ , then  $A(\Pi)$  is one of the following:  $A_l^{(1)}$  ( $l \geq 1$ ),  $B_l^{(1)}$  ( $l \geq 3$ ),  $C_l^{(1)}$  ( $l \geq 2$ ),  $D_l^{(1)}$  ( $l \geq 4$ ),  $E_l^{(1)}$  ( $l = 6, 7, 8$ ),  $F_4^{(1)}$  ( $l = 4$ ),  $G_2^{(1)}$  ( $l = 2$ ),  $A_{2l}^{(2)}$  ( $l \geq 1$ ),  $A_{2l-1}^{(2)}$  ( $l \geq 3$ ),  $D_{l+1}^{(2)}$  ( $l \geq 2$ ),  $E_6^{(2)}$  ( $l = 4$ ) or  $D_4^{(3)}$  ( $l = 2$ ) (see [6]).

Let  $R$  be an extended affine root system. Define subsets  $R_{\text{sh}}$ ,  $R_{\text{lg}}$  and  $R_{\text{ex}}$  of  $R$  by

$$R_{\text{sh}} = \{\alpha \in R \mid (\alpha, \alpha) \leq (\beta, \beta) \text{ for all } \beta \in R\},$$

$R_{\text{ex}} = R \cap \pi^{-1}(2\pi(R_{\text{sh}}))$  and  $R_{\text{lg}} = R \setminus (R_{\text{sh}} \cup R_{\text{ex}})$ . We have

$$R = R_{\text{sh}} \cup R_{\text{lg}} \cup R_{\text{ex}} \text{ (disjoint union).}$$

Notice that  $R_{\text{ex}} = \emptyset$  if and only if the irreducible finite root system  $\pi(R)$  is reduced. For a subset  $S$  of  $R$ , let  $S_{\text{sh}} = S \cap R_{\text{sh}}$ ,  $S_{\text{lg}} = S \cap R_{\text{lg}}$ ,  $S_{\text{ex}} = S \cap R_{\text{ex}}$ .

Let  $W$  denote the subgroup of  $\text{GL}(V)$  generated by  $\{s_\alpha \mid \alpha \in R\}$ . For a subset  $S$  of  $R$ , let  $W_S$  be the subgroup of  $W$  generated by  $\{s_\mu \mid \mu \in S\}$ .

### 1.4 Affine root systems

By checking out directly or by using essentially the same argument as that in [8], we have the following.

**Theorem 1.2 ([8]).** (1) *Let  $(R, V)$  be an affine root system of rank  $= l$ . Then there exists a base  $\Pi = \{\alpha_0, \dots, \alpha_l\}$  of  $R$  such that  $\pi(\Pi \setminus \{\alpha_0\})$  is a base of the irreducible finite root system  $\pi(R)$  and such that  $\pi(\alpha_0)$  is the lowest element among  $\pi(R)_{\text{sh}}$ ,  $\pi(R)_{\text{lg}}$  or  $\pi(R)_{\text{ex}}$  with respect to  $\pi(\Pi \setminus \{\alpha_0\})$ . Moreover, there exists a unique  $\delta = \delta(\Pi) \in \alpha_0 + (\mathbb{N}\alpha_1 \oplus \dots \oplus \mathbb{N}\alpha_l)$  such that  $\mathbb{Z}\delta = \mathbb{Z}R \cap V^0$ .*

(2) *Let  $A = A(\Pi) = (a_{ij})_{0 \leq i, j \leq l}$  be a generalized Cartan matrix of affine-type. Let  $\bar{\tau}$  be the subset of  $\{0, \dots, l\}$  formed by  $i \in \{0, \dots, l\}$  such that*

$$(a_{ij}, a_{ji}) \in \{(-2, -1), (-2, -2), (-4, -1), (0, 0)\}$$

for  $j \in \{0, \dots, l\} \setminus \{i\}$ . Let  $\tau$  be a subset of  $\bar{\tau}$ . Then there exist a rank- $l$  affine root system  $R$  and a base  $\Pi = \{\alpha_0, \dots, \alpha_l\}$  of  $R$  having the following properties.

$$(2\text{-i}) \quad a_{ij} = (\alpha_i^\vee, \alpha_j).$$

$$(2\text{-ii}) \quad R \cap \mathbb{R}\alpha_i = \begin{cases} \{\alpha_i, -\alpha_i\} & i \notin \tau, \\ \{\alpha_i, 2\alpha_i, -\alpha_i, -2\alpha_i\} & i \in \tau. \end{cases}$$

The following lemma is well-known, but we state it for later use. Let  $W_{\text{Fin}} = W_{\Pi \setminus \{\alpha_0\}}$  and  $R_{\text{Fin}} = \cup_{i=1}^l W_{\text{Fin}} \cdot \alpha_i$ . Then  $R_{\text{Fin}}$  is an irreducible finite root system. Moreover, we have

$$(1.7) \quad R \subset (\mathbb{Z}\delta + R_{\text{Fin}}) \cup (\mathbb{Z}\delta + 2R_{\text{Fin}}).$$

**Lemma 1.1.** (cf. [6], [7], [8]) *Let  $R$  be an affine root system and let  $\Pi$  be a base of  $R$ . Let  $f \in V^*$  (the dual space of  $V$ ) be such that  $0 \notin f(\mathbb{Z}\Pi \setminus \{0\})$ . Let  $R^{f,+} = \{\alpha \in R \mid f(\alpha) > 0\}$ . Let  $\Pi^f$  be a subset of  $R$  formed by the elements  $\beta \in R^{f,+}$  satisfying the condition that there exist no  $\beta_1, \dots, \beta_r \in R^{f,+}$  with  $r \geq 2$  such that  $\beta = \beta_1 + \dots + \beta_r$ . Then  $|\Pi^f| = l + 1$  and there exist  $w \in W$  and  $\varepsilon \in \{1, -1\}$  such that  $\Pi^f = \varepsilon w(\Pi)$ . In particular, for any two bases  $\Pi_1$  and  $\Pi_2$  of  $R$ , there exist  $w \in W$  and  $\varepsilon \in \{1, -1\}$  such that  $\Pi_1 = \varepsilon w(\Pi_2)$ .*

**Proof.** We may assume  $f(\delta) > 0$ ; otherwise we replace  $f$  with  $-f$ . Let  $R^+ = R \cap \mathbb{Z}_+\Pi$ . By (1.7), since  $R_{\text{Fin}}$  is a finite set, we see that  $R^{f,+} \cap (-R^+)$  is a finite set. If  $|R^{f,+} \cap (-R^+)| = 0$ , then  $R^{f,+} \subset R^+$ , which implies  $R^{f,+} = R^+$  and  $\Pi^f = \Pi$ . Assume that  $r = |R^{f,+} \cap (-R^+)| > 0$ . Then there exists  $\alpha \in \Pi$  such that  $\alpha \notin R^{f,+}$  (and  $-\alpha \in R^{f,+}$ ). Since  $s_\alpha(R^+ \setminus \{\alpha, 2\alpha\}) = R^+ \setminus \{\alpha, 2\alpha\}$ , we have  $|s_\alpha(R^{f,+}) \cap (-R^+)| = |R^{f,+} \cap s_\alpha(-R^+)| = r - 1$  or  $r - 2$ . Define  $f_2 \in V^*$  by  $f_2 = f \circ s_\alpha$ . Then  $R^{f_2,+} = s_\alpha(R^{f,+})$  and  $\Pi^{f_2} = s_\alpha(\Pi^f)$ . By the repetition of this process, we have  $w \in W$  such that  $w(\Pi^f) = \Pi$ .  $\square$

## 1.5 Affine root systems of rank $\leq 2$

**Lemma 1.2.** (cf. [8, Appendixes 1 and 2]) *Let  $(R, V)$  be an affine root system of rank  $= l$ . Let  $\Pi' = \{\alpha_1, \dots, \alpha_l\}$  be a subset of  $R$  such that  $\{\pi(\alpha_1), \dots, \pi(\alpha_l)\}$  is a base of  $\pi(R)$ . Let  $R'_{\text{Fin}} = \cup_{i=1}^l W_{\Pi'} \cdot \alpha_i$ .*

(1) *Let  $\delta'$  be such that  $\mathbb{Z}\delta' = \mathbb{Z}R \cap V^0$ . Then there exists a unique  $\alpha_0 = \alpha_0(R, \Pi', \delta') \in R$  such that  $\Pi = \{\alpha_0\} \cup \Pi'$  is a base of  $R$  and  $\delta' = \delta(\Pi)$ . Moreover,  $\delta' = \alpha_0 - \nu \in \alpha_0 + \mathbb{N}\Pi'$ , where  $\nu$  is a unique element of  $R'_{\text{Fin}} \cup 2R'_{\text{Fin}}$  such that  $\pi(\nu)$  is the lowest element among  $\pi(W_{\Pi'} \cdot \alpha_0)$  with respect to  $\pi(\Pi')$ .*

(2) Assume that  $l \leq 2$  and that for  $1 \leq i \leq l$ ,  $2\alpha_i \in R$  if there exists  $\alpha'_i \in R$  with  $2\alpha'_i \in R$  and  $\pi(\alpha'_i) = \pi(\alpha_i)$ . Then one and only one of the following possibilities holds. We also use the notation of (1) and give  $\alpha_0$ ,  $\delta = \delta(\Pi)$ ,  $A = A(\Pi)$  and  $\tau = \{i | 2\alpha_i \in R, 0 \leq i \leq l\}$ .

(i) The case of  $l = 1$ .

$(A_1^{(1)})$   $\alpha_0 = \delta - \alpha_1$ ,  $R = \cup_{\varepsilon \in \{\pm 1\}} (\varepsilon\alpha_1 + \mathbb{Z}\delta)$ ,  $A$  is  $A_1^{(1)}$ ,  $\tau = \emptyset$ .

$(A_2^{(2)})$   $\alpha_0 = \delta - 2\alpha_1$ ,  $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + (2\mathbb{Z} + 1)\delta))$ ,

$A$  is  $A_2^{(2)}$ ,  $\tau = \emptyset$ .

$(B^{(1)}(0, 1))$   $\alpha_0 = \delta - 2\alpha_1$  and  $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + \mathbb{Z}\delta))$ ,  $A$  is  $A_2^{(2)}$ ,  $\tau = \{1\}$ .

$(C^{(2)}(2))$   $\alpha_0 = \delta - \alpha_1$ ,  $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + 2\mathbb{Z}\delta))$ ,  $A$  is  $A_1^{(1)}$ ,  $\tau = \{0, 1\}$ .

$(A^{(4)}(0, 2))$   $\alpha_0 = \delta - \alpha_1$ ,  $R = \cup_{\varepsilon \in \{\pm 1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + 4\mathbb{Z}\delta))$

$A$  is  $A_1^{(1)}$ ,  $\tau = \{1\}$ .

In other words,

$$R = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha_1 + \mathbb{Z}\delta) \cup (2\varepsilon\alpha_1 + g_1\delta)),$$

where  $g_1$  is  $\emptyset$ ,  $2\mathbb{Z} + 1$ ,  $\mathbb{Z}$ ,  $2\mathbb{Z}$  or  $4\mathbb{Z}$ .

(ii) The case of  $l = 2$ . We assume  $(\alpha_1, \alpha_1) \leq (\alpha_2, \alpha_2)$ .

$(A_2^{(1)})$   $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$ ,  $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_1) + \mathbb{Z}\delta)$ ,  $A$  is  $A_2^{(1)}$ ,  $\tau = \emptyset$ ,  $(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$(C_2^{(1)})$   $\alpha_0 = \delta - (2\alpha_1 + \alpha_2)$ ,  $R = \cup_{i=1}^2 \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_i) + \mathbb{Z}\delta)$ ,  $A$  is  $C_2^{(1)}$ ,  $\tau = \emptyset$ ,  $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$(G_2^{(1)})$   $\alpha_0 = \delta - (3\alpha_1 + 2\alpha_2)$ ,  $R = \cup_{i=1}^2 \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (w(\alpha_i) + \mathbb{Z}\delta)$ ,  $A$  is  $G_2^{(1)}$ ,  $\tau = \emptyset$ ,  $3(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$(A_4^{(2)})$   $\alpha_0 = \delta - (2\alpha_1 + 2\alpha_2)$ ,  $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup (w(2\alpha_1) + (2\mathbb{Z} + 1)\delta))$ ,  $A$  is  $A_4^{(2)}$ ,  $\tau = \emptyset$ ,  $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$(D_3^{(2)})$   $\alpha_0 = \delta - (\alpha_1 + \alpha_2)$ ,  $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 2\mathbb{Z}\delta))$ ,  $A$  is  $D_3^{(2)}$ ,  $\tau = \emptyset$ ,  $2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$(D_4^{(3)})$   $\alpha_0 = \delta - (2\alpha_1 + \alpha_2)$ ,  $R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + 3\mathbb{Z}\delta))$ ,  $A$  is  $D_4^{(3)}$ ,  $\tau = \emptyset$ ,  $3(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2)$ .

$$\begin{aligned}
& (B^{(1)}(0, 2)) \alpha_0 = \delta - (2\alpha_1 + 2\alpha_2), R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup \\
& (w(2\alpha_1) + \mathbb{Z}\delta)), A \text{ is } A_4^{(2)}, \tau = \{1\}, 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2). \\
& (A^{(2)}(0, 3)) \alpha_0 = \delta - (2\alpha_1 + \alpha_2), R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((\cup_{i=1}^2 w(\alpha_i) + \mathbb{Z}\delta) \cup \\
& (w(2\alpha_1) + 2\mathbb{Z}\delta)), A \text{ is } C_2^{(1)}, \tau = \{1\}, 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2). \\
& (C^{(2)}(3)) \alpha_0 = \delta - (\alpha_1 + \alpha_2), R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} ((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + \\
& 2\mathbb{Z}\delta) \cup (w(2\alpha_1) + 2\mathbb{Z}\delta)), A \text{ is } D_3^{(2)}, \tau = \{0, 1\}, 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2). \\
& (A^{(4)}(0, 4)) \alpha_0 = \delta - (\alpha_1 + \alpha_2), R = \cup_{w \in W_{\{\alpha_1, \alpha_2\}}} (((w(\alpha_1) + \mathbb{Z}\delta) \cup (w(\alpha_2) + \\
& 2\mathbb{Z}\delta) \cup (w(2\alpha_1) + 4\mathbb{Z}\delta))), A \text{ is } D_3^{(2)}, \tau = \{1\}, 2(\alpha_1, \alpha_1) = (\alpha_2, \alpha_2).
\end{aligned}$$

In other words,

$$R = \bigcup_{w \in W_{\{\alpha_1, \alpha_2\}}} \left( \bigcup_{i=1}^2 ((w(\alpha_i) + \mathbb{Z}k_i\delta) \cup (w(2\alpha_i) + g_i k_i \delta)) \right),$$

where  $k_1 = 1$ ,  $k_2 \in \{1, -(\alpha_1^\vee, \alpha_2)\} \subset \{1, 2, 3\}$ ,  $g_2 = \emptyset$  and

$$g_1 \in \begin{cases} \{\emptyset\} & \text{if } (\alpha_1^\vee, \alpha_2) \neq -2, \\ \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1\} & \text{if } (\alpha_1^\vee, \alpha_2) = -2 \text{ and } k_2 = 1, \\ \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\} & \text{if } (\alpha_1^\vee, \alpha_2) = -2 \text{ and } k_2 = 2. \end{cases}$$

Moreover, we also have

$$R \cap (\mathbb{R}\alpha_i \oplus \mathbb{R}\delta) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha_i + \mathbb{Z}k_i\delta) \cup (2\varepsilon\alpha_i + g_i k_i \delta))$$

for  $i \in \{1, 2\}$ .

(The symbol in each parenthesis is the notation used in [7, Table 1-4] for the Dynkin diagram for the pair  $(A, \tau)$ .)

**Proof.** (1) Notice that  $\Pi' \cup \{\delta'\}$  is a basis of  $\mathbb{Z}R$ . Let  $f \in V^*$  be such that  $0 \notin f(\mathbb{Z}R \setminus \{0\})$ ,  $f(\alpha_i) > 0$  ( $1 \leq i \leq l$ ) and  $f(\delta') > 5|f(\mu)|$ , where  $\mu$  is the lowest element of  $R'_{\text{Fin}}$  with respect to  $\Pi'$ . Recall  $\Pi^f$  from Lemma 1.1. Since  $R \subset \mathbb{Z}\delta' + (R'_{\text{Fin}} \cup 2R'_{\text{Fin}})$ , we have  $\Pi' \subset \Pi^f$ . We see that there exists  $\alpha_0 \in R$  such that  $\Pi' \cup \{\alpha_0\} = \Pi^f$ . Since  $\Pi^f$  is also a basis of  $\mathbb{Z}R$ , we have  $\alpha_0 \in \pm\delta' + \mathbb{Z}\Pi'$ . It is clear that  $\pi(\alpha_0) = \pi(\nu)$ . (The reason is the following. Assume  $\pi(\alpha_0) \neq \pi(\nu)$ . Then there exist  $w \in W_{\Pi'}$ ,  $n \in \mathbb{Z}$  and  $\gamma \in \mathbb{Z}_+\Pi' \setminus \{0\}$  such that  $w(\alpha_0) = \nu + n\delta'$  and  $\alpha_0 = w^{-1}(\nu) + n\delta' = \nu + \gamma + n\delta' = \gamma + w(\alpha_0)$ .)

Then  $\alpha_0 - \gamma = w(\alpha_0) \in R$ . This is contradiction since  $\Pi^f$  is a base of  $R$ .) Hence  $\alpha_0 - \nu \in \{\delta', -\delta'\}$ . Notice that  $-\nu \in \mathbb{N}\Pi'$ . Since  $f(\delta') > 0$  and  $f(\alpha_0 - \nu) > 0$ , we have  $\delta' = \alpha_0 - \nu$ .

We show the uniqueness. Let  $\alpha'_0$  be as in the statement. Since  $\{\alpha'_0\} \cup \Pi'$  is also a basis of  $\mathbb{Z}R$ , we have  $\alpha'_0 \in \pm\alpha_0 + \mathbb{Z}\Pi'$ . Since  $\delta' \in \mathbb{N}\alpha'_0 + \mathbb{N}\Pi'$ , we have  $\alpha'_0 \in \alpha_0 + \mathbb{Z}\Pi'$ . Since  $\{\alpha_0\} \cup \Pi'$  is a base of  $R$ ,  $\alpha'_0 \in \alpha_0 + \mathbb{Z}_+\Pi'$ . Since  $\{\alpha'_0\} \cup \Pi'$  is a base of  $R$ , we have  $\alpha'_0 = \alpha_0$ . This completes the proof of (1).

(2) Using (1), Theorem 1.2 and Lemma 1.1, we can easily prove the statement (2).  $\square$

## 1.6 Some facts about affine root systems

Here we give lemmas which shall be used afterward.

**Lemma 1.3.** *Let  $R$  be an affine root system. Let  $\Pi$  be a base of  $R$ . Let  $S$  be a subset of  $\Pi$ . Then*

$$(1.8) \quad (R \setminus 2R) \cap \mathbb{R}S = \{w(\alpha) \mid \alpha \in S, w \in W_S\}.$$

**Proof.** It is clear that for (1.8), RHS is included in LHS. Define  $f \in V^*$  by  $f(\alpha) = 1$  ( $\alpha \in \Pi$ ). Let  $\beta \in (R \setminus 2R) \cap \mathbb{R}S$ . We may assume  $f(\beta) > 0$ . By (1.2),  $\beta \in \mathbb{Z}_+S \cap (R \setminus 2R)$ . Since  $(\beta, \beta) > 0$ , there exists  $\gamma \in S$  such that  $(\beta, \gamma) > 0$ . By (1.1), we may assume  $\beta \notin \mathbb{R}\gamma$ . Then  $s_\gamma(\beta) = \beta - (\gamma^\vee, \beta)\gamma \in \mathbb{Z}_+S \cap (R \setminus 2R)$  and  $0 < f(s_\gamma(\beta)) = f(\beta) - (\gamma^\vee, \beta) < f(\beta)$ . Hence, by induction on  $f(\beta)$ , we can prove the lemma.  $\square$

**Lemma 1.4.** *Let  $R$  be an affine root system. Let  $\Pi$  be a base of  $R$ .*

(1) *For every  $\beta \in R \setminus 2R$ , there exist  $\alpha \in \Pi$  and  $w \in W$  such that  $\beta = w(\alpha)$ . Moreover, if  $\mu \in (\mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi) \setminus (R \cup \mathbb{Z}\delta)$ , then there exists  $w \in W$  such that*

$$w(\mu) \in ((\cup_{\alpha \in \Pi} \mathbb{N}\alpha) \setminus R) \cup (\mathbb{Z}\Pi \setminus (\mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi)).$$

(2) *Let  $\alpha, \beta \in \Pi$  be such that  $\alpha \neq \beta$  and  $(\alpha, \alpha) = (\beta, \beta)$ . Then  $\alpha \notin W.\beta$  if and only if one of the following occurs.*

(2-i)  *$A(\Pi)$  is  $C_l^{(1)}$  and  $\{\alpha, \beta\} = \Pi_{\text{lg}}$ .*

(2-ii)  *$A(\Pi)$  is  $D_{l+1}^{(2)}$  and  $\{\alpha, \beta\} = \Pi_{\text{sh}}$ .*

**Proof.** The statement (1) follows from the same argument as in Proof of Lemma 1.3 with  $\Pi$  in place of  $S$ .

The statement (2) can be checked directly.  $\square$

## 2 Elliptic root systems of rank $\geq 2$

### 2.1 Maps $k$ and $g$ and construction of $R(\Pi, k, g)$

In this section, we always assume  $(R, V)$  to be an elliptic root system of rank  $l \geq 2$ . Let  $\Pi \cup \{a\}$  be a fundamental-set of  $R$ . Let  $(\Pi_{\text{sh}})'$  be the subset of  $\Pi$  formed by  $\alpha \in \Pi$  satisfying the condition that either  $(\alpha, \beta) = 0$  or  $(\beta, \beta) = 2(\alpha, \alpha)$  holds for every  $\beta \in \Pi \setminus \{\alpha\}$ . Then  $(\Pi_{\text{sh}})' \subset \Pi_{\text{sh}}$  and, more precisely, we have

$$(\Pi_{\text{sh}})' = \begin{cases} \Pi_{\text{sh}} & \text{if } l = 2 \text{ and } A(\Pi) \text{ is } C_2^{(1)}, A_4^{(2)} \text{ or } D_3^{(2)}, \\ \Pi_{\text{sh}} & \text{if } l \geq 3 \text{ and } A(\Pi) \text{ is } B_l^{(1)}, A_{2l}^{(2)} \text{ or } D_{l+1}^{(2)}, \\ \emptyset & \text{otherwise.} \end{cases}$$

**Theorem 2.1.** *Let  $R$  be an elliptic root system such that  $l \geq 2$  (where  $l = \dim \mathbb{R}\pi(R)$ ). Let  $\Pi \cup \{a\}$  be a fundamental-set of  $R$ . Let  $(\Pi_{\text{sh}})'$  be as above. Then there exist maps  $k : \Pi \rightarrow \{1, 2, 3, 4\}$  and  $g : \Pi \rightarrow \mathcal{P}$  satisfying the following.*

(i)

$$(2.1) \quad R = \bigcup_{w \in W_{\Pi}} \left( \bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right)$$

and

$$(2.2) \quad (\alpha + \mathbb{R}a) \cap R = \alpha + \mathbb{Z}k(\alpha)a, \quad (2\alpha + \mathbb{R}a) \cap R = 2\alpha + g(\alpha)k(\alpha)a$$

for  $\alpha \in \Pi$ .

(ii)  $\min\{k(\alpha) \mid \alpha \in \Pi\} = 1$ .

(iii) *If  $\alpha, \beta \in \Pi$  are such that  $(\beta^\vee, \alpha) = -1$  and  $(\alpha^\vee, \beta) = -m$  with  $m \in \{1, 2, 3\}$ , then  $k(\beta) = k(\alpha)$  or  $k(\beta) = mk(\alpha)$ . (See Remark 2.1 (1) below.)*

(iv)  $g(\alpha) = \emptyset$  if  $\alpha \notin (\Pi_{\text{sh}})'$ .

(v)  $g(\alpha) \in \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z}+1\}$  if  $\alpha \in (\Pi_{\text{sh}})'$  and there exists  $\beta \in \Pi \setminus \{\alpha\}$  such that  $(\alpha, \beta) \neq 0$  and  $k(\alpha) = k(\beta)$  (notice  $(\alpha^\vee, \beta) = -2$  and  $(\beta^\vee, \alpha) = -1$ ).

(vi)  $g(\alpha) \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\}$  if  $\alpha \in (\Pi_{\text{sh}})'$  and there exists  $\beta \in \Pi \setminus \{\alpha\}$  such that  $(\alpha, \beta) \neq 0$  and  $2k(\alpha) = k(\beta)$  (notice  $(\alpha^\vee, \beta) = -2$  and  $(\beta^\vee, \alpha) = -1$ ).

*Remark 2.1.* Before we give the proof of Theorem 2.1, we remark the following.

(1) Since  $l \geq 2$ , it follows that  $(\beta^\vee, \alpha) = -1$  and  $(\alpha^\vee, \beta) \in \{-1, -2, -3\}$  for  $\alpha, \beta \in \Pi$  with  $\alpha \neq \beta$ ,  $(\alpha, \beta) \neq 0$  and  $(\alpha, \alpha) \leq (\beta, \beta)$ . The image  $k(\Pi)$  for each  $\Pi$  is explicitly indicated in (2.5) in the proof of Theorem 2.1. In the cases where  $g(\alpha) \neq \emptyset$  for some  $\alpha \in \Pi$ , all the possible values of  $k$  and  $g$  are given in Tables 4.1-4.4.

(2) By (iii) and (iv) of Theorem 2.1, we conclude that if  $\alpha \in \Pi$  is such that  $g(\alpha) \neq \emptyset$ , then  $(\beta^\vee, \alpha) = -1$ ,  $(\alpha^\vee, \beta) = -2$  and  $k(\beta) \in \{k(\alpha), 2k(\alpha)\}$  for  $\beta \in \Pi \setminus \{\alpha\}$  with  $(\alpha, \beta) \neq 0$ .

(3) If  $g(\alpha) \neq \emptyset$  for some  $\alpha \in \Pi$ , then  $A(\Pi)$  is  $C_2^{(1)}$  ( $l = 2$ ),  $B_l^{(1)}$  ( $l \geq 3$ ),  $A_{2l}^{(2)}$  ( $l \geq 2$ ) or  $D_{l+1}^{(2)}$  ( $l \geq 2$ ) since  $(\Pi_{\text{sh}})' \neq \emptyset$  (see (iii) of Theorem 2.1).

**Proof of Theorem 2.1.** We first notice that  $\mathbb{Z}R = (\bigoplus_{\alpha \in \Pi} \mathbb{Z}\alpha) \oplus \mathbb{Z}a$ . Since  $R$  cannot be included in  $\mathbb{Z}\Pi$ , there exists  $\gamma \in R$  such that  $\gamma \in ma + \mathbb{Z}\Pi$  for some  $m \in \mathbb{Z} \setminus \{0\}$ . We may assume  $\gamma \in R \setminus 2R$ . By Lemma 1.4 (1), there exist  $\alpha \in \Pi$  and  $w \in W_\Pi$  such that  $w(\gamma) = \alpha + ma$ . Notice  $\alpha + ma \in R$ . For  $\beta \in \Pi$  with  $\beta \neq \alpha$  and  $(\alpha, \beta) \neq 0$ , we have  $s_\alpha s_{\alpha+ma}(\beta) \in R$  and

$$s_\alpha s_{\alpha+ma}(\beta) = s_\alpha(\beta - (\alpha^\vee, \beta)(\alpha + ma)) = \beta - (\alpha^\vee, \beta)ma.$$

Hence

$$(2.3) \quad (\alpha' + (\mathbb{Z} \setminus \{0\})a) \cap R \neq \emptyset$$

for all  $\alpha' \in \Pi$ .

Let  $\alpha, \beta \in \Pi$  be such that  $\alpha \neq \beta$ ,  $(\alpha, \beta) \neq 0$  and  $(\alpha, \alpha) \leq (\beta, \beta)$ . Let  $z_{\alpha, \beta} = -(\alpha^\vee, \beta)$ . Since  $l \geq 2$ , we see that  $(\beta^\vee, \alpha) = -1$  and  $z_{\alpha, \beta} \in \{1, 2, 3\}$ . Let  $V_{\alpha, \beta} = \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a$  and let  $R_{\alpha, \beta} = R \cap V_{\alpha, \beta}$ . By (2.3),  $(R_{\alpha, \beta}, V_{\alpha, \beta})$  is a rank-two affine root system. Notice that  $V_{\alpha, \beta}^0 = \mathbb{R}a$ . Since  $\pi_a(R) \subset \mathbb{Z}_+ \pi_a(\Pi) \cup \mathbb{Z}_- \pi_a(\Pi)$ , we have  $\pi_a(R_{\alpha, \beta}) \subset \mathbb{Z}_+ \{\pi_a(\alpha), \pi_a(\beta)\} \cup \mathbb{Z}_- \{\pi_a(\alpha), \pi_a(\beta)\}$ , which implies that  $\{\pi_a(\alpha), \pi_a(\beta)\}$  is a base of the rank-two irreducible finite root system  $\pi_a(R_{\alpha, \beta})$ . By (2.3) and Lemma 1.2 (2) (ii), we have the following.

- (1) There exists  $m_{\alpha, \beta} \in \mathbb{N}$  such that  $V_{\alpha, \beta}^0 \cap \mathbb{Z}R_{\alpha, \beta} = \mathbb{Z}m_{\alpha, \beta}a$ .
- (2) There exist  $y_{\alpha, \beta} \in \{1, z_{\alpha, \beta}\}$  and  $G_{\alpha, \beta} \in \mathcal{P}$  such that

$$\begin{cases} G_{\alpha, \beta} = \emptyset & \text{if } z_{\alpha, \beta} \neq 2, \\ G_{\alpha, \beta} \in \{\emptyset, \mathbb{Z}, 2\mathbb{Z}, 2\mathbb{Z} + 1\} & \text{if } z_{\alpha, \beta} = 2 \text{ and } y_{\alpha, \beta} = 1, \\ G_{\alpha, \beta} \in \{\emptyset, 2\mathbb{Z}, 4\mathbb{Z}\} & \text{if } z_{\alpha, \beta} = y_{\alpha, \beta} = 2 \end{cases}$$

and

$$\begin{cases} R_{\alpha,\beta} = \bigcup_{w \in W_{\{\alpha,\beta\}}} \left( (w(\alpha) + \mathbb{Z}m_{\alpha,\beta}a) \cup (w(2\alpha) + G_{\alpha,\beta}m_{\alpha,\beta}a) \right. \\ \quad \left. \cup (w(\beta) + \mathbb{Z}y_{\alpha,\beta}m_{\alpha,\beta}a) \right), \\ R_{\alpha,\beta} \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1,-1\}} ((\varepsilon\alpha + \mathbb{Z}m_{\alpha,\beta}a) \cup (2\varepsilon\alpha + G_{\alpha,\beta}m_{\alpha,\beta}a)), \\ R_{\alpha,\beta} \cap (\mathbb{R}\beta \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1,-1\}} (\varepsilon\beta + \mathbb{Z}y_{\alpha,\beta}m_{\alpha,\beta}a). \end{cases}$$

Moreover, we notice that

$$R \cap (\mathbb{R}\gamma \oplus \mathbb{R}a) = R_{\alpha,\beta} \cap (\mathbb{R}\gamma \oplus \mathbb{R}a)$$

for  $\gamma \in \{\alpha, \beta\}$ .

By the previous paragraph, we conclude that there exist maps  $k : \Pi \rightarrow \mathbb{N}$  and  $g : \Pi \rightarrow \mathcal{P}$  satisfying the condition that

$$(2.4) \quad R \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1,-1\}} ((\varepsilon\alpha + \mathbb{Z}k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a))$$

( $\alpha \in \Pi$ ) and satisfying the conditions (iii)-(vi) of the statement of this theorem. By Lemma 1.4 (1), we see that for every  $\mu \in R$ , there exist  $w \in W_\Pi$ ,  $\alpha \in \Pi$  and  $u \in \{1, 2\}$  such that  $\pi_a(w(\mu)) = \pi_a(u\alpha)$ , which implies that  $w(\mu) \in u\alpha + \mathbb{Z}a$ . By (2.4), we see that  $R$  has the same expression as that in (2.1). By (2.4), we see that  $k$  and  $g$  also satisfy (2.2). Hence  $k$  and  $g$  satisfy the condition (i). Since  $a \in \mathbb{Z}R$ , by (2.1) and (iii), we conclude that

$$(2.5) \quad k(\Pi) = \begin{cases} \{1\} \text{ if } A(\Pi) \text{ is } A_l^{(1)}, D_l^{(1)} \text{ or } E_l^{(1)}, \\ \{1\} \text{ or } \{1, 3\} \text{ if } A(\Pi) \text{ is } G_2^{(1)} \text{ or } D_4^{(3)}, \\ \{1\}, \{1, 2\} \text{ or } \{1, 2, 4\} \text{ if } A(\Pi) \text{ is } A_{2l}^{(2)}, \\ \{1\} \text{ or } \{1, 2\} \text{ otherwise.} \end{cases}$$

Hence  $k(\Pi) \subset \{1, 2, 3, 4\}$  and  $k$  satisfies the condition (ii). This completes the proof.  $\square$

**Definition 2.1.** Let  $l \geq 2$ . Let  $A = A(\Pi) = (a_{ij})_{0 \leq i, j \leq l}$  be a generalized Cartan matrix of affine-type. Let  $V$  be an  $l + 2$ -dimensional  $\mathbb{R}$ -linear space with a basis  $\{\alpha_0, \dots, \alpha_l, a\}$ . Let  $(, )$  be a symmetric bilinear form on  $V$  such that  $(a, a) = 0$ ,  $(a, \alpha_i) = 0$ ,  $(\alpha_i, \alpha_i) > 0$  and  $(\alpha_i^\vee, \alpha_j) = a_{ij}$ , where  $\alpha_i^\vee = \frac{2\alpha_i}{(\alpha_i, \alpha_i)}$ . Let  $\Pi = \{\alpha_0, \dots, \alpha_l\}$ . Let  $k : \Pi \rightarrow \{1, 2, 3, 4\}$  and  $g : \Pi \rightarrow \mathcal{P}$

be maps satisfying the same conditions as those in (ii)-(vi) of the statement of Theorem 2.1. For such  $\Pi$ ,  $k$  and  $g$ , set

$$(2.6) \quad R = R(\Pi, k, g) = \bigcup_{w \in W_\Pi} \left( \bigcup_{\alpha \in \Pi} ((w(\alpha) + \mathbb{Z}k(\alpha)a) \cup (w(2\alpha) + g(\alpha)k(\alpha)a)) \right),$$

where  $W_\Pi$  is the subgroup of  $\text{GL}(V)$  generated by  $\{s_{\alpha_i} | 0 \leq i \leq l\}$ ;  $s_{\alpha_i}$  are defined by  $s_{\alpha_i}(v) = v - (\alpha_i^\vee, v)\alpha_i$  ( $v \in V$ ).

Let  $\pi_a : V \rightarrow V/\mathbb{R}a$  be the canonical map.

The following is a converse of Theorem 2.1.

**Theorem 2.2.** *Keep the notation as in Definition 2.1. Then  $(R(\Pi, k, g), V)$  is an elliptic root system. Moreover,  $\Pi \cup \{a\}$  is a fundamental-set of  $R(\Pi, k, g)$ .*

The proof of the theorem shall be given in Subsection 3.3.

**Lemma 2.1.** *Keep the notation as in Definition 2.1.*

(1) *Let  $f_i : \Pi \rightarrow 2^{\mathbb{Z}}$  ( $i \in \{1, 2\}$ ) be maps satisfying the condition that  $f_i(\alpha) = f_i(\beta)$  if  $\alpha, \beta \in \Pi$  are such that  $w(\alpha) = \beta$  for some  $w \in W_\Pi$ . Let*

$$\mathcal{R} = \bigcup_{w \in W_\Pi} \left( \bigcup_{\alpha \in \Pi} ((w(\alpha) + f_1(\alpha)a) \cup (w(2\alpha) + f_2(\alpha)a)) \right).$$

Then

$$(2.7) \quad \mathcal{R} \cap (\mathbb{R}\alpha \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + f_1(\alpha)a) \cup (2\varepsilon\alpha + f_2(\alpha)a))$$

for  $\alpha \in \Pi$ . Moreover,

$$(2.8) \quad \mathcal{R} \cap (\mathbb{R}w(\alpha) \oplus \mathbb{R}a) = \bigcup_{\varepsilon \in \{1, -1\}} ((\varepsilon w(\alpha) + f_1(\alpha)a) \cup (\varepsilon w(2\alpha) + f_2(\alpha)a))$$

for  $\alpha \in \Pi$  and  $w \in W_\Pi$ . Furthermore, for a subset  $S$  of  $\Pi$ , we have

$$(2.9) \quad \mathcal{R} \cap (\mathbb{R}S \oplus \mathbb{R}a) = \bigcup_{w \in W_S} \left( \bigcup_{\alpha \in S} ((w(\alpha) + f_1(\alpha)a) \cup (w(2\alpha) + f_2(\alpha)a)) \right).$$

(2)  *$k(\alpha) = k(\beta)$  and  $g(\alpha) = g(\beta)$  if  $\alpha, \beta \in \Pi$  are such that  $w(\alpha) = \beta$  for some  $w \in W_\Pi$ . In particular,  $R(\Pi, k, g)$  has the same properties as those in (1) with  $k(\alpha)\mathbb{Z}$  and  $k(\alpha)g(\alpha)$  in place of  $f_1(\alpha)$  and  $f_2(\alpha)$  respectively.*

**Proof.** (1) The equalities (2.7) and (2.8) follow immediately from the definition of  $f_i$ . The equality (2.9) follows from Lemma 1.3 and (2.7).

(2) By Lemma 1.4 (2), for  $\alpha, \beta \in \Pi$ , we see that  $\beta \in W_\Pi \alpha$  if and only if  $\alpha = \beta$  or there exist  $\gamma_1, \dots, \gamma_r \in \Pi$  such that  $\gamma_1 = \alpha$  and  $\gamma_r = \beta$  and  $(\gamma_p^\vee, \gamma_{p+1}) = (\gamma_{p+1}^\vee, \gamma_p) = -1$  ( $1 \leq p \leq r-1$ ). Notice that  $k(\alpha) = k(\beta)$  and  $g(\alpha) = g(\beta)$  ( $= \emptyset$ ) for  $\alpha, \beta \in \Pi$  with  $(\alpha^\vee, \beta) = (\beta^\vee, \alpha) = -1$ . This completes the proof.  $\square$

## 2.2 Notation $T_l, T_l^{\text{nm}}$ and $\mathfrak{g}(\Pi, k, g)$ for any $(\Pi, k, g) \in T_l$

**Definition 2.2.** Let  $l \geq 2$ .

(1) Let  $T_l$  be the set of the triples  $(\Pi, k, g)$ , where  $\Pi = \{\alpha_0, \dots, \alpha_l\}$ ,  $k : \Pi \rightarrow \{1, 2, 3, 4\}$  and  $g : \Pi \rightarrow \mathcal{P}$  are those in Definition 2.1.

(2) Let  $(\Pi, k, g) \in T_l$ . For the  $(\Pi, k, g)$ , we also let  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  be the  $\mathbb{C}$ -Lie superalgebra defined in the same way as in Definition 1.2 (we have not given a proof of Theorem 2.2). We also define  $\mathcal{E} = \mathcal{E}(\Pi, k, g)$ ,  $\mathfrak{h} = \mathfrak{h}(\Pi, k, g)$  and  $\mathfrak{g}_\mu$  in the same way as that for the Lie superalgebra of Definition 1.2.

(3) Define subsets  $T_l^{\text{rm}}, T_l^{\text{nm}}, T_l^{\text{r}}$  and  $T_l^{\text{nr}}$  of  $T_l$  by

$$\begin{aligned} T_l^{\text{rm}} &= \{(\Pi, k, g) \in T_l \mid g(\alpha) = \emptyset \text{ for all } \alpha \in \Pi\}, \\ T_l^{\text{nm}} &= T_l \setminus T_l^{\text{rm}}, \\ T_l^{\text{r}} &= \{(\Pi, k, g) \in T_l \mid g(\alpha) \in \{\emptyset, 2\mathbb{Z} + 1\} \text{ for all } \alpha \in \Pi\}, \\ T_l^{\text{nr}} &= T_l \setminus T_l^{\text{r}}. \end{aligned}$$

*Remark 2.2.* (1) Let  $l \geq 2$ . Let  $J_l$  be the set of the isomorphism classes of the rank- $l$  elliptic root systems. For a rank- $l$  elliptic root system  $R$ , we denote by  $[R]$  ( $\in J_l$ ) the isomorphism class represented by  $R$ . By Theorem 2.2, we can define a map  $\mathcal{F}_l : T_l \rightarrow J_l$  by  $\mathcal{F}_l((\Pi, k, g)) = [R(\Pi, k, g)]$ . By Theorem 2.1,  $\mathcal{F}_l$  is a surjective map.

(2) Let  $J_l^{\text{rm}} = \mathcal{F}_l(T_l^{\text{rm}})$ ,  $J_l^{\text{r}} = \mathcal{F}_l(T_l^{\text{r}})$  and  $J_l^{\text{nr}} = \mathcal{F}_l(T_l^{\text{nr}})$ . We see that  $J_l^{\text{r}}$  (respectively,  $J_l^{\text{nr}}$ ) is the set of the isomorphism classes of the reduced (respectively, non-reduced) elliptic root systems of rank  $= l$  ( $\geq 2$ ). We also see that  $J_l^{\text{rm}}$  is the set of the isomorphism classes  $[R]$  of the reduced elliptic root systems  $(R, V)$  satisfying the condition that there exists a one-dimensional subspace  $G$  of  $V^0$  such that  $G \cap \mathbb{Z}R \neq \{0\}$  and  $(\pi_G(R), V/G)$  is a reduced affine root system, where  $\pi_G : V \rightarrow V/G$  is the canonical map. Notice that this assumption is the same as in [10, §5 A)]. We call this  $R$  a *reduced-marked* elliptic root system. Notice that  $J_l^{\text{rm}} \subset J_l^{\text{r}}$  and, as mentioned in Remark 1.1 (2),  $|J_l^{\text{r}}/J_l^{\text{rm}}| = 2$ .

### 3 Proof of Theorem 1.1 (1)

#### 3.1 Rank one subalgebras

In this section, we begin with the following lemma.

**Lemma 3.1.** *Let  $l \geq 2$  and  $(\Pi, k, g) \in T_l^{\text{nm}}$ . Let  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  be as in Definition 2.2 (2). Then we have  $h_\sigma \neq 0$  for  $\sigma \in \mathcal{E}(\Pi, k, g) \setminus \{0\}$ , where  $h_\sigma \in \mathfrak{h} \subset \mathfrak{g}$ . In particular,  $\dim \mathfrak{h} = l + 4$ .*

Proof of this lemma shall be given in Subsection 4.8.

Keep the notation as above. For a subset  $S$  of  $\Pi$ , let  $\mathfrak{g}^S$  be the sub-Lie superalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and  $E_\mu, E_{-\mu}, E_{\mu^*}, E_{-\mu^*}$  ( $\mu \in S$ ). For  $\sigma \in \mathcal{E}$ , let  $\mathfrak{g}_\sigma^S = \mathfrak{g}_\sigma \cap \mathfrak{g}^S$ .

Let  $\mu \in \mathcal{B}$ . By (SR1-9), we see that  $E_\mu \in \mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  is locally nilpotent, so, for  $x \in \mathbb{C} \setminus \{0\}$ , we can define  $n_{xE_\mu} \in \text{Aut}(\mathfrak{g})$  by

$$n_{xE_\mu} = \begin{cases} \exp \text{ad} x E_\mu \exp \text{ad}(-x^{-1} E_{-\mu}) \exp \text{ad} x E_\mu & \text{if } p(\mu) = 0, \\ \exp(\frac{1}{4} \text{ad}[x E_\mu, x E_\mu]) \exp(\frac{1}{4} \text{ad}[x^{-1} E_{-\mu}, x^{-1} E_{-\mu}]) \exp(\frac{1}{4} \text{ad}[x E_\mu, x E_\mu]) & \text{if } p(\mu) = 1. \end{cases}$$

Notice that

$$(3.1) \quad n_{xE_\mu}^{\pm 1}(h_\sigma) = h_{s_\mu(\sigma)} \quad \text{and} \quad n_{xE_\mu}^{\pm 1}(\mathfrak{g}_\sigma) = \mathfrak{g}_{s_\mu(\sigma)} \quad (\sigma \in \mathcal{E}).$$

Notice that for  $\mu, \nu \in \mathcal{B}_+$  with  $(\mu^\vee, \nu) = -1$ , we have

$$(3.2) \quad \begin{cases} n_{xE_\mu}(E_{\pm\nu}) = \pm x^{\pm 1} [E_{\pm\mu}, E_{\pm\nu}], \\ n_{xE_\nu}(E_{\pm\mu}) = (\pm x)^{\mp(\nu^\vee, \mu)} \frac{1}{(-(\nu^\vee, \mu))!} (\text{ad} E_{\pm\nu})^{-(\nu^\vee, \mu)} E_{\pm\mu}, \end{cases}$$

where we notice that  $(\nu^\vee, \mu) = -2$  if  $p(\nu) = 1$ . Let

$$(3.3) \quad n_\mu = n_{E_\mu}.$$

**Lemma 3.2.** *Let  $l \geq 2$  and  $(\Pi, k, g) \in T_l^{\text{nm}}$ . Let  $\alpha \in \Pi$  and  $R_\alpha = R(\Pi, k, g) \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$ . Then we have the following.*

(1)  $R_\alpha = \cup_{\varepsilon \in \{1, -1\}} ((\varepsilon\alpha + k(\alpha)a) \cup (2\varepsilon\alpha + g(\alpha)k(\alpha)a))$  and  $\mathbb{Z}R_\alpha \cap V^0 = \mathbb{Z}k(\alpha)a$ .

(2)  $R_\alpha$  is a rank-one affine root system with a base  $\{\alpha, -\alpha^*\}$ . Moreover,  $-\alpha^* = \alpha_0(R_\alpha, \{\alpha\}, -k(\alpha)a)$  (see Lemma 1.2 (1)) and we may identify  $\alpha$  and  $\alpha^*$  with  $\alpha_1$  and  $-\alpha_0$  of Lemma 1.2 (2) (i) respectively.

(3) We have

$$\mathfrak{g}^{\{\alpha\}} = \mathfrak{h} \bigoplus \left( \bigoplus_{\mu \in R_\alpha} \mathfrak{g}_\mu^{\{\alpha\}} \right) \bigoplus \left( \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{mk(\alpha)a}^{\{\alpha\}} \right)$$

and  $\dim \mathfrak{g}_\mu^{\{\alpha\}} = 1$  ( $\mu \in R_\alpha$ ).

**Proof.** The statement (1) follows immediately from Lemma 2.1, especially (2.7). The statement (2) follows immediately from the statement (1) and Lemma 1.2 (2) (i). The statement (3) can be proved by the statement (2), (SR1-5), Lemma 3.1 and the well-known argument in the proof of [6, Corollary 5.12] (see also [7, Proposition 1.6]). (Notice that for  $\alpha \in \Pi$ , since  $[E_\alpha, E_{-\alpha}] = h_{\alpha^\vee} \neq 0$  and  $[E_{\alpha^*}, E_{-\alpha^*}] = h_{(\alpha^*)^\vee} \neq 0$ , we have  $E_{\pm\alpha} \neq 0$  and  $E_{\pm\alpha^*} \neq 0$ .)  $\square$

## 3.2 Rank two subalgebras

In Subsections 3.2 and 3.3, we keep the notation of Definitions 3.1 and 3.2 below.

**Definition 3.1.** Let  $l \geq 2$  and  $(\Pi, k, g) \in T_l^{\text{norm}}$ . Let  $R = R(\Pi, k, g)$  be as in (2.6). Notice that  $A(\Pi)$  is  $C_2^{(1)}$  ( $l = 2$ ),  $B_l^{(1)}$  ( $l \geq 3$ ),  $A_{2l}^{(2)}$  ( $l \geq 2$ ) or  $D_{l+1}^{(2)}$  ( $l \geq 2$ ) (see Remark 2.1 (3)). Let  $\alpha, \beta \in \Pi$  be such that  $\alpha \neq \beta$ ,  $(\alpha, \beta) \neq 0$  and  $(\alpha, \alpha) \leq (\beta, \beta)$ . Notice that  $(\beta^\vee, \alpha) = -1$ . Let  $V_{\alpha, \beta} = \mathbb{R}\alpha \oplus \mathbb{R}\beta \oplus \mathbb{R}a$  ( $\subset V$ ) and let  $R_{\alpha, \beta} = R \cap V_{\alpha, \beta}$ . Notice that it follows from Lemma 2.1, especially (2.9), that

$$(3.4) \quad R_{\alpha, \beta} = \cup_{w \in W_{\{\alpha, \beta\}}} (\cup_{\mu \in \{\alpha, \beta\}} ((w(\mu) + \mathbb{Z}k(\mu)a) \cup (2w(\mu)\alpha + g(\mu)k(\mu)a))).$$

Define  $\gamma \in R_{\alpha, \beta}$  by

$$\begin{aligned} (A_2^{(1)}) \quad & \gamma = s_\alpha(-\beta^*) \text{ if } (\alpha, \alpha) = (\beta, \beta), k(\alpha) = k(\beta), g(\alpha) = g(\beta) = \emptyset. \\ (C_2^{(1)}) \quad & \gamma = s_\alpha(-\beta^*) \text{ if } 2(\alpha, \alpha) = (\beta, \beta), k(\alpha) = k(\beta), g(\alpha) = g(\beta) = \emptyset. \\ (D_3^{(2)}) \quad & \gamma = s_\beta(-\alpha^*) \text{ if } 2(\alpha, \alpha) = (\beta, \beta), 2k(\alpha) = k(\beta), g(\alpha) = g(\beta) = \emptyset. \\ (A_4^{(2)}) \quad & \gamma = s_\beta(-\alpha^*) \text{ if } 2(\alpha, \alpha) = (\beta, \beta), k(\alpha) = k(\beta), g(\alpha) = 2\mathbb{Z} + 1, \\ & g(\beta) = \emptyset. \\ (B^{(1)}(0, 2)) \quad & \gamma = s_\beta(-\alpha^*) \text{ if } 2(\alpha, \alpha) = (\beta, \beta), k(\alpha) = k(\beta), g(\alpha) = \mathbb{Z}, \\ & g(\beta) = \emptyset. \end{aligned}$$

$(A^{(2)}(0, 3)) \gamma = s_\alpha(-\beta^*)$  if  $2(\alpha, \alpha) = (\beta, \beta)$ ,  $k(\alpha) = k(\beta)$ ,  $g(\alpha) = 2\mathbb{Z}$ ,  $g(\beta) = \emptyset$ .

$(C^{(2)}(3)) \gamma = s_\beta(-\alpha^*)$  if  $2(\alpha, \alpha) = (\beta, \beta)$ ,  $2k(\alpha) = k(\beta)$ ,  $g(\alpha) = 2\mathbb{Z}$ ,  $g(\beta) = \emptyset$ .

$(A^{(4)}(0, 4)) \gamma = s_\beta(-\alpha^*)$  if  $2(\alpha, \alpha) = (\beta, \beta)$ ,  $2k(\alpha) = k(\beta)$ ,  $g(\alpha) = 4\mathbb{Z}$ ,  $g(\beta) = \emptyset$ .

Let  $\Pi' = \{\alpha, \beta, \gamma\}$ .

**Lemma 3.3.**  $R_{\alpha, \beta}$  is a rank-two affine root system with a base  $\Pi'$ . Moreover, we may identify  $\alpha$ ,  $\beta$  and  $\gamma$  with  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_0$  of Lemma 1.2 (2) (ii) respectively.

This follows immediately from (3.4) and Lemma 1.2 (2) (ii).

**Definition 3.2.** Let  $l \geq 2$ . Let  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  with  $(\Pi, k, g) \in T_l^{\text{norm}}$  (see Definition 2.2 (2)). Let  $\gamma = s_\zeta(-\xi^*)$  be the same expression as that of Definition 3.1, where  $\{\zeta, \xi\} = \{\alpha, \beta\}$ . We define elements  $E_\gamma$  and  $E_{-\gamma}$  of  $\mathfrak{g}$  by

$$E_{\pm\gamma} = (\pm 1)^{p(\xi^*)} n_\zeta(E_{\mp\xi^*}).$$

Notice that  $\mathbb{Z}R_{\alpha, \beta} \cap V^0 = \mathbb{Z}k(\xi)a$  and  $\gamma = \alpha_0(R_{\alpha, \beta}, \{\alpha, \beta\}, -k(\xi)a)$  (see Lemma 1.2 (1)).

We write  $X \sim Y$  to mean that  $X = zY$  for some  $z \in \mathbb{C} \setminus \{0\}$ .

**Lemma 3.4.** *We have*

$$(3.5) \quad \begin{cases} (\text{ad}E_{\pm\mu})^{1-(\mu^\vee, \nu)} E_{\pm\nu} = 0 & \text{if } \mu, \nu \in \Pi' \text{ with } \mu \neq \nu, \\ [E_\mu, E_{-\nu}] = \delta_{\mu\nu} h_{\mu^\vee} & \text{if } \mu, \nu \in \Pi'. \end{cases}$$

**Proof.** If  $R_{\alpha, \beta}$  is  $A_2^{(1)}$ ,  $C_2^{(1)}$  or  $D_3^{(2)}$ , the equalities (3.5) can be proved by the same calculations as those in [15, §2.3]. If  $R_{\alpha, \beta}$  is  $A^{(2)}(0, 3)$  (respectively,  $C^{(2)}(3)$  or  $A^{(4)}(0, 4)$ ), the equalities (3.5) can be proved by calculations similar to those for  $C_2^{(1)}$  (respectively,  $D_3^{(2)}$ ).

Assume that  $R_{\alpha, \beta}$  is  $A_4^{(2)}$  or  $B^{(1)}(0, 2)$ . By (3.1), we have  $[E_\gamma, E_{-\gamma}] = h_{s_\beta(-\alpha^*)^\vee} = h_{\gamma^\vee}$ . By (3.2) and (SR7), we have

$$E_\gamma = n_\beta(E_{-\alpha^*}) \sim [E_{-\beta}, [E_{-\beta}, E_{-\alpha^*}]] \sim [E_{-\beta}, [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]].$$

Hence, by (SR5),  $[E_\gamma, E_{-\beta}] \sim [E_{-\beta}, [E_{-\beta}, [E_{-\beta}, E_{-\alpha^*}]]] = 0$ . By (3.2) and (SR5,7), since  $n_\beta(E_{\pm\beta}) \sim E_{\mp\beta}$ , we have

$$\begin{aligned}
[E_\gamma, E_{-\alpha}] &\sim [[E_{-\beta}, [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]], E_{-\alpha}] \\
&\sim [[E_{-\beta}, E_{-\alpha}], [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]] \\
&\sim [[E_{-\beta}, E_{-\alpha}], [E_{-\beta}, E_{-\alpha^*}]] \\
&\sim n_\beta([E_{-\alpha}, [E_\beta, [E_{-\beta}, [E_{-\beta}, E_{-\alpha^*}]]]]) \\
&\sim n_\beta([E_{-\alpha}, [E_{-\beta}, E_{-\alpha^*}]]) \\
&\sim n_\beta([E_{-\alpha}, [E_{-\alpha}, [E_{-\alpha}, E_{-\beta^*}]]]) \\
&= 0.
\end{aligned}$$

Similarly we have  $[E_{-\gamma}, E_\beta] = 0$  and  $[E_{-\gamma}, E_\alpha] = 0$ . Since the  $E_\lambda$  ( $\lambda \in \Pi' \cup -\Pi'$ ) are locally nilpotent, we have all the equalities in (3.5).  $\square$

**Lemma 3.5.** *Let  $\mathfrak{g}'$  be the sub-Lie superalgebra of  $\mathfrak{g}$  generated by  $\mathfrak{h}$  and  $E_\mu, E_{-\mu}$  ( $\mu \in \Pi'$ ). Then  $\mathfrak{g}' = \mathfrak{g}^{\{\alpha, \beta\}}$ .*

**Proof.** It is clear that  $\mathfrak{g}' \subset \mathfrak{g}^{\{\alpha, \beta\}}$ . We show  $\mathfrak{g}^{\{\alpha, \beta\}} \subset \mathfrak{g}'$ . Let  $\zeta$  and  $\xi$  be as in Definition 3.2. Since  $E_{\mp\xi^*} = (\pm 1)^{p(\xi^*)} n_\zeta^{-1}(E_{\pm\gamma}) \in \mathfrak{g}'$ , it suffices to prove that  $E_{\pm\zeta^*} \in \mathfrak{g}'$ . By (SR6,7) and (3.2), we have  $n_\xi(E_{\pm\zeta^*}) = n_{\xi^*}(E_{\pm\zeta})$ . Hence  $E_{\pm\zeta^*} = n_\xi^{-1} n_{\xi^*}(E_{\pm\zeta}) \in \mathfrak{g}'$ . This completes the proof.  $\square$

**Lemma 3.6.** (1) *We have*

$$\mathfrak{g}^{\{\alpha, \beta\}} = \mathfrak{h} \bigoplus \left( \bigoplus_{\mu \in R_{\alpha, \beta}} \mathfrak{g}_\mu^{\{\alpha, \beta\}} \right) \bigoplus \left( \bigoplus_{m \in \mathbb{Z} \setminus \{0\}} \mathfrak{g}_{mk(\alpha)a}^{\{\alpha, \beta\}} \right)$$

and  $\dim \mathfrak{g}_\mu^{\{\alpha, \beta\}} = 1$  ( $\mu \in R_{\alpha, \beta}$ ).

(2) *We have*

$$\mathfrak{g}_\mu^{\{\alpha, \beta\}} = \mathfrak{g}_\mu^{\{\alpha\}} \quad (\mu \in R_\alpha) \quad \text{and} \quad \mathfrak{g}_\nu^{\{\alpha, \beta\}} = \mathfrak{g}_\nu^{\{\beta\}} \quad (\nu \in R_\beta)$$

and

$$\mathfrak{g}_{mk(\alpha)a}^{\{\alpha, \beta\}} = \begin{cases} \mathfrak{g}_{mk(\alpha)a}^{\{\alpha\}} + \mathfrak{g}_{mk(\alpha)a}^{\{\beta\}} & \text{if } mk(\alpha) \in \mathbb{Z}k(\beta), \\ \mathfrak{g}_{mk(\alpha)a}^{\{\alpha\}} & \text{if } mk(\alpha) \notin \mathbb{Z}k(\beta) \end{cases}$$

for  $m \in \mathbb{Z} \setminus \{0\}$ . (See Lemma 3.2 for  $R_\alpha$  and  $R_\beta$ .)

**Proof.** The statement (1) follows from Lemmas 3.1, 3.3, 3.4, 3.5 and the well-known argument in the proof of [6, Corollary 5.12] (see also [7, Proposition 1.6]).

We prove (2). Recall Lemma 3.2. Notice the following.

- (i)  $\mathfrak{g}^{\{\alpha, \beta\}}$  is generated by  $\mathfrak{h}$  and  $\mathfrak{g}_\lambda^{\{\mu\}}$  ( $\mu \in \{\alpha, \beta\}$  and  $\lambda \in R_\mu$ ).
- (ii)  $R_\alpha = R_{\alpha, \beta} \cap (\mathbb{R}\alpha \oplus \mathbb{R}a)$  and  $R_\beta = R_{\alpha, \beta} \cap (\mathbb{R}\beta \oplus \mathbb{R}a)$ .
- (iii) For  $\mu \in \{\alpha, \beta\}$  and  $\lambda \in R_\mu$ , we have  $\mathfrak{g}_\lambda^{\{\alpha, \beta\}} = \mathfrak{g}_\lambda^{\{\mu\}}$  (since  $\dim \mathfrak{g}_\lambda^{\{\alpha, \beta\}} = \dim \mathfrak{g}_\lambda^{\{\mu\}} = 1$ ).

By (1) and the fact that  $k(\beta) \in \mathbb{N}k(\alpha)$ , the statement (2) holds.  $\square$

### 3.3 Proof of Theorem 1.1 (1) and Theorem 2.2

Let  $S$  be a subset of  $\Pi$ . Let  $m_S = \min\{k(\alpha) | \alpha \in S\}$ . Define sub-Lie superalgebras  $\mathfrak{n}^{S,+}$ ,  $\mathfrak{n}^{S,-}$ ,  $\mathfrak{l}^{S,+}$  and  $\mathfrak{l}^{S,-}$  of  $\mathfrak{g}^S$  by

$$\mathfrak{n}^{S,\pm} = \bigoplus_{\mu \in (\mathbb{Z}_\pm S \oplus \mathbb{Z}m_S a) \setminus \mathbb{Z}m_S a} \mathfrak{g}_\mu^S$$

and

$$\mathfrak{l}^{S,\pm} = \bigoplus_{r \in \mathbb{Z}_\pm \setminus \{0\}} \mathfrak{g}_{rm_S a}^S.$$

Let  $\mathfrak{n}^\pm = \mathfrak{n}^{\Pi,\pm}$  and  $\mathfrak{l}^\pm = \mathfrak{l}^{\Pi,\pm}$ .

**Lemma 3.7.** (1)  $\mathfrak{g} = \mathfrak{n}^+ \oplus \mathfrak{l}^+ \oplus \mathfrak{h} \oplus \mathfrak{l}^- \oplus \mathfrak{n}^-$ .

(2)  $\mathfrak{n}^\pm$  is generated by  $\mathfrak{n}^{\{\alpha\},\pm}$  ( $\alpha \in \Pi$ ). Namely, letting  $\mathfrak{n}^{(1),\pm} = \bigoplus_{\alpha \in \Pi} \mathfrak{n}^{\{\alpha\},\pm}$  and  $\mathfrak{n}^{(i),\pm} = [\mathfrak{n}^{(1),\pm}, \mathfrak{n}^{(i-1),\pm}]$  ( $i \geq 2$ ), we have  $\mathfrak{n}^\pm = \sum_{i=1}^{\infty} \mathfrak{n}^{(i),\pm}$ .

(3)  $\mathfrak{l}^\pm = \sum_{\alpha \in \Pi} \mathfrak{l}^{\{\alpha\},\pm}$ .

**Proof.** We first notice that  $\mathfrak{g}$  is generated by  $\mathfrak{h}$  (see also Lemma 3.1) and  $\mathfrak{n}^{\{\alpha\},+}$ ,  $\mathfrak{n}^{\{\alpha\},-}$  ( $\alpha \in \Pi$ ) since  $E_{\pm\alpha}$ ,  $E_{\pm\alpha^*} \in \mathfrak{n}^{\{\alpha\},\pm}$ . Since  $\{w(\alpha) \in V | \alpha \in \Pi, w \in W_\Pi\} \subset \mathbb{Z}_+\Pi \cup \mathbb{Z}_-\Pi$ , we have

$$(3.6) \quad R \subset (\mathbb{Z}_+\Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi \oplus \mathbb{Z}a).$$

Let  $\alpha \in \Pi$ . By Lemma 3.2, we see that

$$\begin{cases} [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{n}^{\{\alpha\},\pm}] \subset \mathfrak{n}^{\{\alpha\},\pm}, \\ [\mathfrak{n}^{\{\alpha\},+}, \mathfrak{n}^{\{\alpha\},-}] \subset \mathfrak{n}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{h} \oplus \mathfrak{l}^{\{\alpha\},-} \oplus \mathfrak{n}^{\{\alpha\},-} (= \mathfrak{g}^{\{\alpha\}}), \\ [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},-}] \subset \mathfrak{n}^{\{\alpha\},\pm} \end{cases}$$

(notice that  $[\mathfrak{g}_{\pm\alpha+r_1k(\alpha)a}^{\{\alpha\}}, \mathfrak{g}_{\mp 2\alpha+r_2k(\alpha)a}^{\{\alpha\}}] \subset \mathfrak{n}^{\{\alpha\},\mp}$  ( $r_1 \in \mathbb{Z}$  and  $r_2 \in g(\alpha)$ )).

Let  $\alpha, \beta \in \Pi$  be such that  $\alpha \neq \beta$ . We first assume that  $(\alpha, \beta) \neq 0$  and  $(\alpha, \alpha) \leq (\beta, \beta)$ . By (3.6), we have

$$(3.7) \quad (\mathbb{Z}_{\pm}\alpha \oplus \mathbb{Z}_{\mp}\beta \oplus \mathbb{Z}a) \cap R_{\alpha,\beta} = \emptyset,$$

where the definition of  $R_{\alpha,\beta}$  can be found in Definition 3.1. By Lemma 3.6 and (3.7), we see that

$$\begin{cases} [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{n}^{\{\beta\},\mp}] = \{0\}, \\ [\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{l}^{\{\beta\},+} \oplus \mathfrak{l}^{\{\beta\},-}] \subset \mathfrak{n}^{\{\alpha\},\pm}, \\ [\mathfrak{n}^{\{\beta\},\pm}, \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},-}] \subset \mathfrak{n}^{\{\beta\},\pm}. \end{cases}$$

Next, we assume  $(\alpha, \beta) = 0$ . By (SR5), we have  $[\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{n}^{\{\beta\},\mp}] = \{0\}$ ,  $[\mathfrak{n}^{\{\alpha\},\pm}, \mathfrak{l}^{\{\beta\},+} \oplus \mathfrak{l}^{\{\beta\},-}] = \{0\}$  and  $[\mathfrak{n}^{\{\beta\},\pm}, \mathfrak{l}^{\{\alpha\},+} \oplus \mathfrak{l}^{\{\alpha\},-}] = \{0\}$ .

Let  $\mathfrak{i}^{\pm} = \sum_{\alpha \in \Pi} \mathfrak{l}^{\{\alpha\},\pm}$ . By the above arguments, we have

$$[\mathfrak{n}^{(1),+}, \mathfrak{n}^{(1),-}] \subset \mathfrak{n}^{(1),+} \oplus \mathfrak{i}^+ \oplus \mathfrak{h} \oplus \mathfrak{i}^- \oplus \mathfrak{n}^{(1),-}.$$

Hence

$$[\mathfrak{n}^{(1),\mp}, \mathfrak{n}^{(i),\pm}] \subset (\mathfrak{n}^{(1),+} \oplus \mathfrak{i}^+ \oplus \mathfrak{h} \oplus \mathfrak{i}^- \oplus \mathfrak{n}^{(1),-}) + \sum_{j=2}^i \mathfrak{n}^{(j),\pm}.$$

Then we see that the statement of this lemma holds and that  $\mathfrak{i}^{\pm} = \mathfrak{l}^{\pm}$ .  $\square$

**Lemma 3.8.** *Let  $\alpha \in \Pi$ . Then we have  $\dim \mathfrak{g}_{\mu} = 1$  for  $\mu \in R_{\alpha}$  and we have  $\dim \mathfrak{g}_{\lambda} = 0$  for  $\lambda \in (\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus (R_{\alpha} \cup \mathbb{Z}a)$ . (See Lemma 3.2 for  $R_{\alpha}$ .)*

**Proof.** The statement follows immediately from Lemmas 3.2 and 3.7 (especially Lemma 3.7 (2)).  $\square$

The following lemma includes Theorem 1.1 (1) (see Definition 2.2 (2)).

**Lemma 3.9.** *Let  $l \geq 2$  and  $(\Pi, k, g) \in T_l^{\text{norm}}$ . Then  $\mathfrak{g} = \mathfrak{g}(\Pi, k, g)$  has the same root space decomposition as in Theorem 1.1 (1).*

**Proof.** By Theorem 1.2 (2) and Lemma 1.4 (1), we see that  $\pi_a(R)$  is a rank- $l$  affine root system with a base  $\pi_a(\Pi)$ . Let  $\mu \in R$ . By Lemma 1.4 (1), there exist  $\alpha \in \Pi$  and  $w \in W_\Pi$  such that  $w(\mu) \in R_\alpha$ . By Lemma 3.8,  $\dim \mathfrak{g}_{w(\mu)} = 1$ . Express  $w$  as  $w = s_{\beta_1} \cdots s_{\beta_r}$  with some  $\beta_1, \dots, \beta_r \in \Pi$ . It follows from (3.1) that  $\dim \mathfrak{g}_\mu = \dim n_{\beta_1} \cdots n_{\beta_r}(\mathfrak{g}_\mu) = \dim \mathfrak{g}_{s_{\beta_1} \cdots s_{\beta_r}(\mu)} = \dim \mathfrak{g}_{w(\mu)} = 1$ . Let  $\mu \in \mathbb{Z}R \setminus (R \cup (\mathbb{Z}\delta \oplus \mathbb{Z}a))$ . By Lemma 1.4 (1), there exists  $w \in W$  such that

$$w(\mu) \in \left( \bigcup_{\alpha \in \Pi} ((\mathbb{Z}\alpha \oplus \mathbb{Z}a) \setminus R_\alpha) \right) \cup ((\mathbb{Z}\Pi \oplus \mathbb{Z}a) \setminus ((\mathbb{Z}_+\Pi \oplus \mathbb{Z}a) \cup (\mathbb{Z}_-\Pi \oplus \mathbb{Z}a))).$$

Using  $n_\alpha$ 's again, by Lemmas 3.7 and 3.8, we have  $\dim \mathfrak{g}_\mu = 0$ . By these facts and Lemmas 3.1 and 3.7, the statement holds.  $\square$

**Proof of Theorem 1.1 (1).** This follows immediately from Lemma 3.9 and Theorem 2.1.  $\square$

**Proof of Theorem 2.2.** If  $(\Pi, k, g) \in T_l^{\text{rm}}$ , the theorem follows from [10, (6.4)-(6.6)]. Assume  $(\Pi, k, g) \in T_l^{\text{norm}}$ . By Lemma 3.9,  $R = R(\Pi, k, g)$  is the set of the non-isotropic roots of  $\mathfrak{g}(\Pi, k, g)$ . Let  $W$  be the subgroup of  $\text{GL}(V)$  generated by  $\{s_\mu | \mu \in R\}$ . By (2.6) and Lemma 3.2 (2),  $W$  is generated by  $\{s_\mu | \mu \in \mathcal{B}_+\}$ . Using  $n_\mu$ 's, by (3.1), we see that  $R$  satisfies (AX4). By Theorem 1.2 (2) and Lemma 1.4 (1),  $\pi_a(R)$  is an affine root system. Hence  $R$  satisfies (AX3) and (AX5). It is clear that  $R$  satisfies (AX1) and (AX2). It is also clear that  $(, )$  is positive semi-definite and  $\dim V^0 = 2$ , where  $V^0 = \{v \in V | (v, v) = 0\}$ .

By Lemma 3.2 (1), we see that  $\Pi \cup \{a\}$  is a fundamental-set. This completes the proof.  $\square$

## 4 Proof of Lemma 3.1 and Theorem 1.1 (2), (3)

One of the purposes of this section is to show Lemma 3.1.

## 4.1 Contragredient Lie superalgebras

We first recall the definition of the contragredient Lie superalgebras [7]. Let  $\bar{I}$  be a finite set. Let  $\bar{I}^{\text{odd}}$  be a subset of  $\bar{I}$ . Define a map  $\bar{p} : \bar{I} \rightarrow \{0, 1\}$  by  $\bar{p}(i) = 1$  ( $i \in \bar{I}^{\text{odd}}$ ) and  $\bar{p}(j) = 0$  ( $j \in \bar{I} \setminus \bar{I}^{\text{odd}}$ ). Let  $\bar{A} = (\bar{a}_{ij})_{i, j \in \bar{I}}$  be an  $|\bar{I}| \times |\bar{I}|$  matrix such that there exist  $\bar{e}_i \in \mathbb{C} \setminus \{0\}$  ( $i \in \bar{I}$ ) satisfying the condition that  $\bar{D}^{-1}\bar{A}$  is a symmetric matrix, where  $\bar{D}$  is the diagonal matrix  $\text{diag}(\bar{e}_1, \dots, \bar{e}_{|\bar{I}|})$ . Such  $\bar{A}$  is called a symmetrizable matrix. For the triple  $(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  of these  $\bar{A}$ ,  $\bar{I}$  and  $\bar{I}^{\text{odd}}$ , we define a  $\mathbb{C}$ -Lie superalgebra  $\bar{\mathfrak{G}} = \bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  in the following way. We define a  $\mathbb{C}$ -Lie superalgebra  $\bar{\mathfrak{G}}' = \bar{\mathfrak{G}}'(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  by generators

$$\bar{h}'_i, \bar{t}'_i, \bar{E}'_i, \bar{F}'_i \ (i \in \bar{I})$$

with parities

$$p(\bar{h}'_i) = p(\bar{t}'_i) = 0, \quad p(\bar{E}'_i) = p(\bar{F}'_i) = \bar{p}(i)$$

and defining relations

$$\begin{cases} [\bar{h}'_i, \bar{h}'_j] = [\bar{h}'_i, \bar{t}'_j] = [\bar{t}'_i, \bar{t}'_j] = 0, \\ [\bar{h}'_i, \bar{E}'_j] = \bar{a}_{ij}\bar{E}'_j, [\bar{t}'_i, \bar{E}'_j] = \delta_{ij}\bar{E}'_j, \\ [\bar{h}'_i, \bar{F}'_j] = -\bar{a}_{ij}\bar{F}'_j, [\bar{t}'_i, \bar{F}'_j] = -\delta_{ij}\bar{F}'_j, \\ [\bar{E}'_i, \bar{F}'_j] = \delta_{ij}\bar{h}'_i. \end{cases}$$

Let  $\bar{\mathfrak{H}}'$  be the sub-Lie superalgebra of  $\bar{\mathfrak{G}}'$  generated by  $\bar{h}'_i, \bar{t}'_i$  ( $i \in \bar{I}$ ). By a well-known argument (see [6, §1.3]), we see that  $\{\bar{h}'_i, \bar{t}'_i | i \in \bar{I}\}$  is a basis of  $\bar{\mathfrak{H}}'$ , i.e.,  $\dim \bar{\mathfrak{H}}' = 2|\bar{I}|$ . Let  $\mathfrak{r}$  be the ideal of  $\bar{\mathfrak{G}}'$  maximal among the ideals  $\mathfrak{r}'$  such that  $\mathfrak{r}' \cap \bar{\mathfrak{H}}' = \{0\}$ . Then we define  $\bar{\mathfrak{G}} = \bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  as the quotient Lie superalgebra  $\bar{\mathfrak{G}}'/\mathfrak{r}$ . The Lie superalgebra  $\bar{\mathfrak{G}}$  is a *contragredient Lie superalgebra* (see [7]). Let  $\bar{\pi} : \bar{\mathfrak{G}}' \rightarrow \bar{\mathfrak{G}}$  be the canonical map. We denote  $\bar{\pi}(\bar{\mathfrak{H}}')$ ,  $\bar{\pi}(\bar{h}'_i)$ ,  $\bar{\pi}(\bar{t}'_i)$ ,  $\bar{\pi}(\bar{E}'_i)$  and  $\bar{\pi}(\bar{F}'_i)$  by  $\bar{\mathfrak{H}}$ ,  $\bar{h}_i$ ,  $\bar{t}_i$ ,  $\bar{E}_i$  and  $\bar{F}_i$  respectively.

## 4.2 Properties of contragredient Lie superalgebras

By a well-known argument (see [7]), we have the following.

**Lemma 4.1.**  $\dim \bar{\mathfrak{H}} = 2|\bar{I}|$ , i.e.,  $\{\bar{h}_i, \bar{t}_i | i \in \bar{I}\}$  is a basis of  $\bar{\mathfrak{H}}$ .

Let  $\bar{\mathfrak{E}} = \bar{\mathfrak{H}}^*$  (the dual space of  $\bar{\mathfrak{H}}$ ). Define a basis  $\{\bar{\alpha}_i, \bar{\gamma}_i | i \in \bar{I}\}$  of  $\bar{\mathfrak{E}}$  by  $\bar{\alpha}_i(\bar{h}_j) = \bar{a}_{ji}$ ,  $\bar{\alpha}_i(\bar{t}_j) = \bar{\gamma}_i(\bar{h}_j) = \delta_{ij}$  and  $\bar{\gamma}_i(\bar{t}_j) = 0$ . Define a non-degenerate symmetric bilinear form  $(, ) : \bar{\mathfrak{E}} \times \bar{\mathfrak{E}} \rightarrow \mathbb{C}$  by  $(\bar{\alpha}_i, \bar{\alpha}_j) = \bar{a}_{ij}/\bar{\epsilon}_i$ ,  $(\bar{\gamma}_i, \bar{\alpha}_j) = \delta_{ij}/\bar{\epsilon}_i$  and  $(\bar{\gamma}_i, \bar{\gamma}_j) = 0$ . For  $\bar{\sigma} \in \bar{\mathfrak{E}}$ , denote by  $h_{\langle \bar{\sigma} \rangle}$  the element of  $\bar{\mathfrak{H}}$  satisfying the property that  $\bar{\tau}(h_{\langle \bar{\sigma} \rangle}) = (\bar{\tau}, \bar{\sigma})$  ( $\bar{\tau} \in \bar{\mathfrak{H}}$ ). Then we have  $\bar{h}_i = \bar{\epsilon}_i h_{\langle \bar{\alpha}_i \rangle}$  and  $\bar{t}_i = \bar{\epsilon}_i h_{\langle \bar{\gamma}_i \rangle}$ .

The following is also proved by a well-known argument (see [6, Theorem 2.2]).

**Lemma 4.2.** *There exists a super-symmetric invariant form  $(, ) : \bar{\mathfrak{G}} \times \bar{\mathfrak{G}} \rightarrow \mathbb{C}$  such that  $(h_{\langle \bar{\sigma} \rangle}, h_{\langle \bar{\tau} \rangle}) = (\bar{\sigma}, \bar{\tau})$  and  $(\bar{E}_i, \bar{F}_j) = \delta_{ij}\bar{\epsilon}_i$ .*

By the same argument as in the proof of [14, Theorem 4.1.1], we have the following.

**Lemma 4.3.** *We have the following.*

- (1)  $(\text{ad } \bar{E}_i)^{1-\bar{a}_{ij}} \bar{E}_j = 0$  if  $\bar{p}(i) = 0$ ,  $\bar{a}_{ii} = 2$  and  $\bar{a}_{ij} \in \mathbb{Z}_-$  or if  $\bar{p}(i) = 1$ ,  $\bar{a}_{ii} = 2$  and  $\bar{a}_{ij} \in 2\mathbb{Z}_-$ .
- (2)  $[\bar{E}_i, \bar{E}_j] = 0$  if  $\bar{a}_{ij} = \bar{a}_{ji} = 0$ .
- (3)  $[\bar{E}_j, [[\bar{E}_i, \bar{E}_j], \bar{E}_r]] = 0$  if  $i \neq j \neq r \neq i$ ,  $\bar{p}(j) = 1$ ,  $\bar{a}_{jj} = \bar{a}_{ir} = \bar{a}_{ri} = 0$  and  $\bar{a}_{ji} + \bar{a}_{jr} = 0$ .
- (4)  $[[\bar{E}_i, \bar{E}_j], \bar{E}_r] = [[\bar{E}_i, \bar{E}_r], \bar{E}_j]$  if  $i \neq j \neq r \neq i$ ,  $\bar{p}(i) = 0$ ,  $\bar{p}(j) = \bar{p}(r) = 1$ ,  $\bar{a}_{ij} = \bar{a}_{ji} = \bar{a}_{ir} = \bar{a}_{ri} = -1$  and  $\bar{a}_{jr} = \bar{a}_{rj} = 2$ .
- (5) *The same formulas as those in (1)-(4) with  $\bar{F}_i$ 's in place of  $\bar{E}_i$ 's hold.*

Let  $\mathbb{C}[t, t^{-1}]$  be the Laurent polynomial algebra. Let

$$\mathfrak{L} = \mathfrak{L}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}}) = \bar{\mathfrak{G}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}v \oplus \mathbb{C}w.$$

We view  $\mathfrak{L}$  as a Lie superalgebra in the following way. The parity of  $X \otimes t^n$  is the same as the one of  $X$  for a homogeneous element  $X$  of  $\bar{\mathfrak{G}}$ ; the parities of  $v$  and  $w$  are 0. The Lie super-bracket of  $\mathfrak{L}$  is given by

$$\begin{aligned} & [X \otimes t^m + a_1v + b_1w, Y \otimes t^n + a_2v + b_2w] \\ &= [X, Y] \otimes t^{m+n} + m\delta_{m+n,0}(X, Y)v + b_1nY \otimes t^n - b_2mX \otimes t^m \end{aligned}$$

for homogeneous elements  $X, Y$  of  $\bar{\mathfrak{G}}$ . We shall also denote by  $(, )$  the super-symmetric invariant form on  $\mathfrak{L}$  defined by

$$(4.1) \quad (X \otimes t^m + a_1v + b_1w, Y \otimes t^n + a_2v + b_2w) = \delta_{m+n,0}(X, Y) + a_1b_2 + b_1a_2.$$

### 4.3 Notation $T_l^{\text{norm}}(A)$ and numbering of $\alpha_0, \dots, \alpha_l$

Let  $l \geq 2$ . If  $A$  is an  $(l+1) \times (l+1)$  generalized Cartan matrix of affine-type, let  $T_l^{\text{norm}}(A)$  be the subset of  $T_l^{\text{norm}}$  formed by the elements  $(\Pi, k, g) \in T_l^{\text{norm}}$  such that  $A(\Pi) = A$ . Then we have

$$T_l^{\text{norm}} = \begin{cases} T_l^{\text{norm}}(B_l^{(1)}) \cup T_l^{\text{norm}}(A_{2l}^{(2)}) \cup T_l^{\text{norm}}(D_{l+1}^{(2)}) & \text{if } l \geq 3, \\ T_2^{\text{norm}}(C_2^{(1)}) \cup T_2^{\text{norm}}(A_4^{(2)}) \cup T_2^{\text{norm}}(D_3^{(2)}) & \text{if } l = 2. \end{cases}$$

(See Remark 2.1 (3)).

Let  $(\Pi = \{\alpha_0, \dots, \alpha_l\}, k, g) \in T_l^{\text{norm}}$  with  $l \geq 2$ . From now on, we assume that  $(\alpha_i, \alpha_i) \leq (\alpha_{i+1}, \alpha_{i+1})$  and  $(\alpha_i, \alpha_{i+1}) \neq 0$  for  $1 \leq i \leq l-1$  and assume that  $(\alpha_0, \alpha_0) \geq (\alpha_1, \alpha_1)$ . We define a total order on  $\mathcal{P}$  by  $4\mathbb{Z} < 2\mathbb{Z} < \mathbb{Z} < 2\mathbb{Z} + 1 < \emptyset$ . We also assume that if  $(\alpha_i, \alpha_i) = (\alpha_j, \alpha_j)$  and  $i < j$ , then  $g(\alpha_i) > g(\alpha_j)$  or  $g(\alpha_i) = g(\alpha_j)$  and  $k(\alpha_i) \geq k(\alpha_j)$ .

### 4.4 Unfolding for $D_{l+1}^{(2)}$

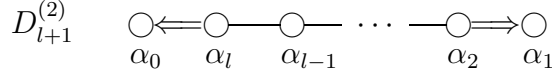


Fig. 4.1

In this subsection, we always let  $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$  with  $l \geq 2$ . Then  $\Pi_{\text{sh}} = \{\alpha_0, \alpha_1\}$ ,  $\Pi_{\text{lg}} = \{\alpha_i \mid 2 \leq i \leq l\}$  and  $\Pi_{\text{ex}} = \emptyset$ . Moreover, we see that  $k(\Pi)$  is  $\{1\}$  or  $\{1, 2\}$  (see (2.5)). Recall that  $g(\beta) = \emptyset$  for  $\beta \in \Pi_{\text{lg}}$  and that  $k(\beta_1) = k(\beta_2)$  for  $\beta_1, \beta_2 \in \Pi_{\text{lg}}$ . Moreover, we recall that for  $\alpha \in \Pi_{\text{sh}}$  and  $\beta \in \Pi_{\text{lg}}$ , it follows that  $(k(\alpha), g(\alpha), k(\beta))$  is  $(1, \emptyset, 1)$ ,  $(1, \emptyset, 2)$ ,  $(2, \emptyset, 2)$ ,  $(1, 2\mathbb{Z} + 1, 1)$ ,  $(2, 2\mathbb{Z} + 1, 2)$ ,  $(1, \mathbb{Z}, 1)$ ,  $(2, \mathbb{Z}, 2)$ ,  $(1, 2\mathbb{Z}, 1)$ ,  $(1, 2\mathbb{Z}, 2)$ ,  $(2, 2\mathbb{Z}, 2)$  or  $(1, 4\mathbb{Z}, 2)$  (see also Table 4.1). Let  $\beta$  denote an element of  $\Pi_{\text{lg}}$  and define a map  $\vartheta : \Pi_{\text{sh}} \rightarrow \{1, 2, 3\}$  by

$$\vartheta(\alpha) = \begin{cases} 1 & \text{if } g(\alpha) = \emptyset \text{ and } k(\alpha) = k(\beta), \\ 1 & \text{if } g(\alpha) = \mathbb{Z} \text{ (this implies } k(\alpha) = k(\beta)), \\ 2 & \text{if } g(\alpha) = \emptyset \text{ and } 2k(\alpha) = k(\beta)(= 2), \\ 2 & \text{if } g(\alpha) = 2\mathbb{Z} \text{ and } 2k(\alpha) = k(\beta)(= 2), \\ 3 & \text{if } g(\alpha) = 2\mathbb{Z} \text{ and } k(\alpha) = k(\beta), \\ 3 & \text{if } g(\alpha) = 2\mathbb{Z} + 1 \text{ (this implies } k(\alpha) = k(\beta)), \\ 3 & \text{if } g(\alpha) = 4\mathbb{Z} \text{ (this implies } 2k(\alpha) = k(\beta)(= 2)). \end{cases}$$

We have given  $\vartheta$  in order to define  $k^\vee$  below.

We define a map  $k^\vee : \Pi \rightarrow \{1, 2, 3, 4\}$  as follows.

(i) If  $\vartheta(\alpha_0) = 3$  or  $\vartheta(\alpha_1) = 3$  (i.e.,  $3 \in \vartheta(\Pi_{\text{sh}})$ ), let

$$(4.2) \quad k^\vee(\alpha') = \begin{cases} 2 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 1, \\ 4 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 2, \\ 2 & \text{if } \alpha' \in \Pi_{\text{sh}}, \vartheta(\alpha') = 3 \text{ and } g(\alpha') \in \{2\mathbb{Z}, 2\mathbb{Z} + 1\}, \\ 3 & \text{if } \alpha' \in \Pi_{\text{sh}}, \vartheta(\alpha') = 3 \text{ and } g(\alpha') = 4\mathbb{Z}, \\ 2 & \text{if } \alpha' \in \Pi_{\text{lg}}. \end{cases}$$

(ii) If  $1 \leq \vartheta(\alpha_0), \vartheta(\alpha_1) \leq 2$  (i.e.,  $3 \notin \vartheta(\Pi_{\text{sh}})$ ), let

$$k^\vee(\alpha') = \begin{cases} 1 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 1, \\ 2 & \text{if } \alpha' \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha') = 2, \\ 1 & \text{if } \alpha' \in \Pi_{\text{lg}}. \end{cases}$$

*Remark 4.1.* The situation  $4 \in k^\vee(\Pi)$  occurs only when  $\vartheta(\Pi_{\text{sh}}) = \{2, 3\}$  (see (4.2)). More precisely, for  $\alpha \in \Pi$ ,  $k^\vee(\alpha) = 4$  holds if and only if  $\alpha \in \Pi_{\text{sh}}$ ,  $\vartheta(\alpha) = 2$  and  $\vartheta(\alpha') = 3$  for  $\alpha' \in \Pi_{\text{sh}} \setminus \{\alpha\}$ . See also Table 4.1 and Fig. 4.10 of Remark 4.2 and (v-1,2) of Example 4.1.

Define a subset  $\widehat{I} = \widehat{I}(\Pi, k, g)$  of  $\Pi \times \{1, 2, 3, 4\}$  by

$$\widehat{I} = \widehat{I}(\Pi, k, g) = \{(\alpha, x) \in \Pi \times \{1, 2, 3, 4\} \mid 1 \leq x \leq k^\vee(\alpha)\}.$$

We define an  $|\widehat{I}| \times |\widehat{I}|$  matrix  $\widehat{A} = \widehat{A}(\Pi, k, g) = (\widehat{a}_{(\alpha,x),(\beta,y)})_{(\alpha,x),(\beta,y) \in \widehat{I}}$  in the following way (see also Remark 4.2 and Example 4.1). For  $\alpha, \beta \in \Pi$ , we let  $\widehat{A}_{\alpha,\beta}$  denote the  $k^\vee(\alpha) \times k^\vee(\beta)$  submatrix  $(\widehat{a}_{(\alpha,x),(\beta,y)})_{1 \leq x \leq k^\vee(\alpha), 1 \leq y \leq k^\vee(\beta)}$  of  $\widehat{A}$ . For  $r \in \mathbb{N}$ , let  $e(r) = (e(r)_{ij})_{1 \leq i,j \leq r}$  denote the  $r \times r$  identity matrix, i.e.,  $e(r)_{ii} = 1$  and  $e(r)_{ij} = 0$  ( $i \neq j$ ).

(i) Let  $\alpha \in \Pi (= \Pi_{\text{sh}} \cup \Pi_{\text{lg}})$ . Then we define

$$\widehat{A}_{\alpha,\alpha} = \begin{cases} 2e(k^\vee(\alpha)) & \text{if } \alpha \in \Pi_{\text{lg}}, \\ 2e(k^\vee(\alpha)) & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) \in \{\emptyset, \mathbb{Z}\}, \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z} \text{ and } \vartheta(\alpha) = 3, \\ \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2, \\ \begin{pmatrix} 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}}, g(\alpha) = 2\mathbb{Z}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4, \\ \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) = 2\mathbb{Z} + 1, \\ \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & -1 \\ 1 & -1 & 0 \end{pmatrix} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } g(\alpha) = 4\mathbb{Z}. \end{cases}$$

(ii) Assume  $\alpha, \beta \in \Pi$  to be such that  $\alpha \neq \beta$  and  $(\alpha, \beta) = 0$ . Then we define both  $\widehat{A}_{\alpha,\beta}$  and  $\widehat{A}_{\beta,\alpha}$  to be the zero matrices.

(iii) Assume  $\beta_1, \beta_2 \in \Pi_{\text{lg}}$  to be such that  $\beta_1 \neq \beta_2$  and  $(\beta_1, \beta_2) \neq 0$ . Notice  $k^\vee(\beta_1) = k^\vee(\beta_2)$ . Then we let  $\widehat{A}_{\beta_1,\beta_2} = \widehat{A}_{\beta_2,\beta_1} = -e(k^\vee(\beta_1))$ .

(iv) Assume  $\alpha \in \Pi_{\text{sh}}$  and  $\beta \in \Pi_{\text{lg}}$  to be such that  $(\alpha, \beta) \neq 0$ . Then  $\widehat{A}_{\alpha,\beta}$  and  $\widehat{A}_{\beta,\alpha}$  are defined by the following.

(iv-i) Assume  $\vartheta(\alpha) = 1$ . Notice  $k^\vee(\alpha) = k^\vee(\beta) \in \{1, 2\}$ . Then we let  $\widehat{A}_{\alpha,\beta} = -2e(k^\vee(\alpha))$  and  $\widehat{A}_{\beta,\alpha} = -e(k^\vee(\alpha))$ .

(iv-ii) Assume  $\vartheta(\alpha) = 2$ . Notice that  $(k^\vee(\alpha), k^\vee(\beta))$  is  $(2, 1)$  or  $(4, 2)$ . We let

$$\widehat{A}_{\beta,\alpha} = \begin{cases} \begin{pmatrix} -1 & -1 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (2, 1), \\ \begin{pmatrix} -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (4, 2), \end{cases}$$

and we let  $\widehat{A}_{\alpha,\beta} = {}^t(\widehat{A}_{\beta,\alpha})$  (the transpose matrix of  $\widehat{A}_{\beta,\alpha}$ ).

(iv-iii) Assume  $\vartheta(\alpha) = 3$ . Notice  $k^\vee(\alpha) \in \{2, 3\}$  and  $k^\vee(\beta) = 2$ . Then we let

$$\widehat{A}_{\beta,\alpha} = \begin{cases} -e(2) & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (2, 2), \\ \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} & \text{if } (k^\vee(\alpha), k^\vee(\beta)) = (3, 2), \end{cases}$$

and  $\widehat{A}_{\alpha,\beta} = {}^t(\widehat{A}_{\beta,\alpha})$ .

**Lemma 4.4.** *The matrix  $\widehat{A}$  is symmetrizable.*

This can be checked directly.

We define a subset  $\widehat{I}^{\text{odd}}$  of  $\widehat{I}$  by

$$\widehat{I}^{\text{odd}} = \{(\alpha, x) \in \widehat{I} \mid \widehat{a}_{(\alpha,x),(\alpha,x)} = 0\} \cup \{(\alpha, x) \in \widehat{I} \mid g(\alpha) = \mathbb{Z}\}.$$

Now we define a Lie superalgebra  $\widehat{\mathfrak{L}} = \widehat{\mathfrak{L}}(\Pi, k, g)$  to be  $\mathfrak{L}(\widehat{A}, \widehat{I}, \widehat{I}^{\text{odd}})$ .

*Remark 4.2.* *Dynkin diagrams of  $\bar{\mathfrak{G}}(\widehat{A}(\Pi, k, g), \widehat{I}(\Pi, k, g), \widehat{I}^{\text{odd}})$ . Let  $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$  with  $l \geq 2$ . In Table 4.1, we give the list of  $k, g, \vartheta, k^\vee$  in the case where  $A(\Pi)$  is  $D_{l+1}^{(2)}$ . We let  $\beta$  denote an element of  $\Pi_{lg}$  and we notice  $\{\alpha_0, \alpha_1\} = \Pi_{\text{sh}}$ . Each number (r),  $i \leq r \leq vi$ , in the rightmost column of the table means the Dynkin diagram  $\widehat{\Gamma}$  which shall be described in (r) in Fig. 4.6-4.11 after the table is given; (r)\* means the one obtained from (r) by replacing  $\alpha_1, \alpha_i$  ( $2 \leq i \leq l$ ),  $\alpha_0$  by  $\alpha_0, \alpha_{l-i+2}, \alpha_1$  respectively.*

$k(\alpha_0)$	$g(\alpha_0)$	$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$	$\vartheta(\alpha_0)$	$\vartheta(\alpha_1)$	$k^\vee(\alpha_0)$	$k^\vee(\beta)$	$k^\vee(\alpha_1)$	$\widehat{\Gamma}$
1	$\emptyset$	1	1	$2\mathbb{Z} + 1$	1	3	2	2	2	(iv)*
1	$\emptyset$	2	2	$2\mathbb{Z} + 1$	2	3	4	2	2	(v)*
1	$\emptyset$	1	1	$\mathbb{Z}$	1	1	1	1	1	(i)
1	$\emptyset$	2	2	$\mathbb{Z}$	2	1	2	1	1	(ii)
1	$\emptyset$	1	1	$2\mathbb{Z}$	1	3	2	2	2	(iv)*
1	$\emptyset$	2	1	$2\mathbb{Z}$	2	2	2	1	2	(iii)
2	$\emptyset$	2	1	$2\mathbb{Z}$	1	2	1	1	2	(ii)*
1	$\emptyset$	2	2	$2\mathbb{Z}$	2	3	4	2	2	(v)*
1	$\emptyset$	2	1	$4\mathbb{Z}$	2	3	4	2	3	(v)*
2	$\emptyset$	2	1	$4\mathbb{Z}$	1	3	2	2	3	(iv)*
1	$2\mathbb{Z} + 1$	1	1	$2\mathbb{Z} + 1$	3	3	2	2	2	(vi)
1	$2\mathbb{Z} + 1$	1	1	$\mathbb{Z}$	3	1	2	2	2	(iv)
1	$2\mathbb{Z} + 1$	1	1	$2\mathbb{Z}$	3	3	2	2	2	(vi)
2	$2\mathbb{Z} + 1$	2	1	$2\mathbb{Z}$	3	2	2	2	4	(v)
2	$2\mathbb{Z} + 1$	2	1	$4\mathbb{Z}$	3	3	2	2	3	(vi)
1	$\mathbb{Z}$	1	1	$\mathbb{Z}$	1	1	1	1	1	(i)
1	$\mathbb{Z}$	1	1	$2\mathbb{Z}$	1	3	2	2	2	(iv)*
2	$\mathbb{Z}$	2	1	$2\mathbb{Z}$	1	2	1	1	2	(ii)*
2	$\mathbb{Z}$	2	1	$4\mathbb{Z}$	1	3	2	2	3	(iv)*
1	$2\mathbb{Z}$	1	1	$2\mathbb{Z}$	3	3	2	2	2	(vi)
1	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$	2	2	2	1	2	(iii)
2	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$	3	2	2	2	4	(v)
1	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$	2	3	4	2	3	(v)*
2	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$	3	3	2	2	3	(vi)
1	$4\mathbb{Z}$	2	1	$4\mathbb{Z}$	3	3	3	2	3	(vi)

Table 4.1

The Dynkin diagram  $\widehat{\Gamma}$  of  $\bar{\mathfrak{G}}(\widehat{A}(\Pi, k, g), \widehat{I}(\Pi, k, g), \widehat{I}^{\text{odd}})$  is given in the following way. In  $\widehat{\Gamma}$ , the symbol  $i, x$  means  $(\alpha_i, x) (\in \widehat{I}(\Pi, k, g))$ . We call the dot of  $\widehat{\Gamma}$  corresponding to  $(\alpha_i, x)$  the  $(\alpha_i, x)$ -th dot. As for the  $(\alpha_i, x)$ -th dot

of  $\widehat{\Gamma}$ , we mean

$$\begin{cases} (\alpha_i, x) \notin \widehat{I}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 2 & \text{for (1) of Fig. 4.2,} \\ (\alpha_i, x) \in \widehat{I}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 2 & \text{for (2) of Fig. 4.2,} \\ (\alpha_i, x) \in \widehat{I}^{\text{odd}} \text{ and } \widehat{a}_{(\alpha_i, x), (\alpha_i, x)} = 0 & \text{for (3) of Fig. 4.2.} \end{cases}$$

If  $\alpha_i \in \Pi_{\text{sh}}$  and  $\vartheta(\alpha_i) = 1$ , then

$$(4) \text{ of Fig. 4.2 means } \begin{cases} (1) \text{ of Fig. 4.2} & \text{if } g(\alpha_i) = \emptyset, \\ (2) \text{ of Fig. 4.2} & \text{if } g(\alpha_i) = \mathbb{Z}. \end{cases}$$

$$(1) \quad \bigcirc_{i, x} \quad (2) \quad \bullet_{i, x} \quad (3) \quad \otimes_{i, x} \quad (4) \quad \odot_{i, x}$$

Fig. 4.2

If there exist no lines between the  $(\alpha_i, x)$ -th dot and the  $(\alpha_j, y)$ -th dot, we mean that  $\widehat{a}_{(\alpha_i, x), (\alpha_j, y)} = \widehat{a}_{(\alpha_j, y), (\alpha_i, x)} = 0$ . Otherwise we mean

$$(\widehat{a}_{(\alpha_i, x), (\alpha_j, y)}, \widehat{a}_{(\alpha_j, y), (\alpha_i, x)}) = \begin{cases} (-1, -1) & \text{for (1) of Fig. 4.3,} \\ (-1, -2) & \text{for (2) of Fig. 4.3,} \\ (-1, -1) & \text{for (3) of Fig. 4.3,} \\ (1, 1) & \text{for (4) of Fig. 4.3,} \\ (2, 2) & \text{for (5) of Fig. 4.3.} \end{cases}$$

$$(1) \quad \bigcirc_{i, x} \text{---} \bigcirc_{j, y} \quad (2) \quad \bigcirc_{i, x} \text{====} \odot_{j, y} \quad (3) \quad \bigcirc_{i, x} \text{---} \otimes_{j, y} \quad (4) \quad \otimes_{i, x} \text{---} \otimes_{j, y} \quad (5) \quad \otimes_{i, x} \text{====} \otimes_{j, y}$$

Fig. 4.3

If  $\alpha_i \in \Pi_{\text{sh}}$  and  $\vartheta(\alpha_i) = 2$ , then

$$(1) \text{ of Fig. 4.4 means } \begin{cases} (2) \text{ of Fig. 4.4} & \text{if } g(\alpha_i) = \emptyset, \\ (3) \text{ of Fig. 4.4} & \text{if } g(\alpha_i) = 2\mathbb{Z}. \end{cases}$$

$$(1) \quad \times_{i, x} \text{====} \times_{i, y} \quad (2) \quad \bigcirc_{i, x} \quad \bigcirc_{i, y} \quad (3) \quad \otimes_{i, x} \text{====} \otimes_{i, y}$$

Fig. 4.4

If  $\alpha_i \in \Pi_{\text{sh}}$  and  $\vartheta(\alpha_i) = 3$ , then

$$(1) \text{ of Fig. 4.5 means } \begin{cases} (2) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 2\mathbb{Z}, \\ (3) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 2\mathbb{Z} + 1, \\ (4) \text{ of Fig. 4.5} & \text{if } g(\alpha_i) = 4\mathbb{Z}. \end{cases}$$

$$(1) \quad \times \square \times \quad (2) \quad \otimes \text{---} \otimes \quad (3) \quad \circ \text{---} \circ \quad (4) \quad \otimes \text{---} \otimes \text{---} \circ$$

$$i, 1 \quad i, 2 \quad i, 1 \quad i, 2 \quad i, 1 \quad i, 2 \quad i, 1 \quad i, 3 \quad i, 2$$

Fig. 4.5

Then  $\widehat{\Gamma}$  is given in the following way. In the following, we assume  $\vartheta(\alpha_0) \geq \vartheta(\alpha_1)$ . The cases where  $\vartheta(\alpha_0) < \vartheta(\alpha_1)$  can be treated similarly.

(i) If  $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 1$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.6.

$$\circ \leftarrow \circ \text{---} \circ \text{---} \dots \text{---} \circ \rightarrow \circ$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.6

(ii) If  $\vartheta(\alpha_0) = 2$  and  $\vartheta(\alpha_1) = 1$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.7.

$$\begin{array}{c} 0, 2 \\ \times \\ \parallel \\ \times \end{array} \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \rightarrow \circ$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.7

(iii) If  $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 2$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.8.

$$\begin{array}{c} 0, 2 \\ \times \\ \parallel \\ \times \end{array} \text{---} \circ \text{---} \circ \text{---} \dots \text{---} \circ \begin{array}{c} 1, 2 \\ \times \\ \parallel \\ \times \end{array}$$

$$0, 1 \quad l, 1 \quad l-1, 1 \quad 2, 1 \quad 1, 1$$

Fig. 4.8

(iv) If  $\vartheta(\alpha_0) = 3$  and  $\vartheta(\alpha_1) = 1$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.9.

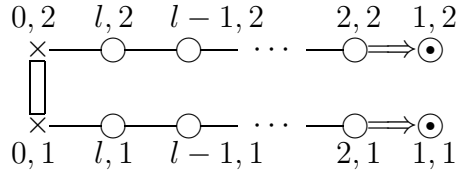


Fig. 4.9

(v) If  $\vartheta(\alpha_0) = 3$  and  $\vartheta(\alpha_1) = 2$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.10.

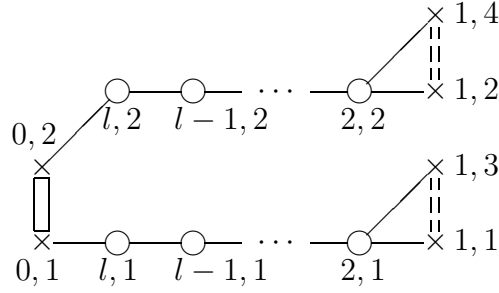


Fig. 4.10

(vi) If  $\vartheta(\alpha_0) = \vartheta(\alpha_1) = 3$ , then  $\widehat{\Gamma}$  is the one of Fig. 4.11.

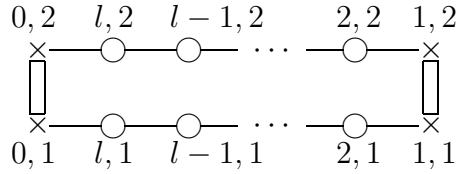


Fig. 4.11

**Example 4.1.** Assume  $l = 3$ . Let  $(\Pi = \{\alpha_0, \alpha_1, \alpha_2, \alpha_3\}, k, g) \in T_l^{\text{nm}}(D_{l+1}^{(2)})$ .

(i) If  $k(\alpha_0) = k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 1$ ,  $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$  and  $g(\alpha_1) = \mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (i) of Fig. 4.12.

(ii) If  $k(\alpha_0) = 1$ ,  $k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 2$ ,  $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = \mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (ii) of Fig. 4.12.

(iii) If  $k(\alpha_0) = k(\alpha_1) = 1$ ,  $k(\alpha_2) = k(\alpha_3) = 2$ ,  $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = 2\mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (iii) of Fig. 4.12.

(iv)\* If  $k(\alpha_0) = k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 1$ ,  $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = 2\mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (iv)\* of Fig. 4.12.

(v-1) If  $k(\alpha_0) = 1$ ,  $k(\alpha_1) = k(\alpha_2) = k(\alpha_3) = 2$ ,  $g(\alpha_0) = g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = 2\mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (v-1) of Fig. 4.12.

(v-2) If  $k(\alpha_0) = k(\alpha_1) = 1$ ,  $k(\alpha_2) = k(\alpha_3) = 2$ ,  $g(\alpha_0) = 2\mathbb{Z}$ ,  $g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = 4\mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (v-2) of Fig. 4.12.

(vi) If  $k(\alpha_0) = k(\alpha_2) = k(\alpha_3) = 2$ ,  $k(\alpha_1) = 1$ ,  $g(\alpha_0) = 2\mathbb{Z} + 1$ ,  $g(\alpha_2) = g(\alpha_3) = \emptyset$ , and  $g(\alpha_1) = 4\mathbb{Z}$ , then  $\widehat{\Gamma}$  is the one of (vi) of Fig. 4.12.

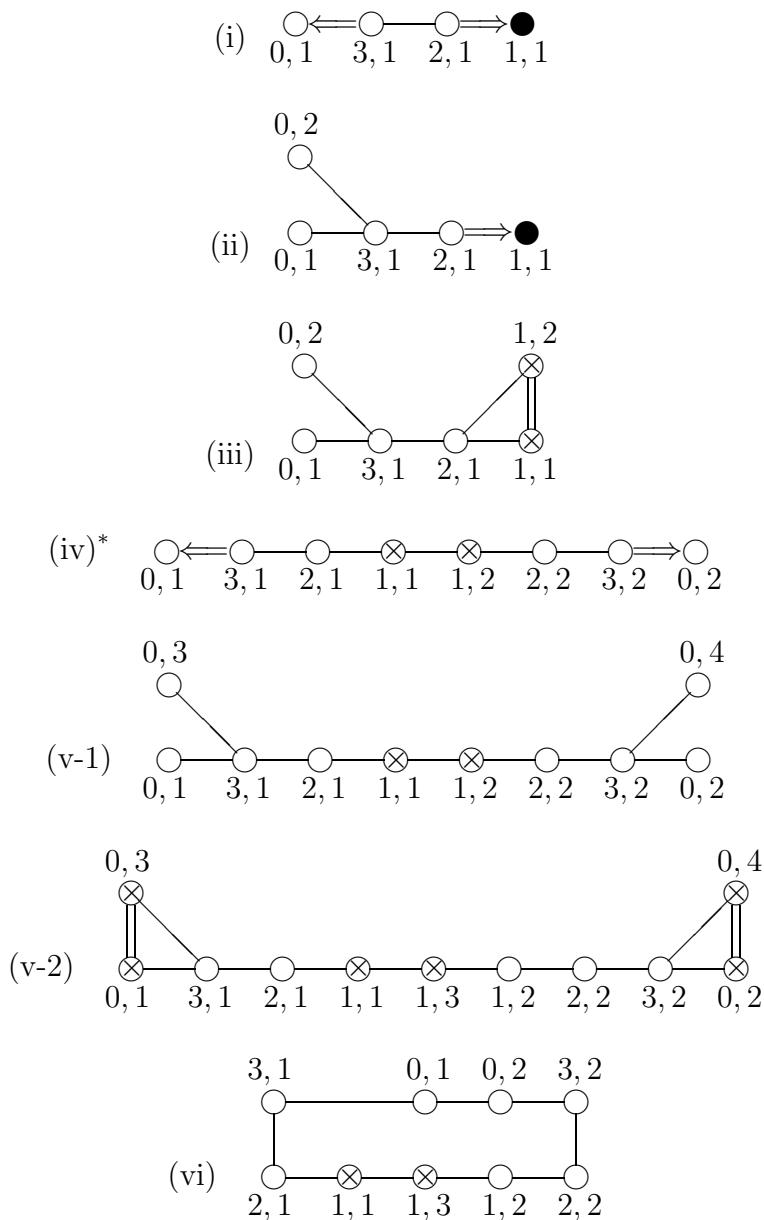


Fig. 4.12

In the following lemma, we let  $\bar{E}_{-(\alpha,x)} = \bar{F}_{(\alpha,x)} \in \widehat{\mathfrak{L}}$  (notice  $\bar{F}_{(\alpha,x)} \in \bar{\mathfrak{G}}(\widehat{A}, \widehat{I}, \widehat{I}^{\text{odd}}) \subset \widehat{\mathfrak{L}}$ ).

**Lemma 4.5.** *Let  $l \geq 2$  and  $(\Pi, k, g) \in T_l^{\text{norm}}(D_{l+1}^{(2)})$ . Let  $\zeta = \exp(\frac{\pi\sqrt{-1}}{4}) = \frac{1+\sqrt{-1}}{\sqrt{2}}$ . Then there exists a unique homomorphism  $\psi_1 : \mathfrak{g}(\Pi, k, g) \rightarrow \widehat{\mathfrak{L}}(\Pi, k, g)$  satisfying the following.*

$$\psi_1(E_{\pm\alpha}) = \begin{cases} \bar{E}_{\pm(\alpha,1)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 1, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1 \text{ and } k^\vee(\alpha) = 1, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1 \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)} + \bar{E}_{\pm(\alpha,3)} + \bar{E}_{\pm(\alpha,4)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4, \\ \sqrt{2}(\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) \in \{2\mathbb{Z} + 1, 2\mathbb{Z}\} \\ & \text{(this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{2}(\bar{E}_{\pm(\alpha,1)} \pm [\bar{E}_{\pm(\alpha,3)}, \bar{E}_{\pm(\alpha,2)}]) & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 4\mathbb{Z} \\ & \text{(this implies } k^\vee(\alpha) = 3\text{),} \end{cases}$$

$$\psi_1(E_{\pm\alpha^*}) = \begin{cases} \bar{E}_{\pm(\alpha,1)} \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 1, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{lg}} \text{ and } k^\vee(\alpha) = 2, \\ \bar{E}_{\pm(\alpha,1)} \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \emptyset \text{ and } k^\vee(\alpha) = 1, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \emptyset \text{ and } k^\vee(\alpha) = 2, \\ \pm\frac{1}{4}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,1)}] \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \mathbb{Z} \text{ and } k^\vee(\alpha) = 1, \\ \frac{\sqrt{-1}}{4}(-[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,1)}] + [\bar{E}_{\pm(\alpha,2)}, \bar{E}_{\pm(\alpha,2)}]) \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 1, g(\alpha) = \mathbb{Z} \text{ and } k^\vee(\alpha) = 2, \\ \pm\sqrt{-1}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm 1} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 2 \text{ (this implies } k(\alpha) = 1\text{),} \\ (\zeta^{\mp 3}\bar{E}_{\pm(\alpha,1)} + \zeta^{\mp 1}\bar{E}_{\pm(\alpha,2)} + \zeta^{\pm 1}\bar{E}_{\pm(\alpha,3)} + \zeta^{\pm 3}\bar{E}_{\pm(\alpha,4)}) \otimes t^{\pm 1} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 2 \text{ and } k^\vee(\alpha) = 4 \text{ (this implies } k(\alpha) = 1\text{),} \\ \pm\sqrt{-2}(-\bar{E}_{\pm(\alpha,1)} + \bar{E}_{\pm(\alpha,2)}) \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 2\mathbb{Z} \text{ (this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{-1}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,2)}] \otimes t^{\pm k(\alpha)} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 2\mathbb{Z} + 1 \text{ (this implies } k^\vee(\alpha) = 2\text{),} \\ \sqrt{2}(\bar{E}_{\pm(\alpha,2)} + \sqrt{-1}[\bar{E}_{\pm(\alpha,1)}, \bar{E}_{\pm(\alpha,3)}]) \otimes t^{\pm 1} & \text{if } \alpha \in \Pi_{\text{sh}}, \vartheta(\alpha) = 3 \text{ and } g(\alpha) = 4\mathbb{Z} \text{ (this implies } k^\vee(\alpha) = 3 \text{ and } k(\alpha) = 1\text{),} \end{cases}$$

$$\psi_1(h_{\alpha^\vee}) = \begin{cases} \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{lg}}, \\ \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha) \in \{1, 2\}, \\ 2 \sum_{x=1}^{k^\vee(\alpha)} \bar{h}_{(\alpha,x)} & \text{if } \alpha \in \Pi_{\text{sh}} \text{ and } \vartheta(\alpha) = 3, \end{cases}$$

and  $\psi_1(h_a) = \frac{k^\vee(\beta)(\beta,\beta)}{(\bar{\alpha}(\beta,1), \bar{\alpha}(\beta,1))} v$  ( $\beta \in \Pi_{\text{lg}}$ ),  $\psi_1(h_{\Lambda_a}) = w$ ,

$$\psi_1(h_{\Lambda_\delta}) = \begin{cases} \bar{t}_{(\alpha_0,1)} + \bar{t}_{(\alpha_0,2)} & \text{if } g(\alpha_0) = 4\mathbb{Z}, \\ \sum_{x=1}^{k^\vee(\alpha_0)} \bar{t}_{(\alpha_0,x)} & \text{otherwise.} \end{cases}$$

In particular,  $\dim \mathfrak{h}(\Pi, k, g) = l + 4$ .

This can be checked directly (see Lemma 4.3).

*Remark 4.3.* The ideas of  $k^\vee$ ,  $\widehat{A}$ ,  $\widehat{I}^{\text{odd}}$  and  $\psi_1$  have been inspired by explicit constructions of the affine Lie superalgebras in [13]. In fact, for  $\alpha, \beta \in \Pi$  with  $\alpha \neq \beta$  and  $(\alpha, \beta) \neq 0$ ,  $\psi_1(\mathfrak{g}^{\{\alpha, \beta\}})$  gives an explicit construction of the rank-two affine Lie superalgebra  $\mathfrak{g}^{\{\alpha, \beta\}}$  similar to that in [13, Theorem 6.11 (and Table 4)].

#### 4.5 From $A_{2l}^{(2)}$ to $D_{l+1}^{(2)}$

Let  $l \geq 2$  and  $(\Pi = \{\alpha_0, \dots, \alpha_l\}, k, g) \in T_l^{\text{norm}}(A_{2l}^{(2)})$ . Notice that  $\Pi_{\text{sh}} = \{\alpha_1\}$ ,  $\Pi_{\text{lg}} = \{\alpha_2, \dots, \alpha_l\}$  and  $\Pi_{\text{ex}} = \{\alpha_0\}$ . We define  $(\Pi' = \{\alpha'_0, \dots, \alpha'_l\}, k', g') \in T_l^{\text{norm}}(D_{l+1}^{(2)})$  to be as follows. Let

$$\begin{cases} \beta'_i = \alpha'_i \ (0 \leq i \leq l) & \text{if } g(\alpha_1) \in \{2\mathbb{Z}, 4\mathbb{Z}\} \\ \beta'_0 = \alpha'_1, \beta'_1 = \alpha'_0, \beta'_i = \alpha'_{l-i+2} \ (2 \leq i \leq l) & \text{if } g(\alpha_1) \in \{2\mathbb{Z} + 1, \mathbb{Z}\} \end{cases}$$

and  $m_0 = 1 + \delta_{k(\alpha_0), 1}$ . Then we let

$$k'(\beta'_j) = \frac{k(\alpha_j)m_0}{1 + \delta_{j,0}}$$

( $0 \leq j \leq l$ ) and

$$g'(\beta'_r) = \begin{cases} 2\mathbb{Z} & \text{if } r = 0, \\ g(\alpha_r) & \text{if } 1 \leq r \leq l. \end{cases}$$

Denote  $a$  and  $\delta$  for  $(\Pi', k', g')$  by  $a'$  and  $\delta'$  respectively. Assume  $(\beta'_1, \beta'_1) = (\alpha_1, \alpha_1)$ . Let

$$\Lambda'_{\delta'} = \begin{cases} \Lambda_{\delta'} & \text{if } \beta'_0 = \alpha'_0, \\ \Lambda_{\delta'} - \frac{l}{2(\alpha'_1, \alpha'_1)} \delta' + \sum_{i=1}^l (l - i + 1) \alpha'_i & \text{if } \beta'_0 = \alpha'_1. \end{cases}$$

**Lemma 4.6.** *Keep the notation as above. Then there exists a homomorphism  $\psi_2 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$  such that  $\psi_2(E_{\pm\alpha_0}) = \pm\frac{1}{4}[E_{\pm\beta'_0}, E_{\pm\beta'_0}]$ ,  $\psi_2(E_{\pm\alpha_0^*}) = \pm\frac{1}{4}[E_{\pm(\beta'_0)^*}, E_{\pm(\beta'_0)^*}]$ ,  $\psi_2(E_{\pm\alpha_i}) = E_{\pm\beta'_i}$ ,  $\psi_2(E_{\pm\alpha_i^*}) = E_{\pm(\beta'_i)^*}$  ( $1 \leq i \leq l$ ),  $\psi_2(h_{\alpha_0^\vee}) = \frac{1}{2}h_{(\beta'_0)^\vee}$ ,  $\psi_2(h_{\alpha_i^\vee}) = h_{(\beta'_i)^\vee}$ , ( $1 \leq i \leq l$ ),  $\psi_2(h_a) = m_0 h_{a'}$ ,  $\psi_2(h_{\Lambda_a}) = \frac{1}{m_0} h_{\Lambda_{a'}}$  and  $\psi_2(h_{\Lambda_\delta}) = \frac{1}{2} h_{\Lambda_{\delta'}}$ .*

*Moreover,  $\dim \mathfrak{h}(\Pi, k, g) = l + 4$ .*

**Proof.** The first statement can be proved directly. The second statement follows from Lemma 4.5 and the fact that  $\psi_2(\mathfrak{h}(\Pi, k, g)) = \mathfrak{h}(\Pi', k', g')$ .  $\square$

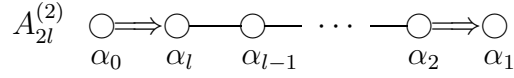


Fig. 4.13

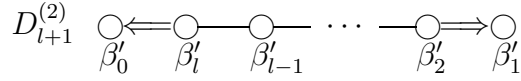


Fig. 4.14

In the following table,  $\beta \in \Pi_{\text{lg}}$  and  $\beta' \in \Pi'_{\text{lg}}$ .

$k(\alpha_0)$	$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$	$k'(\beta'_0)$	$g'(\beta'_0)$	$k'(\beta')$	$k'(\beta'_1)$	$g'(\beta'_1)$
1	1	1	$2\mathbb{Z} + 1$	1	$2\mathbb{Z}$	2	2	$2\mathbb{Z} + 1$
2	1	1	$2\mathbb{Z} + 1$	1	$2\mathbb{Z}$	1	1	$2\mathbb{Z} + 1$
1	1	1	$\mathbb{Z}$	1	$2\mathbb{Z}$	2	2	$\mathbb{Z}$
2	1	1	$\mathbb{Z}$	1	$2\mathbb{Z}$	1	1	$\mathbb{Z}$
1	1	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	2	2	$2\mathbb{Z}$
2	1	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	1	1	$2\mathbb{Z}$
2	2	1	$2\mathbb{Z}$	1	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$
4	2	1	$2\mathbb{Z}$	2	$2\mathbb{Z}$	2	1	$2\mathbb{Z}$
2	2	1	$4\mathbb{Z}$	1	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$
4	2	1	$4\mathbb{Z}$	2	$2\mathbb{Z}$	2	1	$4\mathbb{Z}$

Table 4.2.

## 4.6 From $B_l^{(1)}$ to $D_{l+1}^{(2)}$

Let  $l \geq 3$  and  $(\Pi, k, g) \in T_l^{\text{norm}}(B_l^{(1)})$ . Notice that  $\Pi_{\text{sh}} = \{\alpha_1\}$  and  $\Pi_{\text{lg}} = \{\alpha_0, \alpha_2, \dots, \alpha_l\}$ . We let  $(\Pi' = \{\alpha'_0, \dots, \alpha'_l\}, k', g') \in T_l^{\text{norm}}(D_{l+1}^{(2)})$  be such that  $k'(\alpha'_i) = k(\alpha_i)$  and  $g'(\alpha'_i) = g(\alpha_i)$  ( $0 \leq i \leq l$ ). Denote  $a$  and  $\delta$  for  $(\Pi', k', g')$  by  $a'$  and  $\delta'$  respectively. Assume  $(\alpha'_1, \alpha'_1) = (\alpha_1, \alpha_1)$ .

**Lemma 4.7.** *Keep the notation as above. Then there exists a homomorphism  $\psi_3 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$  such that  $\psi_3(E_{\pm\alpha_0}) = \frac{1}{2}[E_{\pm\alpha'_0}, [E_{\pm\alpha'_0}, E_{\pm\alpha'_1}]]$ ,  $\psi_3(E_{\pm\alpha_0^*}) = \frac{1}{2}[E_{\pm\alpha'_0}, [E_{\pm\alpha'_0}, E_{\pm(\alpha'_1)^*}]]$ ,  $\psi_3(E_{\pm\alpha_i}) = E_{\pm\alpha'_i}$ ,  $\psi_3(E_{\pm\alpha_i^*}) = E_{\pm(\alpha'_i)^*}$  ( $1 \leq i \leq l$ ),  $\psi_3(h_{\alpha_0^\vee}) = h_{(\alpha'_0)^\vee} + h_{(\alpha'_1)^\vee}$ ,  $\psi_3(h_{\alpha_i^\vee}) = h_{(\alpha'_i)^\vee}$  ( $1 \leq i \leq l$ ),  $\psi_3(h_a) = h_{a'}$ ,  $\psi_3(h_{\Lambda_a}) = h_{\Lambda_{a'}}$  and  $\psi_3(h_{\Lambda_\delta}) = \frac{1}{2}h_{\Lambda_{\delta'}}$ .*

Moreover,  $\dim \mathfrak{h}(\Pi, k, g) = l + 4$ .

**Proof.** By (3.1) and (3.2), for  $\gamma' \in \{\alpha'_l, (\alpha'_l)^*\}$ , we have  $n_{\alpha'_0}(E_{\pm\gamma'}) = \frac{1}{2}[E_{\pm\alpha'_0}, [E_{\pm\gamma'}, E_{\pm\gamma'}]]$  and  $[n_{\alpha'_0}(E_{\gamma'}), n_{\alpha'_0}(E_{-\gamma'})] = h_{(\alpha'_0)^\vee} + h_{(\gamma')^\vee}$ . Then the first statement follows from direct calculations and (3.1). The second statement follows from Lemma 4.5 and the fact that  $\psi_3(\mathfrak{h}(\Pi, k, g)) = \mathfrak{h}(\Pi', k', g')$ .  $\square$

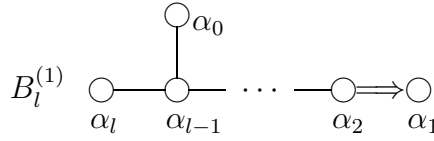


Fig. 4.15

In the following table,  $\beta \in \Pi_{\text{lg}}$ .

$k(\beta)$	$k(\alpha_1)$	$g(\alpha_1)$
1	1	$2\mathbb{Z} + 1$
1	1	$\mathbb{Z}$
1	1	$2\mathbb{Z}$
2	1	$2\mathbb{Z}$
2	1	$4\mathbb{Z}$

Table 4.3.

## 4.7 From $C_2^{(1)}$ to $B_3^{(1)}$ or $F^{(1)}(4)$

Notice that if  $(\Pi, k, g) \in T_2^{\text{nm}}(C_2^{(1)})$  ( $l = 2$ ), then  $\Pi_{\text{sh}} = \{\alpha_1\}$  and  $\Pi_{\text{lg}} = \{\alpha_0, \alpha_2\}$ .

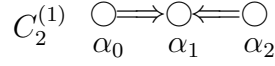


Fig. 4.16

$k(\alpha_0)$	$k(\alpha_1)$	$g(\alpha_1)$	$k(\alpha_2)$
1	1	$2\mathbb{Z} + 1$	1
1	1	$\mathbb{Z}$	1
1	1	$2\mathbb{Z}$	1
2	1	$2\mathbb{Z}$	1
2	1	$2\mathbb{Z}$	2
2	1	$4\mathbb{Z}$	2

Table 4.4

Let  $(\Pi, k, g) \in T_2^{\text{nm}}(C_2^{(1)})$  be such that  $k(\alpha_0) = k(\alpha_2)$ . For such  $(\Pi, k, g)$ , let  $(\Pi' = \{\alpha'_0, \alpha'_1, \alpha'_2, \alpha'_3\}, k', g') \in T_3^{\text{nm}}(B_3^{(1)})$  be such that  $k'(\alpha'_0) = k'(\alpha'_2) = k'(\alpha'_3) = k(\alpha_0)$ ,  $k'(\alpha'_1) = 1$ ,  $g'(\alpha'_0) = g'(\alpha'_2) = g'(\alpha'_3) = \emptyset$  and  $g'(\alpha'_1) = g(\alpha_1)$ . Denote  $a$  and  $\delta$  for  $(\Pi', k', g')$  by  $a'$  and  $\delta'$  respectively. Assume  $(\alpha'_1, \alpha'_1) = (\alpha_1, \alpha_1)$ .

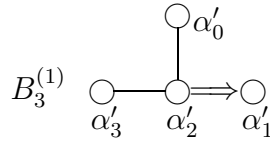


Fig. 4.17

**Lemma 4.8.** *Keep the notation as above. Then there exists a homomorphism  $\psi_4 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{g}(\Pi', k', g')$  such that  $\psi_4(E_{\pm\alpha_1}) = E_{\pm\alpha'_1}$ ,  $\psi_4(E_{\pm\alpha_1^*}) = E_{\pm(\alpha'_1)^*}$ ,  $\psi_4(E_{\pm\alpha_0}) = \pm[E_{\pm\alpha'_0}, E_{\pm\alpha'_2}]$ ,  $\psi_4(E_{\pm\alpha_2}) = \pm[E_{\pm\alpha'_3}, E_{\pm\alpha'_2}]$ ,  $\psi_4(E_{\pm\alpha_0^*}) = \pm[E_{\pm\alpha'_0}, E_{\pm(\alpha'_2)^*}]$ ,  $\psi_4(E_{\pm\alpha_2^*}) = \pm[E_{\pm\alpha'_3}, E_{\pm(\alpha'_2)^*}]$ ,  $\psi_4(h_{\alpha_0^\vee}) = h_{(\alpha'_0)^\vee} + h_{(\alpha'_2)^\vee}$ ,  $\psi_4(h_{\alpha_2^\vee}) = h_{(\alpha'_3)^\vee} + h_{(\alpha'_2)^\vee}$ ,  $\psi_4(h_{\alpha_1^\vee}) = h_{(\alpha'_1)^\vee}$ ,  $\psi_4(h_a) = h_{a'}$ ,  $\psi_4(h_{\Lambda_a}) = h_{\Lambda_{a'}}$  and  $\psi_4(h_{\Lambda_\delta}) = h_{\Lambda_{\delta'}}$ .*

Moreover,  $\dim \mathfrak{h}(\Pi, k, g) = 6$ .

**Proof.** This can be proved directly in a way similar to Proof of Lemma 4.7, together with the formula  $n_{\alpha'_i}(E_{\pm\alpha'_i}) = \pm[E_{\pm\alpha'_i}, E_{\pm\alpha'_i}]$  and  $n_{\alpha'_i}(E_{\pm(\alpha'_2)^*}) = \pm[E_{\pm\alpha'_i}, E_{\pm(\alpha'_2)^*}]$  ( $i \in \{0, 3\}$ ) (see also (3.1) and (3.2)) and Lemma 4.7.  $\square$

Let  $(\Pi, k, g) \in T_2^{\text{nm}}(C_2^{(1)})$  be such that  $k(\alpha_0) \neq k(\alpha_2)$ . We have assumed  $k(\alpha_0) > k(\alpha_1)$ . Then we see that such  $(\Pi, k, g)$  exists uniquely and is such that  $k(\alpha_1) = k(\alpha_2) = 1$ ,  $k(\alpha_0) = 2$ ,  $g(\alpha_1) = 2\mathbb{Z}$  and  $g(\alpha_0) = g(\alpha_2) = \emptyset$ .

Let  $\bar{I} = \{0, 1, 2, 3, 4\}$  and  $\bar{I}^{\text{odd}} = \{1, 2\}$ . Define a  $5 \times 5$  matrix  $\bar{A}$  by

$$\bar{A} = \begin{pmatrix} 2 & -1 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & -1 \\ 0 & 2 & 0 & -2 & -1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -1 & -1 & 0 & 2 \end{pmatrix}.$$

The Dynkin diagram of  $\bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  is given in Fig. 4.18. We notice that  $\bar{\mathfrak{G}}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  is the affine Lie superalgebra  $F^{(1)}(4)$ .

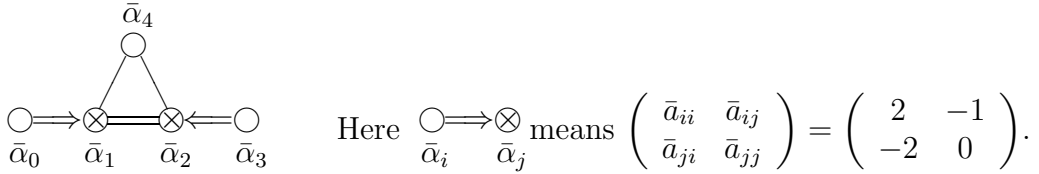


Fig. 4.18

**Lemma 4.9.** *Keep the notation as above. Then there exists a homomorphism  $\psi_5 : \mathfrak{g}(\Pi, k, g) \rightarrow \mathfrak{L}(\bar{A}, \bar{I}, \bar{I}^{\text{odd}})$  such that  $\psi_5(E_{\alpha_2}) = \bar{E}_0 + \bar{E}_3$ ,  $\psi_5(E_{-\alpha_2}) = \bar{F}_0 + \bar{F}_3$ ,  $\psi_5(E_{\alpha_1}) = \bar{E}_1 + \bar{E}_2$ ,  $\psi_5(E_{-\alpha_1}) = \bar{F}_1 + \bar{F}_2$ ,  $\psi_5(E_{\alpha_0}) = \bar{E}_4$ ,  $\psi_5(E_{-\alpha_0}) = \bar{F}_4$ ,  $\psi_5(E_{\alpha_2^*}) = -\sqrt{-1}(\bar{E}_0 - \bar{E}_3) \otimes t$ ,  $\psi_5(E_{-\alpha_2^*}) = \sqrt{-1}(\bar{F}_0 - \bar{F}_3) \otimes t^{-1}$ ,  $\psi_5(E_{\alpha_1^*}) = -\sqrt{-1}(\bar{E}_1 - \bar{E}_2) \otimes t$ ,  $\psi_5(E_{-\alpha_1^*}) = \sqrt{-1}(\bar{F}_1 - \bar{F}_2) \otimes t^{-1}$ ,  $\psi_5(E_{\alpha_0^*}) = \bar{E}_4 \otimes t^2$ ,  $\psi_5(E_{-\alpha_0^*}) = \bar{F}_4 \otimes t^{-2}$ ,  $\psi_5(h_{\alpha_2^\vee}) = \bar{h}_0 + \bar{h}_3$ ,  $\psi_5(h_{\alpha_0^\vee}) = \bar{h}_4$ ,  $\psi_5(h_{\alpha_1^\vee}) = \bar{h}_1 + \bar{h}_2$ ,  $\psi_5(h_{\Lambda_a}) = w$ ,  $\psi_5(h_{\Lambda_\delta}) = \bar{t}_4$  and  $\psi_5(h_a) = \frac{(\alpha_0, \alpha_0)}{(\bar{\alpha}_4, \bar{\alpha}_4)} v$ .*

Moreover,  $\dim \mathfrak{h}(\Pi, k, g) = 6$ .

This can be proved directly (see Lemma 4.3).

## 4.8 Proof of Lemma 3.1 and Theorem 1.1 (2), (3)

**Proof of Lemma 3.1.** The lemma follows immediately from Lemmas 4.5-4.9.  $\square$

**Proof of Theorem 1.1 (2).** Recall the maps  $\psi_i$  ( $1 \leq i \leq 5$ ) from Lemmas 4.5-4.9. Let  $(\Pi, k, g) \in T_l^{\text{nr}}m$  and define a super-symmetric invariant form  $(, )' : \mathfrak{g}(\Pi, k, g) \times \mathfrak{g}(\Pi, k, g) \rightarrow \mathbb{C}$  by

$$(, )' = \begin{cases} (, ) \circ (\psi_1 \times \psi_1) & \text{if } (\Pi, k, g) \in T_l^{\text{nr}}m(D_{l+1}^{(2)}) \ (l \geq 2), \\ (, ) \circ ((\psi_1 \circ \psi_2) \times (\psi_1 \circ \psi_2)) & \text{if } (\Pi, k, g) \in T_l^{\text{nr}}m(A_{2l}^{(2)}) \ (l \geq 2), \\ (, ) \circ ((\psi_1 \circ \psi_3) \times (\psi_1 \circ \psi_3)) & \text{if } (\Pi, k, g) \in T_l^{\text{nr}}m(B_l^{(1)}) \ (l \geq 3), \\ (, ) \circ ((\psi_1 \circ \psi_3 \circ \psi_4) \times (\psi_1 \circ \psi_3 \circ \psi_4)) & \\ \quad \text{if } (\Pi, k, g) \in T_l^{\text{nr}}m(C_2^{(1)}) \ (l = 2) \text{ and } k(\alpha_0) = k(\alpha_1), \\ (, ) \circ (\psi_5 \times \psi_5) & \text{if } (\Pi, k, g) \in T_l^{\text{nr}}m(C_2^{(1)}) \ (l = 2) \text{ and } k(\alpha_0) > k(\alpha_1), \end{cases}$$

where  $(, )$  of RHS is the one of (4.1). Then we see that there exists  $c \in \mathbb{C} \setminus \{0\}$  such that  $c(, )'$  has the same property as in (1.6). This completes the proof.  $\square$

**Proof of Theorem 1.1 (3).** By the same argument as in the proof of [6, Theorem 2.2], we see that for  $\mu \in R$ , there exist  $E'_{\pm\mu} \in \mathfrak{g}'_{\mu}$  such that

$$(4.3) \quad [E'_{\mu}, E'_{-\mu}] = h_{\mu^{\vee}}.$$

By (3-ii), we see that  $h_{\sigma}$  ( $\sigma \in \mathcal{E}$ ) and  $E'_{\pm\mu}$  ( $\mu \in \mathcal{B}_+(\subset R)$ ) satisfy the equalities (SR1-5,8,9) (see also (3.4) and (2.8)). For  $\mu \in \mathcal{B}_+$ , define  $n'_{\mu} \in \text{Aut}(\mathfrak{g})$  in the same way as that for  $n_{\mu}$  (see (3.3)) by using  $E'_{\pm\mu}$  in place of  $E_{\pm\mu}$ . For  $\sigma \in \mathcal{E}$ , let  $\mathfrak{g}'_{\sigma} = \{X' \in \mathfrak{g}' \mid [h_{\mu}, X'] = (\sigma, \mu)X' \ (\mu \in \mathcal{E})\}$ . Let  $(\alpha, \beta, y) \in \mathcal{A}$ . By (3-ii),  $\dim \mathfrak{g}'_{\pm\beta} = 1$ . By (4.3), there exists  $c_{\alpha, \beta} \in \mathbb{C} \setminus \{0\}$  such that  $E'_{\pm\beta} = c_{\alpha, \beta}^{\pm 1} (n'_{\alpha^*})^{-1} n'_{\alpha} (E'_{\pm\beta^*})$ . Since  $n'_{\alpha^*} (E'_{\pm\beta}) = c_{\alpha, \beta}^{\pm 1} n'_{\alpha} (E'_{\pm\beta^*})$ , by the same equalities as those in (3.2), we have  $(\pm 1)^{c(\alpha)+1} c(\alpha) (\text{ad} E'_{\pm\alpha^*})^y E'_{\pm\beta} = c_{\alpha, \beta}^{\pm 1} (\text{ad} E'_{\pm\alpha})^{c(\alpha)y} E'_{\pm\beta^*}$  (see (1.4) for  $c(\alpha)$ ). Then, since  $A(\Pi)$  is not  $A_l^{(1)}$ , we can normalize  $E'_{\pm\mu}$  ( $\mu \in \mathcal{B}_+$ ) by scalar multiples so that they also satisfy the equalities (SR6,7). Then there exists a homomorphism  $f : \mathfrak{g} \rightarrow \mathfrak{g}'$  such that  $f(E_{\pm\mu}) = E'_{\pm\mu}$  ( $\mu \in \mathcal{B}_+$ ) and  $f(h_{\sigma}) = h_{\sigma}$  ( $\sigma \in \mathcal{E}$ ).

We show that  $f$  is surjective. By Lemma 3.2, we see that  $W_{R_\alpha} = W_{\{\alpha, -\alpha^*\}}$ . By (2.6), we see that

$$R(\Pi, k, g) = W\mathcal{B}_+ \cup W\{2\mu \mid \mu \in \mathcal{B}_+, p(\mu) = 1\}$$

and  $W = W_{\mathcal{B}_+}$ . For  $\mu \in \mathcal{B}_+$  with  $p(\mu) = 1$ , we can see  $[[E'_\mu, E'_\mu], [E'_{-\mu}, E'_{-\mu}]] = -8h_{\mu^\vee} \neq 0$ , which implies  $\mathbb{C}[E'_{\pm\mu}, E'_{\pm\mu}] = \mathfrak{g}'_{\pm 2\mu}$ . Using  $n'_\mu$ 's ( $\mu \in \mathcal{B}_+$ ), we conclude that  $\mathfrak{g}'$  is generated by  $\mathfrak{h}$  and  $E'_{\pm\mu}$  ( $\mu \in \mathcal{B}_+$ ). Hence  $f$  is surjective. This completes the proof.  $\square$

*Remark 4.4.* (1) K. Saito [10] introduced a notion of elliptic root bases of the reduced-marked elliptic root systems. K. Saito and D. Yoshii [12] gave defining relations of the simply-laced elliptic Lie algebras in terms of the elliptic root bases. The author [15] gave defining relations of  $\mathfrak{g}(\Pi, k, g)$  with  $(\Pi, k, g) \in T_l^{\text{rm}}$  in terms of the elliptic root base. We can extend the notion to that for  $R(\Pi, k, g)$  with  $(\Pi, k, g) \in T_l^{\text{nrmm}}$  and we can give defining relations written in terms of that notion. The result will appear elsewhere.

(2) Let  $(\Pi, k, g) \in T_l$  with  $l \geq 2$ . Let  $\mathfrak{g}' = [\mathfrak{g}(\Pi, k, g), \mathfrak{g}(\Pi, k, g)]$  and let  $\mathfrak{z}$  be the center of  $\mathfrak{g}'$ . Using our defining relations, we can show that the canonical map from  $\mathfrak{g}'$  to  $\mathfrak{g}'/\mathfrak{z}$  is a universal central extension map. Using the map, we can discuss how our algebras relate with the extended affine Lie algebras in [1] and the (2-variable) toroidal Lie algebras in [9]. The result will appear elsewhere.

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